4 CLASSICAL FILTER RESPONSE

It is good to rub and polish our minds against those of others

In Chap. 2, we considered frequency domain approximations and performed some preliminary filter designs. In Chap. 3, we designed filters on the basis of time domain approximations. The quality of the designs depended largely on our engineering knowledge, intuition, and experience. We now want to obtain filter designs using systematic design procedures. In this chapter, we consider classical transfer functions which have been derived and their frequency and time domain behavior. In the next chapter, we consider optimum transfer functions which have been created to meet certain optimization criteria. In these two chapters we will be considering only low-pass filters. The transformations to be discussed in Chap. 6 allow high-pass, band-pass, band-stop, and all-pass filters to be similarly designed using these results.

We begin by considering the ideal low-pass filter. Since we will see it is nonrealizable, we can only design filters which approximate the ideal behavior. A wide variety of filters have been developed which approximate the ideal frequency response either in magnitude or delay (i.e., phase). Some filters attempt to realize both magnitude and delay responses simultaneously and are called transitional types. The filters considered in this chapter are classical filters. By this we mean that their gain expressions are directly related to well-known classical polynomials or curves. A wide variety of classical filters will be investigated. Their magnitude, delay, and step responses will be tabulated. In addition, a series of nomographs will be introduced which facilitate the rapid determination of the required filter order to meet arbitrary frequency domain requirements. This chapter will forge an indispensable link in the chain of understanding for the engineer. It will also give him a wide variety of standard filter response forms. From these, he may select the one best suited to meet his particular requirements.

4.1 IDEAL LOW-PASS FILTERS

The magnitude response of a general low-pass filter has the form shown in Fig. 4.1.1. In quantitatively describing magnitude responses, the following terminology is often used:

1. **Passband**: Frequencies over which response is within $M_p$ dB of its maximum value ($\omega < \omega_p$).
2. **Stopband**: Frequencies over which response is attenuated at least $M_s$ dB from its maximum value ($\omega > \omega_s$).
3. **Transition Band**: Intermediate frequencies lying between passband and stopband ($\omega_p < \omega < \omega_s$).
4. **Band-edge Frequency** (Cutoff Frequency): Frequency $\omega_p$ marking edge of passband.
5. **Stopband Frequency**: Frequency $\omega_s$ marking edge of stopband.
6. **$M_p$ Bandwidth**: Equals band-edge frequency.
7. **3 dB Bandwidth (3 dB Frequency):** Maximum frequency at which response is down 3 dB from its maximum value (\( \omega = \omega_{3\text{dB}} = B \)).

An ideal low-pass filter would have infinite rejection in its stopband, a transition band of zero width, and constant gain in its passband. The ideal low-pass filter response having unity gain and unity bandwidth is shown in Fig. 1.5.2a. It is assumed to have linear phase of slope \(-\tau_o\) (i.e., constant delay) through the passband. Thus, the gain equals

\[
H(j\omega) = e^{-j\omega\tau_o}, \ |\omega| < 1 \quad \text{and} \quad 0, \ |\omega| > 1
\]  

(4.1.1)

No ratio of finite degree polynomials (i.e., a rational function) will realize the filter gain magnitude exactly. In general, however, the higher the degree of the gain function, the better will be the approximation. This chapter shall be concerned with the various approaches for approximating the ideal low-pass filter.

### 4.2 MAXIMALLY FLAT MAGNITUDE FILTERS

Suppose that we approximate the squared magnitude of the ideal filter by the ratio of two polynomials of finite degree where

\[
|H(j\omega)|^2 = \frac{1 + a_1\omega^2 + a_2\omega^4 + \ldots + a_n\omega^{2n}}{1 + b_1\omega^2 + b_2\omega^4 + \ldots + b_n\omega^{2n}}
\]  

(4.2.1)

Since the magnitude function is always even, the two polynomials must also be even. Since gain \(|H|\) approaches zero as \(\omega\) approaches infinity, then order \(m < n\). To obtain as “square” a magnitude characteristic as possible, we want to choose the \(a_i\) and \(b_i\) coefficients so as many derivatives of \(|H|^2\) equal zero at \(\omega = 0\) as possible. Expanding \(|H|^2\) in a Maclaurin series gives

\[
|H(j\omega)|^2 = G(0) + G'(0)\omega + G''(0)\omega^2/2! + G'''(0)\omega^3/3! + \ldots
\]  

(4.2.2)

where the derivatives equal

\[
G^{(n)}(0) = d^n|H(j\omega)|^2/d\omega^n \bigg|_{\omega=0}
\]  

(4.2.3)

Therefore, we obtain a maximally flat magnitude (MFM) characteristic by setting \(G^{(n)}(0) = 0\) starting with \(n = 1\) and following with as many consecutive terms as possible. By long division of \(|H|^2\) given by Eq. 4.2.1, then

\[
|H(j\omega)|^2 = 1 + (a_1 - b_2)\omega^2 + [(a_2 - b_2) - b_2(a_1 - b_1)]\omega^4 + \ldots
\]  

(4.2.4)

Equating coefficients in Eqs. 4.2.2 and 4.2.5, \((n - 1)\) terms may be set equal to zero. Thus, we set

\[
b_n\omega^{2n}
\]  

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\[
a_1
\]

Therefore,

\[|H|\]
SEC. 4.3 BUTTERWORTH FILTERS

\[ a_i = b_i, \text{ } i = 1, 2, \ldots, m; \quad b_i = 0, \text{ } i = m + 1, \ldots, n - 1; \quad \text{and} \quad b_n \neq 0 \quad (4.2.5) \]

Therefore, the denominator polynomial of \(|H|^2\) equals the numerator polynomial plus the term \(b_n \omega^{2n}\) for an MFM characteristic.

**EXAMPLE 4.2.1** Determine the coefficients so that

\[ H(s) = \frac{1}{1 + b_1 s + b_2 s^2} \quad (4.2.6) \]

is an MFM gain function with a 3 dB corner frequency of 1 rad/sec.

**Solution** Since the squared magnitude of the gain function equals

\[ |H(\omega)|^2 = \frac{1}{(1 - b_2 \omega^2)^2 + b_1^2 \omega^2} = \frac{1}{1 + (b_1^2 - 2b_2)\omega^2 + b_2^2 \omega^4} \quad (4.2.7) \]

we set

\[ b_1 = \sqrt{2}b_2, \quad b_2 \neq 0 \quad (4.2.8) \]

\(b_2\) must be chosen so that the 3 dB corner frequency is unity. Setting \(|H(1)|^2 = \frac{1}{2}\) requires \(b_2 = 1\) so \(b_1 = \sqrt{2}\). Thus, the poles lie on a circle of unity radius and have a \(\phi = 0.707\) or a \(\pm 45^\circ\) angle from the negative-real axis. This result is substantiated by inspection of Fig. 2.4.1.

**EXAMPLE 4.2.2** Determine the coefficients so that

\[ H(s) = \frac{1 + a_1 s}{1 + b_1 s + b_2 s^2} \quad (4.2.9) \]

is an MFM gain function with a 3 dB corner frequency of 1 rad/sec.

**Solution** Since the squared magnitude of the gain function equals

\[ |H(\omega)|^2 = \frac{1 + a_1^2 \omega^2}{1 + (b_1^2 - 2b_2)\omega^2 + b_2^2 \omega^4} \quad (4.2.10) \]

for an MFM gain, we set

\[ b_1^2 - 2b_2 = a_1^2, \quad b_2 \neq 0 \quad (4.2.11) \]

Normalizing the 3 dB frequency to unity requires

\[ b_2^2 \omega^4/(1 + a_1^2 \omega^2) \quad |\omega = 1 \quad \text{or} \quad a_1^2 = b_2^2 - 1 \quad (4.2.12) \]

Thus, we have obtained two equations (Eqs. 4.2.11 and 4.2.12) in three unknowns \((a_1, b_1\) and \(b_2\). Therefore, we can specify one of these parameters independently. Root locus analysis shows that the primary difference in these combinations is the form of their transition band responses.

4.3 BUTTERWORTH FILTERS

Butterworth filters\(^1\) are the class of MFM filters which have all of their transmission zeros at infinity. From Eq. 4.2.5, this requires

\[ a_1 = a_2 = \ldots = a_m = 0, \quad b_{m+1} = b_{m+2} = \ldots = b_{n-1} = 0, \quad \text{and} \quad b_n \neq 0 \quad (4.3.1) \]

Therefore, the Butterworth filter has a gain with a squared magnitude of

\[ |H(\omega)|^2 = \frac{1}{1 + \omega^{2n}} \quad (4.3.2) \]
where the 3 dB frequency is normalized to unity by setting $b_n = 1$. Sometimes $|H|^2$ is called a Butterworth function. The denominator polynomials which are derived from it are called the Butterworth polynomials; thus, its network realizations or implementations are called Butterworth filters. The pole locations for $H(s)$ are determined using analytic continuation. This consists of noting that the squared magnitude function equals

$$H(s)H(-s) = |H(s)|^2 \big|_{s=j\omega} = |H(j\omega)|^2 \big|_{\omega = s/-j} = \frac{1}{1 + (-j\omega)^{2n}}$$

(4.3.3)

Therefore, we find the product by analytic continuation and select the left-half-plane pole pattern for $H(s)$; the right-half-plane poles constitute $H(-s)$. From Eq. 4.3.2, then

$$H(s)H(-s) = \frac{1}{1 + (-s)^{2n}} = \frac{1}{1 + (-1)^{n}s^{2n}}$$

(4.3.4)

Thus, the poles of $H(s)H(-s)$ satisfy $s_k^2 = -1$ for $n$ even and $+1$ for $n$ odd. The poles have magnitudes $|s_k| = 1$ and angles of $\pi/n$

$$\arg s_k = \frac{1}{2n}\pi n, \quad n \text{ even}$$

$$= \pm \frac{2\pi}{2n}, \quad n \text{ odd}$$

(4.3.5)

They are symmetrically distributed around a unit circle with a separation of $\pi/n$ radians. The first pole is located $\pi/2n$ radians from the real axis for $n$ even, and on the real axis for $n$ odd. The left-half-plane pole locations for $n = 2, 3, 4,$ and $5$ are shown in Fig. 4.3.1. The transfer function for the Butterworth filter equals

$$H(s) = \frac{1}{1 + b_1s + b_2s^2 + \ldots + b_{n-1}s^{n-1} + b_ns^n}$$

(4.3.6)

where the denominator $D(s)$ equals the product of the pole terms so

$$D(s) = \frac{n/2}{k=1} (s^2 + 2\cos \theta_k s + 1), \quad n \text{ even}$$

$$= (s + 1)^{1/2} (s^2 + 2\cos \theta_k s + 1), \quad n \text{ odd}$$

(4.3.7)

where the $\theta_k$'s are given by Eq. 4.3.5. The denominator polynomials for Butterworth filters of various orders are listed in Table 4.3.1. For simplicity, these are often called Butterworth polynomials. When the order of the filter is specified, the required polynomial is simply taken from the table. Both factored and nonfactored forms are listed for later computational convenience.

Since the Butterworth filter has a gain of

$$|H(j\omega)| = \frac{1}{\sqrt{1 + \omega^{2n}}}$$

(4.3.8)
Table 4.3.1 Butterworth filter poles.

<table>
<thead>
<tr>
<th>Pole Index</th>
<th>Pole Expression</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$p_1 = s + 1$</td>
</tr>
<tr>
<td>2</td>
<td>$p_2 = s^2 + 1.4142s + 1 = (s + 0.7071)^2 + 0.7071^2$</td>
</tr>
<tr>
<td>3</td>
<td>$p_3 = s^3 + 2.0000s^2 + 2.0000s + 1 = (s + 1.0000)(s + 0.5000)^2 + 0.8660^2$</td>
</tr>
<tr>
<td>4</td>
<td>$p_4 = s^4 + 2.6131s^3 + 3.4142s^2 + 2.6131s + 1 = (s + 0.3827)^2 + 0.9239^2 + (s + 0.9239)^2 + 0.3827^2$</td>
</tr>
<tr>
<td>5</td>
<td>$p_5 = s^5 + 3.2361s^4 + 5.2361s^3 + 5.2361s^2 + 3.2361s + 1$</td>
</tr>
<tr>
<td></td>
<td>$= (s + 1.0000)(s + 0.3090)^2 + (s + 0.9511)^2 + (s + 0.9090)^2 + 0.8578^2$</td>
</tr>
<tr>
<td>6</td>
<td>$p_6 = s^6 + 3.8637s^5 + 7.4641s^4 + 9.1416s^3 + 9.4641s^2 + 3.8637s + 1$</td>
</tr>
<tr>
<td></td>
<td>$= (s + 0.2588)^2 + 0.9659^2 + (s + 0.7071)^2 + 0.7071^2 + (s + 0.9659)^2 + 0.2588^2$</td>
</tr>
<tr>
<td>7</td>
<td>$p_7 = s^7 + 4.4940s^6 + 10.0978s^5 + 14.5918s^4 + 14.5918s^3 + 10.0978s^2 + 4.4940s + 1$</td>
</tr>
<tr>
<td></td>
<td>$= (s + 1.0000)(s + 0.2325)^2 + 0.9749^2 + (s + 0.6235)^2 + 0.7618^2 + (s + 0.9010)^2 + 0.4339^2$</td>
</tr>
<tr>
<td>8</td>
<td>$p_8 = s^8 + 5.1258s^7 + 13.1371s^6 + 21.8462s^5 + 25.6884s^4 + 21.8462s^3 + 13.1371s^2 + 5.1258s + 1$</td>
</tr>
<tr>
<td></td>
<td>$= (s + 0.1951)^2 + 0.9898^2 + (s + 0.5563)^2 + 0.8315^2 + (s + 0.8315)^2 + 0.5556^2$</td>
</tr>
<tr>
<td></td>
<td>$= (s + 0.9808)^2 + 0.1951^2$</td>
</tr>
<tr>
<td>9</td>
<td>$p_9 = s^9 + 5.7586s^8 + 16.5817s^7 + 31.1634s^6 + 41.9864s^5 + 41.9864s^4 + 31.1634s^3 + 16.5817s^2 + 5.7586s + 1$</td>
</tr>
<tr>
<td></td>
<td>$= (s + 1.0000)(s + 0.1373)^2 + 0.9848^2 + (s + 0.5000)^2 + 0.8660^2 + (s + 0.7660)^2 + 0.6428^2$</td>
</tr>
<tr>
<td></td>
<td>$= (s + 0.9397)^2 + 0.3420^2$</td>
</tr>
<tr>
<td>10</td>
<td>$p_{10} = s^{10} + 6.3925s^9 + 20.4317s^8 + 42.8021s^7 + 64.8824s^6 + 74.2334s^5 + 64.8824s^4 + 42.8021s^3 + 20.4317s^2 + 6.3925s + 1$</td>
</tr>
<tr>
<td></td>
<td>$= (s + 0.1564)^2 + 0.9877^2 + (s + 0.4854)^2 + 0.8910^2 + (s + 0.7071)^2 + 0.7071^2$</td>
</tr>
<tr>
<td></td>
<td>$= (s + 0.8910)^2 + 0.4540^2 + (s + 0.9877)^2 + 0.1564^2$</td>
</tr>
</tbody>
</table>

From Eq. 4.3.2, the magnitude characteristics may be easily plotted as shown in Fig. 4.3.2. Note that all the filters have a 3 dB bandwidth of unity as required. As the filter order increases, the magnitude approximation approaches the ideal magnitude response of Fig. 1.5.2a. Ideally, the response makes an abrupt transition from unity gain to zero gain at $\omega = 1$. However, the approximation has only finite slope at $\omega = 1$. A measure of the "skirt" steepness is the filter selectivity or cutoff rate which, by definition, equals the negative of the gain slope at the band-edge. Thus, the ideal low-pass filter has a band-edge selectivity of infinity. Butterworth filters have band-edge selectivities which equal

$$-\frac{d|H|}{d\omega} \bigg|_{\omega=1} = \frac{n}{2\sqrt{2}} = 0.354n$$  \hspace{1cm} (4.3.9)

which is easily calculated from Eq. 4.3.8. Thus, they have slopes which vary directly with order.

Another useful figure of merit is the shaping factor $S$. The shaping factor equals the ratio of bandwidths at which the magnitude response is at specified attenuation levels. Attenuation levels of 6 dB and 60 dB have become industrial standards. Thus, the 6–60 dB shaping factor equals

$$S_{6 \text{ dB}} = \frac{B_{60 \text{ dB}}}{B_{6 \text{ dB}}}$$  \hspace{1cm} (4.3.10)

where $S \gg 1$. The smaller the shaping factor ratio, the steeper is the magnitude slope in the transition band; therefore, the narrower is the transition band. Ideal low-pass filters have shaping factors of unity. The 6–60 shaping factor for Butterworth filters is easily shown to equal

$$S_{6 \text{ dB}} = \left(\frac{10^6/m - 1}{10^4/m - 1}\right)^{1/2n}$$  \hspace{1cm} (4.3.11)
where $m = 1$. When $m$ stages of an $n$th-order Butterworth filter are cascaded together, the shaping factor is given by the same equation. The product $nm$ is equal to the total number of filter poles, which in turn, is directly related to filter complexity as we shall later see. The shaping factor is reduced, and therefore, out-of-band rejection is improved by using a single $mn$th-order Butterworth filter rather than $nm$ first-order Butterworth filters.

The order of the Butterworth filter is determined by the required attenuation or rejection at one or more specified stopband frequencies. To determine order, let us express the squared magnitude gain as

$$|H(\omega)|^2 = \frac{1}{1 + e^{-2\pi n} S_n(\omega)}$$

(4.3.12)
Fig. 4.3.3 Nomograph for Butterworth filters. (From J. N. Hallberg, "Filters and nomographs," M.S.E.E. Directed Research, California State Univ., Long Beach, Jan., 1974.)

which is a standard form for most filters. $S_n(\omega)$ is some well-known classical $n$th-order polynomial, and $\epsilon$ is related to the passband ripple. Then $S_n(\omega) = \omega^n$ for the Butterworth filter. If the gain magnitude equals $|H_1|$ at $\omega_1$ and $|H_2|$ at $\omega_2$, then Eq. 4.3.12 can be manipulated to yield the minimum $n$ required as

$$n \geq \log \left( \frac{(|H_2|^{-2} - 1)^{1/2}}{(|H_1|^{-2} - 1)^{1/2}} \right) = \frac{\ln (\epsilon_2/\epsilon_1)}{\ln (\omega_2/\omega_1)} \tag{4.3.13}$$

The smallest integer $n$ satisfying this relation is chosen. If $k$ frequencies and attenuations are specified, then Eq. 4.3.13 is evaluated at each of the $k$ frequencies and the largest $n$ is chosen from among the $k$ values so obtained.

Generally, more insight can be gained in filter order by using a nomograph (i.e., a graphical aid) rather than an equation (in the Butterworth case, Eq. 4.3.13). Nomographs derive their name from the Greek words nomos (law) and graphein (to write). They are literally graphical representations of equations and were introduced in 1891 by d'Ocagne and recently applied to filters by Kawakami. They form marvelous design aids as we shall soon see. The nomograph for Butter-
worth filters is shown in Fig. 4.3.3. A monograph on nomographs will not be presented here, but we shall utilize their results. The interested engineer will find it enlightening to review the theoretical basis of nomographs.\textsuperscript{6}

The procedure for determining filter order begins by first expressing the required frequency response in the form shown in Fig. 4.3.4a. The pertinent parameters equal:

- \( M_p = \) maximum attenuation in (dB) in the passband
- \( M_s = \) minimum attenuation in (dB) in the stopband
- \( \Omega_s = \) normalized stopband frequency

where

\[
\Omega_s = f_s / f_p = \omega_s / \omega_p
\]

and

\[
f_p \ (or \ \omega_p) = \text{band-edge frequency of passband in Hz (or rad/sec)}
\]

\[
f_s \ (or \ \omega_s) = \text{band-edge frequency of stopband in Hz (or rad/sec)}
\]

This data is then entered on the nomograph as shown in Fig. 4.3.4b. A line is drawn through \( M_p \) (point \( P_1 \)) and \( M_s \) (point \( P_2 \)) until it intersects the graph at \( P_3 \). A horizontal line is then drawn across the graph from \( P_3 \). A vertical line is drawn through the stopband frequency \( \Omega_s \). The horizontal line and vertical lines intersect at \( P_5 \). The minimum required filter order is given by the first curve lying above \( P_5 \). After practice, this procedure will become second-nature to the engineer. Let us now consider several examples to demonstrate the usefulness of nomographs.

**EXAMPLE 4.3.1** Determine the order required for a Butterworth filter to meet the specification shown in Fig. 4.3.5a using the Butterworth nomograph. Here 40 dB and 60 dB of rejection is required at frequencies of twice and three times the 1.25 dB bandwidth, respectively.

**Solution** To use the nomograph, we first write that

\[
\begin{align*}
M_p &= 1.25 \text{ dB at } f = 1 \text{ KHz} \\
M_{s1} &= 40 \text{ dB at } f = 2 \text{ KHz, } \Omega_{s1} = 2 \text{ KHz/1 KHz} = 2 \\
M_{s2} &= 60 \text{ dB at } f = 3 \text{ KHz, } \Omega_{s2} = 3 \text{ KHz/1 KHz} = 3
\end{align*}
\]

(4.3.15)

Entering this data on the nomograph as shown in Fig. 4.3.4b, we find \( n_1 \approx 8 \) and \( n_2 \approx 7 \). Thus, an eighth-order Butterworth filter is required.

**EXAMPLE 4.3.2** A low-pass filter having the gain characteristics shown in Fig. 4.3.5a must be realized. However, the band-edge frequencies are \( f_p = 2 \text{ KHz, } f_{s1} = 3 \text{ KHz, and } f_{s2} = 4 \text{ KHz} \). Size and cost considerations limit the filter order \( n \ll 8 \). Can a Butterworth filter be used?

**Solution** Since \( M_p = 1.25 \text{ dB and} \)
Fig. 4.3.5 (a) Filter specification for Example 4.3.1 and (b) entry of data onto nomograph.

\[ M_{s1} = 40 \text{ dB at } \Omega_{s1} = 3 \text{ KHz/2 KHz} = 1.5 \]
\[ M_{s2} = 60 \text{ dB at } \Omega_{s2} = 4 \text{ KHz/2 KHz} = 2 \]  \hspace{1cm} (4.3.16)

using the Butterworth filter nomograph, we find \( n_1 \geq 13 \) and \( n_2 \geq 11 \). Therefore, \( n \geq 13 \) and a Butterworth filter cannot be designed to meet the maximum order requirement of 8.

Sometimes filter specifications are drawn differently in the frequency domain than we have thus far shown. One such commonly encountered form is shown in Fig. P4.27. Rather than having a stairstep rolloff, it has a ramp rolloff. This type of specification insures a minimum rate of attenuation rolloff into the stopband. Thus, it is more stringent than that in Fig. 4.3.4. Very often, we assume that the filters being considered have rapid enough rolloffs so that the specification form of Fig. 4.3.4 is adequate. However, when required it is wise to be more explicit and use the form of Fig. P4.27 to eliminate misunderstanding. The filter nomographs are still used to analyze such situations. The only difference is that the ramp requires several data points to be entered on the nomograph.

We mentioned earlier that another benefit of nomographs, in addition to their ease and speed to use, was the great insight they gave to the properties of the filter. Returning to Fig. 4.3.4 for a moment, we see that the nomograph graphically depicts the interrelation between \( M_p \), \( M_s \), \( \Omega_s \), and \( n \). Mathematically, this means that any three of the parameters may be independently specified, and the fourth parameter is thereby fixed or determined. Thus, rather than specifying \( M_p \), \( M_s \), and \( \Omega_s \) and determining \( n \) as in the previous examples, we could instead specify a different combination. For example, we could specify \( M_p \) and \( n \), and look at the \( M_s \) and \( \Omega_s \) values which result. This is illustrated in the following example.

**EXAMPLE 4.3.3** Determine the order required for a Butterworth filter to meet the general specifications: (1) in-band ripple = 0.5 dB for \( \omega \leq 1 \), and (2) 3 dB corner frequency = 1.1. Determine the frequencies at which 20 dB and 60 dB attenuations are obtained.

**Solution** Using \( M_p = 0.5 \text{ dB} \) and \( M_s = 3 \text{ dB} \) at \( \Omega_s = 1.1/1 = 1.1 \), we find that \( n \geq 11 \). We can now determine the frequencies at which we obtain the specified attenuations. For example, using \( n = 11 \) and \( M_p = 0.5 \), then from the nomograph, the 20 dB rejection frequency \( \Omega_{s1} = 1.36 \) when \( M_s = 20 \text{ dB} \). For \( M_s = 60 \text{ dB} \), then the 60 dB rejection frequency \( \Omega_{s2} = 2.05 \).

The delay characteristic of the Butterworth filter is shown in Fig. 4.3.2. The mathematical
which is indeed the case. We see that the delay variation increases with filter order in the pass-band. For $n = 3$, the delay variation is about 1 second (or $50\%$); for $n = 9$, the delay variation is almost 6 seconds (or $100\%$). Even with this amount of delay variation, the Butterworth filter has better delay characteristics than most of the other filters (having fairly narrow transition bands) which we will consider. It is important to remember that the absolute delay value is not of importance in most filters, but rather the delay variation.

It is useful to recall from Sec. 2.10 that when the gain of a filter is expressed in the form of Eq. 2.0.1, then the low-frequency delay is given by Eq. 2.10.34 and the high-frequency is given by Eq. 2.10.36. If the filter has no finite zeros, as is the case in almost all the low-pass filters we will consider, then these delays reduce to

$$\tau(0) = b_1/b_0; \quad \tau(j\omega) = b_{n-1}/b_n\omega^2, \quad \omega \to \infty$$

(4.3.17)

The $b_i$ coefficients are the coefficients of the filter polynomial given by Eq. 4.3.6. The Butterworth filter polynomials were listed in Table 4.3.1. Thus, we can easily substantiate the low- and high-frequency behaviors of delay in Fig. 4.3.2. For example, when $n = 10$, then $\tau(0) = 6.3925$ seconds while $\tau(j\omega) = 6.3925/\omega^2$ seconds as $\omega \to \infty$. These values verify these two asymptotic values in Fig. 4.3.2 (note that $\tau(j2) = 6.3925/2^2 = 1.6 \approx 1.8$). Since Butterworth filters always have $b_0 = b_n$ and $b_1 = b_{n-1}$, the asymptotic delay strengths are always equal. Also note that the asymptotic phase of every all-pole filter approaches $-\pi/2$ radians or $-90^\circ$, as $\omega \to \infty$.

We can also observe that it is the pole pair having minimum damping factor or maximum $Q$ that primarily determines the maximum delay value (this assumes all have the same $\omega_n$). From Eq. 2.10.13, we recall that the maximum delay contributed by a pole pair with damping factor $\xi$ and resonant frequency $\omega_n$ equals approximately

$$\tau_{\text{max}} \approx 1/\xi\omega_n = 1/\alpha, \quad |\xi| \ll 1$$

(4.3.18)

which occurs at frequency $\omega_n$. Thus, using the poles listed in Table 4.3.1, we see $\tau_{\text{max}} \approx 1/(0.1564)(1) = 6.39$ seconds at $\omega = 1$ for $n = 10$. We see that the actual delay is about 12 seconds. The discrepancy is due to the fact that the other Butterworth poles add appreciable delay because their $\xi$’s do not increase rapidly and their $\omega_n$’s are equal. In filters we shall study later, the poles will be distributed on more elliptical or parabolic curvatures rather than circles. This will result
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in s's which do increase rapidly with decreasing \( \omega_n \). Then this approximation for \( \tau_{\text{max}} \) will yield better results.

The unit step response of the Butterworth filter is shown in Fig. 4.3.6. It is composed of gated sinusoids, each having a frequency equal to the imaginary part of the pole, and an attenuation constant equal to the real part of the pole. It may be calculated from Eqs. 4.3.6 and 4.3.7 using a partial fraction expansion as discussed in Sec. 3.2. Generally, however, no insight is gained by writing this equation and a computer is required for plotting the result. It is important to note that the step response is plotted as a function of normalized time \( \omega_{3\text{dB}}t \) (in radians). Normalized time is used since the step response was calculated using the poles of Table 4.3.1 where \( \omega_{3\text{dB}} \) was unity. Using the time scaling theorem of Table 1.8.2, if the poles are scaled by any nonunity \( \omega_{3\text{dB}} \), then the time domain responses (in seconds) are scaled in time by \( \omega_{3\text{dB}}^{-1} \).

We see that the step responses exhibit more overshoot as \( n \) increases. This is due to the damping factors which decrease with increasing order. The delay time increases with filter order, but rise time remains approximately constant. The rise time result agrees with the empirical relation of Valley and Wallman given by Eq. 3.5.13 where \( t_r = 2.2/\omega_{3\text{dB}} = 2.2 \) seconds since all filters have a normalized bandwidth of 1 rad/sec. Notice that the delay value corresponds closely with the low-frequency value of the delay curves. We anticipated this result in Eq. 3.6.9.

EXAMPLE 4.3.4  Determine if a Butterworth filter can be designed to meet the frequency and time requirements shown in Fig. 4.3.7. Design for minimum filter order and draw the block diagram of the filter if it can be realized.

**Solution**  We begin by determining the required filter order from the frequency response. Since \( M_p = 3 \text{ dB}, M_{s1} = 10 \text{ dB} \) at \( \Omega_{s1} = 2 \), and \( M_{s2} = 40 \text{ dB} \) at \( \Omega_{s2} = 4 \), we see from the Butterworth filter nomograph that \( n_1 \geq 2 \) and \( n_2 \geq 4 \). Thus, at least a fourth-order filter is required.

Now we sketch the step response of the filter noting \( n = 4 \). Since the bandwidth \( \omega_{3\text{dB}} = 2\pi(2 \text{ KHz}) \), we must denormalize time by \( \omega_{3\text{dB}}^{-1} \) in Fig. 4.3.7. We now transfer the normalized step response of Fig. 4.3.6 onto Fig. 4.3.7. We see that we have easily met the time domain requirements. Noting the time differences

\[
\Delta t_1 = 5 - 2.3 = 2.7 \text{ sec} \quad (r = 0.5), \quad \Delta t_2 = 7.5 - 3.5 = 4.0 \text{ sec} \quad (r = 0.8), \\
\Delta t_3 = 10 - 7.3 = 2.7 \text{ sec} \quad (r = 1.05) \quad (4.3.19)
\]

we see that we may reduce the time domain specification by \( \Delta t = \min(\Delta t_1, \Delta t_2, \Delta t_3) = 2.7 \) seconds, if we so desire. Alternatively, we could consider expanding the bandwidth—however, here we require the 2 KHz bandwidth shown. The transfer function of the filter equals

\[
H(s) = \frac{1}{s_n^2 + 0.765s_n + 1} \left( s_n^2 + 1.848s_n + 1 \right) \quad (4.3.20)
\]

where \( s_n = s/2\pi(2 \text{ KHz}) \) using Table 4.3.1. The block diagram realization of the filter is shown in Fig. 4.3.8.
| $f_o = 2$ KHz | $f_o = 2$ KHz | $\zeta = 0.924$ | $\zeta = 0.382$ |

**Fig. 4.3.8** Block diagram of Butterworth filter of Example 4.3.4.

We now see the great ease in utilizing classical filter results since we merely need to rewrite the tabulated transfer function. The bulk of the work involves determining the acceptable type of filter (e.g., Butterworth) and its required order. The transfer functions can always be easily expressed using the frequency normalized form involving $s_n$. The engineer will find that using de-normalized forms involving $s$ appear more complicated and will obscure the pertinent information involving the $\omega_n$ and $\zeta$ of each pole pair. It is also important to remember that each design block must be explicitly labelled with (1) order and type, (2) dc gain $H_0$, (3) resonant frequency of pole ($\omega_n$ or $f_n$), and (4) damping factor of pole ($\zeta$). In Chap. 8, we shall utilize such block diagrams in the design of low-pass filters.

One other comment should be made. We shall usually realize $n$th-order filters by cascading $n/2$ (for $n$ even) or $(n + 1)/2$ (for $n$ odd) second-order stages. The primary advantage of cascade design is the ease of both tuning and temperature compensation.

Now that we have analyzed Butterworth filters in great detail, we proceed to consider a variety of other types. Since the development of each filter type follows the same pattern as that of the Butterworth filter, we shall be briefer (but no less complete) in our discussions.

### 4.4 CHEBYSHEV FILTERS

The Butterworth filter has a monotonically decreasing passband and stopband response. In contrast, the Chebyshev filter, sometimes called the equal-ripple or equiripple filter, has a response which ripples throughout the passband between 1 and $(1 + e^2)^{1/4}$ as shown in Fig. 4.4.1. It has a narrower transition band (i.e., faster cutoff) than the Butterworth filter but increased passband delay variation. Of course, both filters have the same asymptotic slope for a given order. Chebyshev filters also exhibit more step response overshoot.

The Chebyshev filter\(^9\) has a magnitude function

\[
|H(j\omega)|^2 = \frac{1}{1 + e^{2n}T_n^2(\omega)} \tag{4.4.1}
\]

where $T_n$ is the $n$th-order Chebyshev (sometimes spelled Tschebyscheff) polynomial\(^9\) and $e < 1$ is the parameter which determines the ripple magnitude. The $n$th-order Chebyshev polynomial (of the first-kind) equals

\[
T_n(\omega) = \cos(n \cos^{-1} \omega), \quad 0 < \omega < 1 \quad \text{and} \quad \cosh(n \cosh^{-1} \omega), \quad \omega > 1 \tag{4.4.2}
\]

$T_n$ is an even function for $n$ odd and an odd function for $n$ even. The polynomials oscillate between $\pm 1$ in the frequency interval $[-1, 1]$ so $|T_n| \leq 1$ for $|\omega| < 1$. Beyond this region, $T_n$ increases monotonically and asymptotically approaches

\[
T_n(\omega) \approx 2^{n-1} \omega^n, \quad |\omega| > 1 \tag{4.4.3}
\]

Since the $T_n$ polynomials can take on negative as well as positive values for $|\omega| < 1$, they cannot be used directly to generate gain functions. Thus, we use the square of $T_n$ which will always be non-negative. Since it is also easy to show from Eq. 4.4.2 that $T_n^2 = (T_{2n} + 1)/2$, we could just as well use Chebyshev polynomials of order $2n$. The parameter $e^2$ is used to limit the in-band varia-
The ripple width is usually specified in dB where ripple (dB) ≈ 4.34ε² so that specifying the maximum amount of ripple determines parameter ε. Ripples of 0.5, 1, 2, and 3 dB have corresponding ε’s of 0.35, 0.51, 0.76, and 1, respectively.

The poles of the Chebyshev filter are determined by first finding the poles of s_k of T(s)T(−s). We then discard those poles lying in the right-half-plane. The poles of T(s)T(−s) must satisfy 1 + ε²T_n²(s_k/j) = 0 from Eq. 4.4.1 using analytic continuation. Solving for T_n² gives

\[ T_n(s_k/j) = \cos [n \cos^{-1}(s_k/j)] = zj/ε \]  

(4.4.5)

Defining the term p_k = u_k + jv_k = cos⁻¹s_k/j, then we can re-express Eq. 4.4.5 as

\[ \cos np_k = \cos nu_k \cosh nv_k - j \sin nu_k \sinh nv_k = zj/ε \]  

(4.4.6)

where we have expanded the cosine of a complex argument. The real and imaginary parts of p_k must therefore satisfy

\[ \cos nu_k \cosh nv_k = 0, \quad \sin nu_k \sinh nv_k = ±j/ε \]  

(4.4.7)

Since \cosh nv_k is positive and nonzero for real v_k, then \cos nu_k must equal zero so u_k = (2k - 1) π/2n for k = 1, 2, ..., 2n. From the second equation, then v_k must satisfy

\[ v_k = v = (1/n) \sinh^{-1}(1/ε) \]  

(4.4.8)

which is a constant dependent upon order n and ripple factor ε. Therefore, the poles s_k equal

\[ s_k = a_k + jω_k = j \cos p_k = j \cos (u_k + jv) = \sin u_k \sinh v + j \cos u_k \cosh v \]  

(4.4.9)

Squaring a_k and ω_k, dividing by their respective hyperbolic terms, and adding these two equations gives

\[ \frac{a_k^2}{\sinh^2 v} + \frac{ω_k^2}{\cosh^2 v} = 1 \]  

(4.4.10)

which is the standard equation for an ellipse. Thus, the poles lie on an ellipse having its axis coinciding with the s-plane axis. The semi-major axis has value cosh v, the semi-minor axis has value sinh v, and the foci are located at ω = ±1.
4.4.2. Alternatively, Weinberg noted that an inner circle of radius \( [\sinh^{-1} (1/e)]/n \) could be used to locate the poles in place of the ellipse. Using this approach, each Chebyshev pole is located at the intersection of a vertical line drawn from each inner circle pole and a horizontal line drawn from each outer circle pole (see Fig. 4.4.2). Note that Chebyshev poles approach Butterworth poles in the limiting case where the in-band ripple approaches zero since \( \tanh v + 1 \) as \( e \to 0 \).

In practice, it is more convenient to utilize tables of Chebyshev filter poles rather than to generate them. The pole locations for Chebyshev filters having in-band ripples of 0.5, 1, 2, and 3 dB are listed in Table 4.4.1 (for other ripples, see Ref. 11). It is important to note that the frequency has been normalized so that the cutoff frequency of 1 rad/sec corresponds to the point where the gain is reduced by the amount of ripple (rather than 3 dB as for the Butterworth filter).

The Chebyshev filter has a gain

\[
|H(\omega)| = \frac{1}{\sqrt{1 + e^2 T_n^2(\omega)}}
\]

(Chebyshev filters having in-band ripples of 0.1 and 1 dB have the magnitude characteristics shown in Fig. 4.4.3 (for other ripples, see Refs. 12 and 13). Note that the magnitude characteristics have been drawn so they all have a 3 dB cutoff frequency of 1 rad/sec. We see that the Chebyshev filter has a more rapid cutoff than the Butterworth filter. In fact, the Chebyshev filter is optimum in the sense that there is no other low-pass filter, with all of its zeros of transmission at infinity and an in-band ripple not exceeding a maximum value, having a faster cutoff rate outside its passband. The price paid for this rapid cutoff rate is delay having large variances within the passband as we shall see.

We can determine the 3 dB bandwidth of Chebyshev filters having unity ripple bandwidth from Eq. 4.4.1. Since \(|H|^2 = \frac{1}{2} \) at the 3 dB frequency, the Chebyshev polynomial must equal \( T_n(\omega_{3dB}) = 1/e \). Solving for \( \omega_{3dB} \), we find

\[
\omega_{3dB} = \cosh [(1/n) \cosh^{-1} (1/e)]
\]

Since the band-edge frequency is unity (for ripples \( \leq 3 \) dB), this equation relates the ratio of the 3 dB and band-edge frequencies. Notice that \( \cosh^{-1} x \) and \( \sinh^{-1} x \) are about equal for \( x \approx 2 \) (or \( e \approx \frac{1}{2} \)). Thus, from Eq. 4.4.12, the 3 dB bandwidth of the Chebyshev filter is approximately \( \cosh \). Another useful approximation for \( \omega_{3dB} \) is obtained by substituting a Maclaurin series for \( \cosh \) into Eq. 4.4.12 which yields

\[
\omega_{3dB} \approx (2/e)^{1/n/2}, \quad e \ll 1
\]

Thus, for fixed ripple parameter \( e \), then the 3 dB bandwidths vary as the \( n \)th root of a constant.

We should make one comment concerning the additional rejection provided by Chebyshev filters over Butterworth filters. If we compare their stopband responses when both have unity ripple bandwidths (where their gains equal \((1 + e^2)^{-1/2}\)), Eqs. 4.3.2 and 4.4.3 can be used to show that the Chebyshev filter provides an additional \( 6(n - 1) \) dB of rejection over the Butterworth
Table 4.1 Chebyshev filter poles.

<table>
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<tr>
<th>n</th>
<th>0.5 dB Ripple</th>
<th>1 dB Ripple</th>
<th>2 dB Ripple</th>
<th>3 dB Ripple</th>
</tr>
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<td></td>
<td>a</td>
<td>b</td>
<td>a</td>
<td>b</td>
</tr>
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filter. If we instead normalize their 3 dB bandwidths to unity (rather than $(1 + e^2)^{1/2}$), their responses become those shown in Fig. 4.4.4. This demonstrates that little is gained by using a Chebyshev filter with small in-band ripple when the 3 dB bandwidth is fixed.

The Chebyshev filter has a band-edge selectivity of $16$

$$-d|H|/d\omega \bigg|_{\omega=1} = \frac{e^{2n'T_n T_n'}}{(1 + e^{2n'T_n})^{3/2}} \bigg |_{\omega=1} = \frac{e^{2n^2}}{(1 + e^{2n})^{3/2}} \tag{4.4.14}$$
from Eq. 4.4.1 where \( T_n(1)T_n'(1) = n^2 \). Note that this is the selectivity at the ripple cutoff frequency and not the 3 dB frequency. It may, of course, be evaluated at other frequencies but the calculation becomes more involved. Comparing the 3 dB ripple results with the Butterworth slopes in Eq. 4.3.9, we verify that the Chebyshev filter has a cutoff which is \( n \) times larger.

We saw in Eq. 4.4.4 that specifying the maximum amount of ripple determines parameter \( \epsilon \). The asymptotic slope or required rejection at a specified stopband frequency determines the required filter order \( n \). Expressing the gain as \( |H_1| \) at \( \omega_1 \) and \( |H_2| \) at \( \omega_2 \), then solving for the filter order \( n \) using Eq. 4.4.2 gives (assuming \( \omega_1 = 1 \))

\[
\begin{align*}
    n & \geq \frac{\cosh^{-1}(\epsilon_2/\epsilon_1)}{\cosh^{-1}(\omega_2)} = \frac{\ln(2\epsilon_2/\epsilon_1)}{\sqrt{2(\omega_2 - 1)}} \\
    & \quad (4.4.15)
\end{align*}
\]

The analogous Butterworth equation was derived in Eq. 4.3.13. The smallest integer satisfying this equation is chosen. This is an inconvenient equation to apply in practice, and much greater insight may be gained by using the nomograph shown in Fig. 4.4.5. The same low-pass filter specifications and data entry format are used as for the Butterworth nomograph.

**EXAMPLE 4.4.1** Determine the order of the Chebyshev filter required and meet the specifications of Example 4.3.1 where \( M_o = 1.25 \text{ dB}, M_{x1} = 40 \text{ dB at } \Omega_{x1} = 2, \) and \( M_{x2} = 60 \text{ dB at } \Omega_{x2} = 3 \).

**Solution** The minimum order filter results when we let \( M_o = 1.25 \text{ dB} \) rather than some lesser value. Entering this data on Fig. 4.4.5 yields \( n_1 \geq 5 \) and \( n_2 \geq 5 \). Thus, a fifth-order Chebyshev filter having an in-band ripple of 1.25 dB can be used. Since we found that an eighth-order Butterworth filter was required, we can reduce the order by three (almost half) using a Chebyshev filter. We should point out that all the nomographs in this chapter have been constructed so they have the same \( M_o, M_{x}, \Omega_{x}, \) and \( \gamma \)-axes. Thus, once we have obtained the \((\gamma, \Omega_{x})\) combinations, they can be used on all the nomographs which helps to simplify the use of nomographs even further. For example, the two data points \((\gamma, \Omega_{x})\) from Example 4.3.1 are \((4.5, 2)\) and \((6.5, 3)\).

**EXAMPLE 4.4.2** A low-pass filter having the rejection of the last example but different stopband frequencies was analyzed in Example 4.3.2 where size and cost limitations required the filter order \( n \leq 8 \). Can a Chebyshev filter meet these requirements?

**Solution** Since \( \Omega_{x1} = 1.5 \) and \( \Omega_{x2} = 2 \), then we find \( n_1 \geq 7 \) and \( n_2 \geq 7 \) from the Chebyshev filter nomograph. Thus, a seventh-order Chebyshev filter can be used to meet the requirement. Since a thirteenth-order Butterworth filter was required, we have again reduced the filter order by about half.

The delay characteristics of the Chebyshev filter are shown in Fig. 4.4.6 for other ripples.
Butterworth poles have real parts reduced by tanh v. This is much less than unity for large ε and n. Thus, the Chebyshev poles have a smaller damping factor than the Butterworth poles and the maximum peaking varies as \(1/\xi \omega_n\) from Eq. 4.3.18.

The unit step responses of the Chebyshev filter for ripples of 0.5, 1, 2, and 3 are shown in Fig. 4.4.7. These responses are based on filters having ripple (not 3 dB) band-edge frequencies of unity. Thus, on a normalized 3 dB bandwidth basis, the time must be increased by \(\omega_{3dB}\) given by Eq. 4.4.12. Due to the higher-Q poles, overshoot increases with increasing filter order \(n\). The delay and overshoot increase with filter order, but not significantly with increasing ripple. The rise time remains approximately constant which agrees with the empirical result of Valley and Wallman when all filters have a normalized 3 dB bandwidth of 1 rad/sec. It is important to note that even-order filters exhibit more percentage overshoot than odd-order filters. This difference is accentuated for large ripple values. Also even-order filters tend to "pull-down" to their final values while odd-order filters tend to "pull-up."
Fig. 4.4.6 Delay characteristics of Chebyshev filters having 0.01, 0.1, and 0.5 dB ripples and unity 3 dB bandwidths.
Fig. 4.4.7 Step responses of Chebyshev filter having ripples of 0.5, 1, 2, and 3 dB and unity ripple bandwidths.
SEC. 4.9  LINEAR PHASE (MAXIMALLY FLAT DELAY) FILTERS

= 1.3965 and that the minimum stopband attenuation $M_s = 38.17$ dB. The poles and zeros equal

\[ p_1, p_1^* = -0.0956 \pm j0.987, \quad p_2, p_2^* = -0.336 \pm j0.471, \quad z_1, z_1^* = \pm j1.493 \quad (4.8.15) \]

Thus, we exceed our minimum stopband attenuation by about 13 dB. Since we used the required $\theta = 49^\circ$, our 25 dB stopband frequency still equals 1.33. This is an important detail to remember. Note that the high-frequency asymptotic rolloff is $-40$ dB/dec. This example illustrates how design tradeoffs are sometimes required to use the tabulated data.

The delay characteristics of third-, fourth-, and fifth-order elliptic filters having 60 dB stopband rejection are shown in Fig. 4.8.4 for others, see Ref. 13. Since the poles of elliptic filters almost fall on an ellipse, they are somewhat similar in form to Chebyshev delay curves. However, they exhibit less delay variation but more delay peaking. Delay increases with in-band ripple and filter order. Negative delay impulses of area $-\pi$ appear at the zero frequencies. Note if the zeros were not purely imaginary but lay off the $j\omega$-axis, this would produce negative delay peaking of non-zero bandwidth which would significantly change the delay curves near the band-edge.

The unit step responses for third- and fifth-order elliptic filters having various in-band ripples ($0.1 - 3.0$ dB) and angles ($10^\circ - 80^\circ$) are shown in Figs. 4.8.5 and 4.8.6. Note that for a constant in-band ripple, the step response depends on angle $\theta$ or, equivalently, the stopband frequency $\Omega_s$. Increasing $\theta$ corresponds to decreasing $\Omega_s$ and to narrowing the transition band. The low-frequency delay and thus the delay time decreases as $\Omega_s$ decreases. The rise time remains relatively constant as long as the 3 dB bandwidth is about constant. The overshoot decreases as $\Omega_s$ decreases. The overshoot is determined primarily by the highest-Q complex pole pair of radius $\omega_n$ in $H(s)$. As $\Omega_s$ decreases, these poles move closer to the imaginary zeros; this reduces the residue value for that pole and, therefore, the overshoot. For fixed $\theta$ or $\Omega_s$, the overshoot increases with filter order. Also note that reducing $\theta$ is equivalent to increasing $\Omega_s$, and the elliptic filter step response approaches that of the corresponding Chebyshev filter.

4.9 LINEAR PHASE (MAXIMALLY FLAT DELAY) FILTERS

So far in this chapter, we have considered a variety of filters. We began with Butterworth filters and found they had fairly good magnitude and delay characteristics. As we progressed into filters having more and more magnitude selectivity, we found that their delay characteristics exhibited increasing delay ripple and delay peaking. Ideally, their passband delay should be constant as we found in Eq. 4.1.1. Equivalently stated, the passband phase should be linear with a value of zero (i.e., zero intercept) at dc ($\omega = 0$). Any deviation from constant delay is called delay distortion, and any deviation from linear phase (but not necessarily zero intercept) is called phase distortion. Thus, Butterworth filters have moderate delay and phase distortion; Chebyshev and elliptic filters have large delay and phase distortion.

Now we want to consider filters that have small delay and phase distortion. In general, such filters are called linear phase, or alternatively, maximally flat delay or constant delay filters. We will see they have fairly constant passband delay. Their step responses will exhibit less overshoot and ringing. However, their magnitude responses will be much less sensitive than the filters previously discussed. It is possible to reduce the passband delay variation and thereby equalize delay of a filter by use of a delay equalizer as discussed in Sec. 2.10. However, use of delay equalizers introduces more complexity and increases the size of a filter; thus, making specification compromises when possible is desirable. We began this chapter by considering maximally flat magnitude responses. Let us now apply this same approach to produce maximally flat delay response.

The general transfer function in ac steady-state of any filter can be expressed as

\[ H(j\omega) = \frac{a(\omega^2) + j\omega b(\omega^2)}{c(\omega^2) + j\omega d(\omega^2)} \quad (4.9.1) \]
Expanding \( x \) in a Maclaurin series as
\[
\tan^{-1} x = x - x^3/3 + x^5/5 - \ldots, \quad |x| < 1
\]
gives
\[
\theta(j\omega) = [(\omega b/a) - (\omega b/a)^3/3 + (\omega b/a)^5/5 - \ldots] - [(\omega d/c) - (\omega d/c)^3/3 + (\omega d/c)^5/5 - \ldots]
\]
Combining coefficients of like powers of \( \omega \) gives
\[
\theta(j\omega) = \omega(b/a) - (d/c) - (\omega^3/3)(b/a)^3 - (d/c)^3 + (\omega^5/5)(b/a)^5 - (d/c)^5 - \ldots
\]
\[
= -a_1 \omega - a_3 \omega^3 - a_5 \omega^5 - \ldots
\]
Since the delay equals
\[
\tau(\omega) = \frac{-d\theta(j\omega)/d\omega}{\omega} = a_1 + 3a_3\omega^2 + 5a_5\omega^4 + \ldots
\]
linear phase or maximally flat delay (MFD) is obtained by setting as many consecutive derivatives of delay equal to zero as possible excluding the first derivative. Thus,
\[
a_1 \neq 0, \quad a_3 = a_5 = \ldots = 0
\]
for a MFD response. As many derivatives may be set to zero as there are degrees of freedom in \( \tau \). If we consider delay directly, the delay may be expressed as
\[
\tau(\omega^2) = \tau(0) \frac{1 + k_2\omega^2 + k_4\omega^4 + \ldots}{1 + l_2\omega^2 + l_4\omega^4 + \ldots}
\]
MFD response is obtained by setting as many coefficients equal as possible where
\[
k_{2i} = l_{2i}, \quad i = 1, 2, 3, \ldots
\]
This is the same coefficient condition as that for the MFM filter discussed in Sec. 4.2.

**EXAMPLE 4.9.1** For a second-order low-pass filter having a gain given by Eq. 4.2.6, determine the coefficient conditions for \( H(s) \) to be a maximally flat delay (MFD) filter.

**Solution** We first express the filter phase as
\[
\theta(j\omega) = \arg H(j\omega) = \tan^{-1} \left( \frac{-b_1\omega}{1 - b_2\omega^2} \right) = \left( \frac{b_1\omega}{1 - b_2\omega^2} \right)^3 + \ldots
\]
Using the binomial expansion
\[
(1 + x)^n = 1 + nx + n(n-1)x^2/2! + \ldots, \quad |x| < 1
\]
then the first two phase terms in Eq. 4.9.10 equal
\[
b_1\omega(1 - b_2\omega^2)^{-1} = b_1\omega[1 + b_2\omega^2 + (b_2\omega^2)^2 + \ldots]
\]
\[
-(b_1\omega^3/3)(1 - b_2\omega^2)^{-3} = -(b_1\omega^3/3)(1 + 3b_2\omega^2 + 6(b_2\omega^2)^2 + \ldots)
\]
Combining like powers of \( \omega \) gives
\[
\theta(j\omega) = b_1\omega + b_1\omega^3[b_2 - b_1^2/3] + b_1b_2\omega^5[b_2 - b_1^2] + \ldots
\]
Thus, to obtain MFD or linear phase, we set as many coefficients of ascending powers of $\omega$ equal to zero as possible where $b_1 \neq 0$. Since two degrees of freedom are available (i.e., $b_1$ and $b_2$), we may set two coefficients equal to zero as

$$b_2 - b_1^2/3 = 0, \quad b_2 - b_1^2 = 0$$  \hspace{1cm} (4.9.14)

These two equations require the trivial solution that $b_1 = b_2 = 0$ which contradicts $b_1 \neq 0$. Thus, we omit Eq. 4.9.15a and rewrite Eq. 4.9.15 as

$$b_1 = \sqrt{3} b_2, \quad b_1 \neq 0$$  \hspace{1cm} (4.9.15)

This is in contrast to the MFM condition of Eq. 4.2.8. Choosing $b_1 = 1$, the second-order MFD gain function has $\omega_\text{p} = \sqrt{3}$ and $\xi = \sqrt{3}/2 = 0.866$. This agrees with the result we obtained in Chap. 2 in Eq. 2.1.2. Recall that the second-order low-pass filter had no delay peaking for $\xi > 0.866$. Note that we determined the delay function as

$$\tau(\omega^2) = b_1 \frac{1 + b_2^2 \omega^2}{1 + (b_1^2 - 2b_2)\omega^2 + b_2^2 \omega^4}$$  \hspace{1cm} (4.9.16)

from Eq. 2.5.5, then an MFD response is obtained by setting the polynomial coefficients of the numerator and denominator equal as

$$b_1^2 = 3b_2, \quad b_2^2 = 0$$  \hspace{1cm} (4.9.17)

which is equivalent to Eq. 4.9.15 so the results agree.

**EXAMPLE 4.9.2** Determine the coefficient conditions to force $H(s)$ of Example 4.2.2 to have maximally flat delay (MFD) and a 3 dB bandwidth of 1 rad/sec. Is the solution unique (i.e., only a single $a_1$ value unlike Example 4.2.2)? Determine the coefficients if possible.

**Solution** Requiring $\omega_\text{3dB} = 1$ rad/sec in Eq. 4.2.9 constrains the gain as $|H(j1)|^2 = \frac{1}{2}$. This requires

$$2a_1^2 = b_1^2 - 2b_2 + b_2^2 - 1 = (b_1 - b_2)^2 - 1$$  \hspace{1cm} (4.9.18)

Expressing the phase of $H$ as

$$\theta(j\omega) = \arg H(j\omega) = \tan^{-1} a_1 \omega - \tan^{-1} \frac{b_1 \omega}{1 - b_2 \omega^2}$$  \hspace{1cm} (4.9.19)

we may expand tan$^{-1} x$ in a Maclaurin series as before as

$$\theta(j\omega) = \left[ a_1 \omega - \frac{1}{3} (a_1 \omega)^3 + \frac{1}{5} (a_1 \omega)^5 - \ldots \right] - \left[ \frac{b_1 \omega}{1 - b_2 \omega^2} - \frac{1}{3} \left( \frac{b_1 \omega}{1 - b_2 \omega^2} \right)^3 + \ldots \right]$$

$$= \omega [a_1 - b_1] + \omega ^3 [b_1^3/3 - b_1 b_2 - a_1^3/3] + \omega ^5 [b_1^3 b_2 - b_1 b_2^2 + a_1^5/5] + \ldots$$  \hspace{1cm} (4.9.20)

Since we have three degrees of freedom (i.e., $a_1$, $b_1$, and $b_2$) and have used one degree of freedom to obtain unity bandwidth, we may set two coefficients of $\omega^n$ to zero as

$$b_1^3/3 - b_1 b_2 - a_1^3/3 = 0, \quad b_1^3 b_2 - b_1 b_2^2 + a_1^5/5 = 0$$  \hspace{1cm} (4.9.21)

These three nonlinear equations in Eqs. 4.9.18 and 4.9.21 must be solved simultaneously for $a_1$, $b_1$, and $b_2$. A unique set of parameters will be obtained.

### 4.10 BESSEL (THOMSON) FILTERS

The Bessel or Thomson filter is the class of linear phase (or MFD) filter having all of its zeros of transmission at infinity. It is analogous to the Butterworth MFM filter. Its transfer function equals

$$H(s) = \frac{a_0}{a_0 + a_1 s + \ldots + a_n s^n} = \frac{P_n(0)}{P_n(s)}$$  \hspace{1cm} (4.10.1)
Thus, to obtain MFD or linear phase, we set as many coefficients of ascending powers of \( \omega \) equal to zero as possible where \( b_1 \neq 0 \). Since two degrees of freedom are available (i.e., \( b_1 \) and \( b_2 \)), we may set two coefficients equal to zero as

\[
b_2 - b_1^2/3 = 0, \quad b_2 - b_1^2/2 = 0
\]

These last two equations require the trivial solution that \( b_1 = b_2 = 0 \) which contradicts \( b_1 \neq 0 \). Thus, we omit Eq. (4.9.15b) and rewrite Eq. (4.9.15a) as

\[
b_1 = \sqrt{3}b_2, \quad b_1 \neq 0
\]

This is in contrast to the MFM condition of Eq. (4.2.8). Choosing \( b_1 = 1 \), the second-order MFD gain function has \( \omega_n = \sqrt{3} \) and \( \xi = \sqrt{3/2} = 0.866 \). This agrees with the result we obtained in Chap. 2 in Eq. 2.10.12. Recall that the second-order low-pass filter had no delay peaking for \( \xi \geq 0.866 \). Note that had we determined the delay function as

\[
\tau(\omega^2) = b_1 \frac{1 + b_2 \omega^2}{1 + (b_1^2 - 2b_2)\omega^2 + b_2^2\omega^4}
\]

from Eq. 2.5.5, then an MFD response is obtained by setting the polynomial coefficients of the numerator and denominator equal as

\[
b_1^2 = 3b_2, \quad b_2^2 = 0
\]

which is equivalent to Eq. (4.9.15) so the results agree.

**EXAMPLE 4.9.2** Determine the coefficient conditions to force \( H(s) \) of Example 4.2.2 to have maximally flat delay (MFD) and a 3 dB bandwidth of 1 rad/sec. Is the solution unique (i.e., only a single \( a_1 \) value unlike Example 4.2.2)? Determine the coefficients if possible.

**Solution** Requiring \( \omega_{3dB} = 1 \) rad/sec in Eq. 4.2.9 constrains the gain as \( |H(j1)|^2 = 1/2 \). This requires

\[
2a_1^2 = b_1^2 - 2b_2 + b_2^2 - 1 = (b_1 - b_2)^2 - 1
\]

Expressing the phase of \( H \) as

\[
\theta(j\omega) = \arg H(j\omega) = \tan^{-1} a_1 \omega - \tan^{-1} \frac{b_1 \omega}{1 - b_2 \omega^2}
\]

we may expand \( \tan^{-1}x \) in a Maclaurin series as before as

\[
\theta(j\omega) = [a_1 \omega - \frac{1}{3}(a_1 \omega)^3 + \frac{1}{5}(a_1 \omega)^5 - \ldots] - \frac{b_1 \omega}{1 - b_2 \omega^2} - \frac{1}{3} \frac{(b_1 \omega)^3}{1 - b_2 \omega^2} + \ldots
\]

\[
= \omega[a_1 - b_1] + \omega^3[b_1^2/3 - b_1 b_2 - a_1^3/3] + \omega^5[b_1^3/5 - b_1 b_2^2 - a_1^5/5] + \ldots
\]

Since we have three degrees of freedom (i.e., \( a_1, b_1, \) and \( b_2 \)) and have used one degree of freedom to obtain unity bandwidth, we may set two coefficients of \( \omega^n \) to zero as

\[
b_1^2/3 - b_1 b_2 - a_1^3/3 = 0, \quad b_1^3/5 - b_1 b_2^2 + a_1^5/5 = 0
\]

These three nonlinear equations in Eqs. 4.9.18 and 4.9.21 must be solved simultaneously for \( a_1, b_1, \) and \( b_2 \). A unique set of parameters will be obtained.

**4.10 BESSEL (THOMSON) FILTERS**

The Bessel or Thomson filter is the class of linear phase (or MFD) filter having all of its zeros...
\[
P_n(s) = s^n B_n(1/s) = \sum_{i=0}^{n} \frac{(2n-i)!}{2^{n-i} (n-i)!} s^i
\]  

where \( B_n(s) \) is the \( n \)-th order Bessel polynomial of the first kind. Unfortunately, the \( P_n(s) \) polynomials are often mistakenly called Bessel polynomials which leads to confusion. The poles of the Bessel filter are listed in Table 4.10.1 (for orders through 31, see Ref. 31). These poles have been frequency normalized to produce a 3 dB bandwidth of unity. The Bessel filter transfer function is derived in the following manner. The transfer function of an ideal constant delay (or linear phase) filter equals

\[
H(s) = \exp(-s \tau_o) \bigg|_{\tau_o=1} = \frac{1}{e^s} = \frac{1}{\sinh s + \cosh s}
\]  

for one second delay. Storch suggested a procedure for obtaining the \( n \)-th order approximation of Eq. 4.10.3. Utilizing the Maclaurin series expansion for \( \sinh s \) and \( \cosh s \) where

\[
\sinh s = s + \frac{s^3}{3!} + \frac{s^5}{5!} + \ldots, \quad \cosh s = 1 + \frac{s^2}{2!} + \frac{s^4}{4!} + \ldots
\]

the continued fraction expansion of \( \cosh s \) is determined as

\[
\cosh s = \frac{\cosh s}{\sinh s} = 1 + \frac{1}{\frac{3}{s} + \frac{1}{\frac{5}{s} + \frac{1}{\ldots}}}
\]  

The \( n \)-th order approximation of \( e^s \) is obtained by truncating the continued fraction expansion at the \( (2n-1)/s \) term as

\[
\frac{\cosh s}{\sinh s} \approx 1 + \frac{1}{\frac{3}{s} + \ldots + \frac{1}{(2n-1)/s}} = \frac{L_n(1/s)}{M_n(1/s)}
\]
and summing numerator and denominator polynomials. For example, the third-order polynomial approximation is found from

\[
\coth s = \frac{\cosh s}{\sinh s} = \frac{1}{s} \left( \frac{1}{s} - \frac{1}{5s} + \frac{1}{15s} \right) = \frac{s^2 + 15}{s^3 + 15s} = \frac{6s^2 + 15}{s^3 + 15s}/s^3 + 15s^2
\]

Therefore, from Eq. 4.10.7, the third-order Bessel filter has a transfer function

\[
H(s) = \frac{s^3B_3(\infty)}{15 + 15s + 6s^2 + s^2} = \frac{P_3(0)}{P_3(s)} = \frac{s^2B_3(s)}{s^2B_3(1/s)}
\]

which has an ideal delay of one second and a dc gain of unity.

Scanlan found a simple method for determining the approximate poles of a Bessel filter. The approximate Bessel filter poles are located on the Butterworth unit circle. Their imaginary parts are separated by 2/n of the diameter beginning and ending with half this value as shown in Fig. 4.10.1. The magnitude and delay characteristics are shown in Fig. 4.10.2. The delay characteristics are maximally flat. The transfer function given by Eq. 4.10.1 can be expressed in a Maclaurin series as

\[
H(s) = e^{-s}e^{\Gamma(s)}
\]

where

\[
\Gamma(s) = \gamma_2 s^2 + \gamma_4 s^4 + \ldots
\]

The \(\gamma_n\) coefficients are given by

\[
\gamma_n = \frac{1}{n} \sum_{k=1}^{n} p_k^{-n}
\]

where the \(p_n\) are the poles of the Bessel filter. Of special importance is \(\gamma_2\) where \(\gamma_2 = 1/2(2n - 1)\). Thus, truncating \(\Gamma\) at the \(s^2\) term for small \(\omega\), then Eq. 4.10.9 becomes

\[
H(j\omega) = \exp \left[ -\omega^2/2(2n - 1) \right] \exp \left[ -j\omega \right]
\]

It has a Gaussian magnitude as we will discuss later and a delay of one second. Thus, the magnitude for small \(\omega\) equals

\[
|H(j\omega)| = \exp \left[ -\omega^2/2(2n - 1) \right] = -10\omega^2/(2n - 1) \ln 10
\]

which improves with increasing \(n\). Setting \(|H| = \frac{1}{2}\), the approximate 3 dB bandwidth of the Bessel filter equals

\[
\text{(4.10.14)}
\]
The filter selectivity at the band-edge equals
\[ \frac{-d|H|/d\omega}{\omega = \omega_3 \text{ dB}} = P_n(0)P_{n/2} \frac{\omega^3}{\omega_3 \text{ dB}} = 0.491 \] (4.10.15)

which is virtually independent of \( n \) for large \( n \) using Eqs. 4.10.13 and 4.10.14. The selectivity of the Bessel filter is less than that of a second-order Butterworth filter. Thus, Bessel filters have very poor selectivity compared with Chebyshev, Papoulis, elliptic, and other narrow transition band filters.

The order of a Bessel filter can be easily determined from the nomograph of Fig. 4.10.3. The very poor selectivity of the Bessel filter is obvious from the nomograph when it is compared with the earlier nomographs. In the stopband, the Bessel filter provides about \((0.4n)^{1.5n}\) less rejection than a Butterworth filter. As we will see later from the peculiar cross-over for \( n = 10 \) and 15, the nomograph confirms that the Bessel filter becomes Gaussian at small \( \omega \) and large \( n \).

**EXAMPLE 4.10.1** Determine the order of a Bessel filter to meet the specification of Example 4.3.1 where \( M_p = 1.25 \text{ dB}, M_{q1} = 40 \text{ dB at } \Omega_{q1} = 2, \) and \( M_{q2} = 60 \text{ dB at } \Omega_{q2} = 3.\)

**Solution** Entering the data on the Bessel filter nomograph, we see that it is impossible to meet this specification.
EXAMPLE 4.10.2 A low-pass Bessel filter must be designed to meet the following specifications: (1) low-frequency gain = 20 dB, (2) 3 dB frequency = 1000 Hz, (3) 20 dB frequency = 2700 Hz, (4) 60 dB frequency = 10 KHz, and (5) minimum high-frequency rolloff = −60 dB/dec. Determine the required order of the filter. Write its transfer function. Draw a cascaded block diagram showing the required design parameters for each stage.

Solution From the rolloff requirement, the order must be at least equal to \( n \geq -60/(-20) = 3 \). From the nomograph, we find \( n_1 \geq 4 \) and \( n_2 \geq 4 \). The transfer function of the filter must, therefore, equal

\[
H(s) = \frac{10[1.431^2][1.604^2]}{[s_n^2 + 2(0.958)(1.431)s_n + 1.431^2][s_n^2 + 2(0.621)(1.604)s_n + 1.604^2]}
\]

(4.10.16)

where \( s_n = s/2\pi(1 \text{ KHz}) \) from Table 4.10.1. The block diagram of the filter is easily drawn in Fig. 4.10.4.
Although the Bessel filter has poor selectivity, it has excellent delay characteristics (MFD) as seen in Fig. 4.10.2. The Bessel filter having the transfer function given by Eq. 4.10.1 has a one second dc delay (and a 3 dB bandwidth given by Eq. 4.10.14). When we normalize the 3 dB bandwidth to unity as was done for the poles listed in Table 4.10.1, then the dc delay is increased by $\omega_{3\text{db}}$ so $\tau(0)$ has the value given by Eq. 4.10.14 which agrees with the figure. Storch showed that delay can be expressed as:

$$\tau(\omega) = 1 - \frac{(2\pi)^n}{2^{2n+1}} \omega^{2n-1} \left[1 + \frac{1}{(2n - 1)} \omega^2 + \frac{2(2n - 2)(2n - 1)}{(2n - 3)} \omega^4 + \ldots \right]$$

(4.10.17)

Since the first $n$ terms in $\omega^0, \omega^2, \ldots, \omega^{2n-2}$ are missing, we see that the first $(n - 1)$ derivatives of $\tau$ are zero at $\omega = 0$. Thus, delay is MFD as required.

The unit step response of the Bessel filter having unity 3 dB bandwidth is shown in Fig. 4.10.5. The rise time is almost constant at two seconds, but the delay increases with filter order. Although overshoot is not apparent from Fig. 4.10.5, it does occur for $n > 2$ but is less than 1%.

### 4.11 EQUIRIPPLE DELAY FILTERS

The equiripple delay filter approximates an MFD response as the Chebyshev filter approximates an MFM response. The equiripple delay filter approximates constant delay over a wider frequency range than does the Bessel filter.

The equiripple delay filter has a delay function

$$\tau(\omega) = \frac{\tau_0}{1 + e^{T_{2n}^o(\omega)}}$$

(4.11.1)

where parameter $e$ is the delay ripple parameter and $T_{2n}$ is a $2n$th-order Chebyshev polynomial. The nominal low-frequency delay is $\tau_0$ seconds. Since $|T_{2n}| < 1$ in $|\omega| < 1$, the low-frequency delay is bounded between $\tau_0/(1 + e)$ and $\tau_0/(1 - e)$ as shown in Fig. 4.11.1. The dc delay equals $\tau_0/(1 + e)$ for $n$ even and $\tau_0/(1 - e)$ for $n$ odd. The filter order $n$ equals the number of delay maxima and minima in the filter passband $|\omega| < 1$ (excluding the minimum at $\omega = 1$). Recall that the delay, phase, and magnitude responses are interrelated as discussed in Sec. 1.17. Thus, the
6 FREQUENCY TRANSFORMATIONS

In the last two chapters, we investigated the behavior of a variety of classical and optimum low-pass filters. In the frequency domain and time domain analyses of Chaps. 2 and 3, we briefly discussed the concept of frequency transformations. Frequency transformations allowed high-pass, band-pass, band-stop, and all-pass filters to be derived from low-pass filters. We now want to explore this important topic in detail for it will allow us to utilize and exploit all the low-pass filter information we have derived in previous chapters for designing other filter types.

We begin by considering frequency scaling and frequency translation which are the simplest forms of frequency transformations. Then we shall consider the frequency transformations which are used to obtain high-pass, band-pass, band-stop, and all-pass filters. The frequency and time domain interrelations will be carefully investigated. In general, we will find that although these relationships are easily expressed mathematically, they require computer evaluation. Other important, but less known, transformations will be introduced for use in certain applications.

6.1 FREQUENCY SCALING

One of the simplest frequency transformations is that of scaling. We saw in Sec. 1.8 that frequency scaling consisted of replacing \( p \) by \( s/\omega_0 \). The \( s \)-plane is therefore equal to the \( p \)-plane when it is expanded by a factor of \( \omega_0 \). Frequency scaling is equivalent to frequency denormalization; it was used extensively in the earlier chapters to obtain maximum simplicity in equations and graphs. If the normalized transfer function equals \( H(p) \), then the denormalized transfer function \( H(s) \) equals

\[
H(s) = H(p) \bigg|_{p = s/\omega_0}
\]  

(6.1.1)

For example, the first- and second-order low-pass filters had frequency normalized gains of

\[
H_1(p) = \frac{1}{p + 1}, \quad H_2(p) = \frac{1}{p^2 + 2\xi p + 1}
\]  

(6.1.2)

Scaling frequency by \( \omega_0 \) yields frequency denormalized gains of

\[
H_1(s) = \frac{1}{s/\omega_0 + 1} = \frac{\omega_0}{s + \omega_0}, \quad H_2(s) = \frac{1}{(s/\omega_0)^2 + 2\xi(s/\omega_0) + 1} = \frac{\omega_0^2}{s^2 + 2\xi \omega_0 s + \omega_0^2}
\]  

(6.1.3)

We saw in Fig. 2.3.4 that the effect of frequency scaling was to shift the magnitude and phase
responses of the filter from unity to $\omega_0$. The amplitudes and shaping of the magnitude and phase responses remain unchanged. We saw that frequency scaling also shifted the delay response of the filter from unity to $\omega_0$. However, the delay was also reduced by a factor of $1/\omega_0$.

Now that we have reviewed the effects of frequency scaling on the ac steady-state response of a filter, let us review its effects on the time domain response. From Table 1.8.2, we see that if the impulse response of the filter equals

$$h(t) = \mathcal{L}^{-1} [H(p)]$$ (6.1.4)

then under frequency scaling, the response becomes

$$f(t) = \mathcal{L}^{-1} [F(s)] = \mathcal{L}^{-1} [H(s/\omega_0)] = |\omega_0| h(\omega_0 t)$$ (6.1.5)

Thus, the impulse response has the same shape but its amplitude is increased by a factor of $|\omega_0|$ and time is contracted by a factor of $1/\omega_0$. The step response equals

$$g(t) = \mathcal{L}^{-1} [\tilde{F}(s)/s] = \mathcal{L}^{-1} [H(s/\omega_0)/s] = r(\omega_0 t)$$ (6.1.6)

Therefore, the step response has the same amplitude and time is contracted by the factor $1/\omega_0$.

6.2 FREQUENCY TRANSLATION

Another basic frequency transformation is that of translation. We saw in Sec. 1.8 that frequency translation consisted of replacing $s$ by $s + \alpha$. The $s$-plane is therefore equal to the $p$-plane when it is translated by $-\alpha$. If the normalized transfer function equals $H(p)$, then the denormalized transfer function $H(s)$ equals

$$H(s) = H(p) \bigg|_{p = s + \alpha}$$ (6.2.1)

For example, if

$$H_1(p) = \frac{1}{p}, \quad H_2(p) = \frac{1}{p^2 + \beta^2}$$ (6.2.2)

then

$$H_1(s) = \frac{1}{s + \alpha}, \quad H_2(p) = \frac{1}{(s + \alpha)^2 + \beta^2}$$ (6.2.3)

The magnitude, phase, and delay responses are modified considerably. However, the impulse (or step) response is easily written. Since

$$h(t) = \mathcal{L}^{-1} [H(p)]$$ (6.2.4)

then

$$f(t) = \mathcal{L}^{-1} [F(s)] = \mathcal{L}^{-1} [H(s + \alpha)] = e^{-\alpha t} h(t)$$ (6.2.5)

using Table 1.8.2. Thus, translating frequency is equivalent to multiplying the impulse (or step) response by $e^{-\alpha t}$.

Now that we have reviewed frequency scaling and translation, let us utilize these results for deriving high-pass, band-pass, band-stop, and all-pass filters from low-pass filters.


\[ H_{HP}(s) = H_{LP}(p) \bigg| p = \omega_0/s \]  

(6.3.1)

Let us now derive this result. We denote the complex frequency variables \( p \) and \( s \) as

\[ p = u + jv, \quad s = \sigma + j\omega \]  

(6.3.2)

In ac steady-state, \( p = jv \) and \( s = j\omega \). To form a high-pass filter from a low-pass filter, we require the magnitude characteristic to be inverted and to exhibit geometric symmetry as shown in Fig. 6.3.1. The low-pass filter behavior at low frequencies becomes the exact high-pass filter behavior at high frequencies and visa versa. Alternatively stated, we require the frequency axis to be inverted such that: (1) \( p = j0 \) maps into \( s = +j\infty \) (or \(-j\infty \)), (2) \( p = j\omega_0 \) maps into \( s = +j\omega_0 \) (or \(-j\omega_0 \)), and (3) \( p = j\infty \) maps into \( s = j0 \). The transformation which produces this inversion is

\[ |\omega| = 1/|v| \quad \text{or} \quad \omega = \pm 1/v \]  

(6.3.3)

Thus, substituting \( \omega = s/j \) and \( v = p/j \) (i.e., using analytic continuation), the general low-pass to high-pass transformation must equal

\[ s/j = +j/p \quad \text{or} \quad s = \mp 1/p \]  

(6.3.4)

### 6.3.1 EFFECT ON THE S-PLANE

Which of the two transformations should be used? In general, we require minimum-phase filters (which have no right-half-plane zeros or poles) to generate minimum-phase filters under the transformation. Inspecting the two transformations of Eq. 6.3.4, and writing

\[ s = |s| \exp \{j \arg s\}, \quad p = |p| \exp \{j \arg p\} \]  

(6.3.5)

then the magnitudes satisfy \( |s| = 1/|p| \) in both cases. Thus, the critical frequencies are located at reciprocal distances from the origin. Their angles equal

\[ \arg s = -\arg p \quad (\text{for} \ s = 1/p), \quad \arg s = -\arg p + \pi \quad (\text{for} \ s = -1/p) \]  

(6.3.6)

In the second case, the angles are negatives of each other. However, in the first case, an additional 180° rotation must be added. Thus, if the low-pass filter was minimum-phase, the associated...
high-pass filter would also be minimum-phase if we used the transformation \( s = 1/p \). However, if we use the other transformation having a negative sign, then the high-pass filter would be non-minimum-phase. Thus, \( s = 1/p \) is the proper transformation.

To allow the frequency-normalized low-pass filter results to be denormalized, we use \( s = \omega_0/p \) so that the magnitude characteristic is inverted about \( \omega = \omega_0 \) rather than \( \omega = 1 \). For a low-pass filter having any pole-zero distribution in the s-plane, the pole-zero distribution of the high-pass filter is obtained by simply inverting the s-plane around \( \omega_0 \). Thus, low-pass poles or zeros at \( p = 0, \infty \), \( \omega_0 e^{j\theta} \), and \( \infty \) become high-pass poles or zeros at \( s = \infty, \omega_0 e^{-j\theta}/\omega_0 \), and \( 0 \). Note that the damping factors and Q's of all the poles and zeros remain unchanged. Also note that if the low-pass filter has \( n \) zeros at infinity, then the transformation produces \( n \) zeros at the origin (reciprocal of infinity) for the high-pass filter. Summarizing, high-pass filter poles and zeros are easily obtained from low-pass filter poles and zeros by simple inversion.

**EXAMPLE 6.3.1** Determine the transfer function for a fourth-order Butterworth high-pass filter having unity high-frequency gain and a 3 dB frequency of 1000 Hz. Draw its block diagram realization.

**Solution** We first determine the transfer function of the equivalent low-pass filter and then use the low-pass to high-pass transformation. From Table 4.3.1, the transfer function of the low-pass Butterworth filter equals

\[
H(p) = \frac{1}{p^2 + 2(0.924)p + 1}\frac{1}{p^2 + 2(0.383)p + 1}
\]  

(6.3.7)

Using the low-pass to high-pass transformation \( p = \omega_0/s = 1/s_n \), then

\[
H(s) = \frac{s_n^2}{s_n^2 + 2(0.924)s_n + 1}\frac{s_n^2}{s_n^2 + 2(0.383)s_n + 1}
\]

(6.3.8)

where \( s_n = s/2\pi(1000 \text{ Hz}) \). The block diagram of the high-pass filter is shown in Fig. 6.3.2. It is usually easier to convert low-pass filter blocks directly into high-pass filter blocks as shown in the next example.

**EXAMPLE 6.3.2** Determine the transfer function for a fourth-order Chebyshev high-pass filter having unity high-frequency gain, a maximum in-band ripple of 3 dB, and a 3 dB cutoff frequency of 1000 Hz. First, use the low-pass to high-pass transformation. Then simply transform the block diagram of the low-pass filter directly.

**Solution** From Table 4.4.1, the fourth-order Chebyshev low-pass filter has a transfer function

\[
H(p) = 0.707\frac{1}{p_{n1}^2 + 2(0.464)p_{n1} + 1}\frac{1}{p_{n2}^2 + 2(0.0896)p_{n2} + 1}
\]

(6.3.9)

**Fig. 6.3.3** Conversion of block diagram realization of (a) fourth-order Chebyshev low-pass filter to that for the (b) equivalent high-pass filter.
where \( p_{n1} = p/0.443 \) and \( p_{n2} = p/0.950 \). Letting \( p = 1/s_n \), the high-pass filter has a transfer function

\[
H(s) = \frac{(s_n/2.26)^2}{(s_n/2.26)^2 + 2(0.464)(s_n/2.26) + 1} \cdot \frac{(s_n/1.05)^2}{(s_n/1.05)^2 + 2(0.0896)(s_n/1.05) + 1}
\]

(6.3.10)

where \( s_n = s/2\pi(1000 \text{ Hz}) \). Notice that we have used the frequency-normalized form so that the corner frequency and damping factor of each stage can be easily identified.

Now we obtain equivalent information directly from the block diagram realization of the low-pass filter which is shown in Fig. 6.3.3a. With this conversion method, the normalized frequencies are first inverted and then denormalized; damping factors and Q's are left unchanged. Thus, we may easily convert the low-pass filter into a high-pass filter realization as shown in Fig. 6.3.3b.

**EXAMPLE 6.3.3** Convert the fourth-order elliptic low-pass filter of Example 4.8.2 to an elliptic high-pass filter having analogous parameters and a band-edge of 1 KHz.

**Solution** The low-pass filter has the block diagram shown in Fig. 6.3.4a assuming it has unity bandwidth. Inverting normalized frequency gives the equivalent high-pass filter frequencies. Relabelling blocks completes the design (note that LPF becomes HPF, but BSF remains BSF under the transformation). To emphasize the frequency inversion, we have labelled the blocks in terms of normalized frequency where \( f_n = f/1000 \text{ Hz} \).

6.3.2 **MAGNITUDE, PHASE, AND DELAY RESPONSES**

Magnitude, phase, and delay responses are easily determined from Eq. 6.3.1. The magnitude responses of high-pass filters are related to those of low-pass filters as

\[
|H_{HP}(j\omega/\omega_0)| = |H_{LP}(-j/\omega)| = |H_{LP}(j/\omega)|
\]

(6.3.11)

(recalling |H| is always an even function). The frequency inversion of the magnitude characteristic is shown in Fig. 6.3.5a. An alternative plot which is more general is shown in Fig. 6.3.6. Here gain magnitudes are drawn over the entire p- and s-planes. We see that \( p = 0 \) where \( H_{LP}(0) = 1 \) maps into \( |s| = \infty \) where \( H_{HP}(\infty) = 1 \). This forms the “drumhead” around the perimeter of \( H_{HP}(s) \). We also see that \( |p| = \infty \) where \( H_{LP}(\infty) = 0 \) maps into the origin \( s = 0 \) where \( H_{HP}(0) = 0 \). This vividly shows the process of complex frequency inversion in the low-pass to high-pass transformation.

The phase responses of high-pass filters can also be easily expressed as

\[
\arg H_{HP}(j\omega/\omega_0) = \arg H_{LP}(-j/\omega) = -\arg H_{LP}(j/\omega)
\]

(6.3.12)

(recalling \( \arg H \) is always an odd function). Thus, the high-pass filter phase is obtained by simply taking the negative of the low-pass filter phase after it is inverted about \( \nu = 1 \) and translated to \( \nu = \omega_0 \). This is also illustrated in Fig. 6.3.5b.

The delays of high-pass filters can also be easily written. Differentiating Eq. 6.3.12 and remembering that \( \nu = -1/\omega \) and \( \tau \) is always an even function, then for \( \omega_0 = 1 \),

\[
\tau_{HP}(j\omega) = -\frac{d \arg H_{HP}(j\omega)}{d\omega} = -\frac{1}{\omega^2} \frac{d \arg H_{LP}(-j/\omega)}{d(-1/\omega)} = \frac{1}{\omega^2} \tau_{LP}(j/\omega)
\]

(6.3.13)
Fig. 6.3.5 Effect of low-pass to high-pass transformation upon (a) magnitude and (b) phase characteristics.

Thus, to obtain the delay of the high-pass filter, we simply invert the low-pass filter delay characteristic about \( \omega = 1 \) and multiply by \( \omega^{-2} \). In general, high-pass and low-pass filters have quite different delay characteristics. On a frequency-denormalized basis, it is easy to show that

\[
\tau_{HiP}(\omega) = \omega^2 \tau_{LP}(1/\omega) = (b_n - 1/b_n - a_m - 1/a_m), \quad \omega \ll \omega_o \tag{6.3.14}
\]

\[
= \tau_{LP}(0)/\omega^2 = (b_1/b_0 - a_1/a_0)/\omega^2, \quad \omega \gg \omega_o
\]

where the gain of the low-pass filter is given in the summation form of Eq. 2.0.1. This is easily proved using Eqs. 2.10.34, 2.10.36, and 6.3.13. Eq. 6.3.14 shows the high-frequency slope of \( \tau_{LP} \) (defined as \( d\tau_{LP}/d(1/\omega^2) \)) equals the dc delay of the high-pass filter and vice versa. From a pole-zero standpoint, letting \( s = \omega/j \) in Eq. 6.3.13 gives

\[
\tau_{HiP}(s) = -\tau_{LP}(-1/s)s^2 = -\tau_{LP}(1/s)s^2 \tag{6.3.15}
\]

Thus, the delay pole-zero diagram of the high-pass filter is equal to an inverted version of the delay pole-zero pattern for the low-pass filter with the addition of two poles at \( s = 0 \).
Fig. 6.3.7 Magnitude and phase characteristics of Butterworth low-pass and high-pass filters.\textsuperscript{1}

Fig. 6.3.8 Pole-zero pattern for (a) gain and (b) delay of fourth-order Butterworth low-pass and high-pass filters of Example 6.3.5. (The conjugate poles are not shown.)

**EXAMPLE 6.3.4** Determine the magnitude and phase responses of Butterworth high-pass filters from the magnitude and phase responses of unity-bandwidth Butterworth low-pass filters shown in Fig. 6.3.7.

**Solution** The magnitude characteristic $H_p$ of the high-pass filter is obtained by rotating the magnitude characteristic $G_n$ of the low-pass filter about $\nu = 1$. To obtain the phase characteristic $\arg H_p$, we rotate the phase characteristic $\arg G_n$ of the low-pass filter about $\nu = 1$ and change its sign.

**EXAMPLE 6.3.5** Relate the delay responses of Butterworth high-pass filters to the delay responses of unity-bandwidth Butterworth low-pass filters shown in Fig. 4.3.2.

**Solution** This problem is easily solved from the pole-zero standpoint. The delay pole-zero configuration for a fourth-order low-pass Butterworth filter is shown in Fig. 6.3.8b. This is obtained using Fig. 2.10.1 and the gain poles-zeros of the Butterworth filter. Inverting the delay pattern and adding two poles at $s = 0$ using Eq. 6.3.15, we obtain the delay pole-zero configuration of the fourth-order high-pass Butterworth filter. Because all poles and zeros lie on a unit circle for any order Butterworth filter, the configuration remains unchanged. Therefore, we can make the immediate observation that the delay characteristic of Butterworth high-pass filters must be identical to that of Butterworth low-pass filters, except perhaps for a scaling constant. To determine the constant, we note that the dc delay of the high-pass filter $\tau_{HP}(0) = b_{n-1}/b_n$ and the dc delay of the low-pass filter $\tau_{LP}(0) = b_1/b_0$ from Eq. 6.3.14. Since Butterworth filters have $b_0 = b_n = 1$ and $b_1 = b_{n-1} = 2.63131$ (see Table 4.3.1), $\tau_{HP}(0) = \tau_{LP}(0) = 2.6131$ and the delay characteristics are identical. Note that since other filter types do not have all poles and zeros on the unit circle, their delay characteristics will not be preserved.

High-pass filters are specified in the frequency domain, the time domain, or in both domains simultaneously. Frequency domain specifications are especially convenient since we can utilize the classical and optimum filters developed in Chaps. 4 and 5. The nomographs of Chap. 4 are again inval
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In general, the frequency domain specifications of high-pass filters have the form shown in Fig. 6.3.9a. The maximum passband ripple is $M_p$ for frequencies greater than $f_o$. In the stopband of the high-pass filter, the minimum rejection is $M_{a1}$ for $f_1 \leq f < f_o$, and so forth. To utilize the design data of earlier chapters, we convert this into equivalent information for a low-pass filter. The conversion consists of simply normalizing frequencies in the high-pass filter relative to $f_o$. We then form the reciprocal normalized frequency which describes frequency in the low-pass filter. In this form, we can now utilize the low-pass filter magnitude responses and their associated nomographs. After the desired filter type has been selected and its order determined, the normalized poles and zeros of the low-pass filter are converted to those for the high-pass filter using the low-pass to high-pass transformation just discussed.

**EXAMPLE 6.3.6** Determine the minimum filter order required to satisfy the high-pass characteristics shown in Fig. 6.3.10a. Consider Butterworth, Chebyshev, and elliptic filters.

**Solution** The stopband frequencies are normalized by the corner frequency as shown in Fig. 6.3.10a. Inverting normalized frequency results in the low-pass filter specification of Fig. 6.3.10b. From the nomographs of Chap. 4, we find:

- Butterworth: $n_1 \geq 17$ and $n_2 \geq 10$ (use $n \geq 17$)
- Chebyshev: $n_1 \geq 8$ and $n_2 \geq 7$ (use $n \geq 8$)
- Elliptic: $n_1 \geq 5$ and $n_2 \geq 5$ (use $n \geq 5$)

As we anticipate, the elliptic type yields the minimum order filter.
EXAMPLE 6.3.7 Touch-Tone telephone systems require two band-splitting filters to separate low- and high-frequency groups as discussed in Example 2.6.3. Determine the required order of Butterworth, Chebyshev, and elliptic low-pass and high-pass filters to obtain at least 40 dB of stopband rejection. Which filter would you recommend using and why?

Solution The desired magnitude characteristics are shown in Fig. 6.3.11 where we have chosen to allow 3 dB of in-band ripple. The maximum frequency from the low group (941 Hz) and the minimum frequency from the high group (1209 Hz) form the band-edges of the passband and stopband. Normalizing these frequencies, we see that the low-pass and high-pass filters will be of identical order. Entering this data on the nomographs of Chap. 4, we find:

Butterworth: \( n > 19 \), Chebyshev: \( n > 8 \), Elliptic: \( n > 5 \)

The elliptic filter is of lowest degree because of its narrow transition band. Due to its low order, it is useful in this application. We must assure that the amplitude limiter (see Fig. 2.7.3) can accept the large step response overshoot and the settling time for the filters.

6.3.3 IMPULSE AND STEP RESPONSES

The step and impulse responses of high-pass filters are easily expressed in terms of their analogous low-pass filter responses. If the low-pass filters have step response \( r_{LP}(t) \) and impulse response \( h_{LP}(t) \) which have Laplace transforms of

\[
H_{LP}(p) = \mathcal{L}[h_{LP}(t)], \quad R_{LP}(p) = \mathcal{L}[r_{LP}(t)] = H_{LP}(p)/p = \mathcal{L}[\int_{-\infty}^{t} h_{LP}(\tau) d\tau] \quad (6.3.16)
\]

then the analogous high-pass filters responses must equal

\[
h_{HP}(t) = \mathcal{L}^{-1}[H_{LP}(p)] \bigg|_{p = 1/s} = \mathcal{L}^{-1}[H_{LP}(1/s)/s], \quad r_{HP}(t) = \mathcal{L}^{-1}[H_{LP}(1/s)/s] \quad (6.3.17)
\]

using the low-pass to high-pass transformation. Note that \( r_{HP} \) is not equal to \( \mathcal{L}^{-1}[R_{LP}] \). It can be shown that if \( f(t) \) has a Laplace transform \( F(s) \), then

\[
\mathcal{L}^{-1}[F(1/s)/s] = \int_{0}^{\infty} J_0(2\sqrt{\tau}) f(\tau) d\tau u_{-1}(t) \quad (6.3.18)
\]

where \( J_0 \) is the zero-order Bessel function of the first kind. Using the time integration theorem from Table 1.8.2, we can write that

\[
\mathcal{L}^{-1}[F(1/s)] = \int_{0}^{\infty} \frac{d}{dt} [J_0(2\sqrt{\tau}) f(\tau)] d\tau u_{-1}(t) + \int_{0}^{\infty} J_0(0)f(\tau) d\tau u_{0}(t)
\]

\[
= -t^{-1} \int_{0}^{\infty} \sqrt{\tau} J_0(2\sqrt{\tau}) f(\tau) d\tau u_{-1}(t) + \int_{0}^{\infty} f(\tau) d\tau u_{0}(t) \quad (6.3.19)
\]
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Fig. 6.3.12 Bessel functions \( J_0(2x^{1/4}) \) and \( x^{1/4} J_1(2x^{1/4}) \).\(^3\)

\( J_1 \) is the first-order Bessel function of the first kind. The impulse response of the high-pass filter must therefore equal

\[
h_{HP}(t) = \mathcal{F}^{-1}[H_{LP}(1/s)] = -t^{-1} \int_0^\infty \sqrt{\tau} J_1(2\sqrt{\tau}) h_{LP}(\tau) \, d\tau \left. U_{-1}(t) + r_{LP}(\infty) \right. U_0(t)
\]

(6.3.20)

The step response of the high-pass filter equals

\[
r_{HP}(t) = \int_{-\infty}^{t} h_{HP}(\tau) \, d\tau = \int_0^\infty J_0(2\sqrt{\tau}) h_{LP}(\tau) \, d\tau \left. U_{-1}(t) \right.
\]

(6.3.21)

These are very useful results. They show that the step (or impulse) response of a high-pass filter is the integral of the product of the low-pass filter impulse response times \( J_0 \) (or \( J_0' \) plus a \( U_0 \) term).

Although \( h_{LP} \) may be a monotonic response, \( J_0 \) or \( J_0' \) can cause \( h_{HP} \) or \( r_{HP} \) to be oscillatory since \( J_0 \) and \( J_0' \) are themselves oscillatory as shown in Fig. 6.3.12. The second term in Eq. 6.3.20 represents an impulse in \( h_{HP} \) at \( t = 0 \) of area \( r_{LP}(\infty) \). This impulse allows \( r_{HP} \) to make an instantaneous transition from a value 0 to \( r_{LP}(\infty) \) at time \( t = 0 \). Although these results are easily written mathematically and are useful conceptually, they require a computer for evaluation in most practical filters. Inverse Laplace transform solutions are more practical and direct although they also require computer evaluation.

The responses of several classical filters are shown in the following figures: the step response of Butterworth high-pass filters is shown in Fig. 6.3.13;\(^4\) the impulse and step responses of Chebyshev high-pass filters having ripples of 0.5 and 2 dB and unity ripple bandwidths are shown in Fig. 6.3.14.

Another useful result concerns the initial and final values of the high-pass filter unit step response. These are easily related to the responses of the analogous low-pass filter as

\[
r_{HP}(0) = \lim_{s \to \infty} sR_{HP}(s) = H_{HP}(\infty) = \lim_{p \to 0^+} pR_{LP}(p) = H_{LP}(0) = r_{LP}(\infty)
\]

\[
r_{HP}(\infty) = \lim_{s \to 0^+} sR_{HP}(s) = H_{HP}(0) = \lim_{p \to \infty} pR_{LP}(p) = H_{LP}(\infty) = r_{LP}(0)
\]

(6.3.22)

Thus, the initial value of the step response of a high-pass filter is equal to the final value of the step response of its analogous low-pass filter, and vice versa. This becomes obvious as soon as we recall
\[ \delta \equiv -r_{\text{HP}}(0) = \lim_{s \to \infty} [sR_{\text{HP}}(s) - r_{\text{HP}}(0)] = \lim_{s \to 0} [H_{\text{HP}}(s) - H_{\text{HP}}(\infty)] \] (6.3.23)

In terms of the analogous low-pass filter from which \(H_{\text{HP}}\) is derived, the sag equals

\[ \delta \equiv -\lim_{s \to 0} [H_{\text{LP}}(s) - r_{\text{LP}}(\infty)] = -\lim_{s \to 0} [H_{\text{LP}}(s) - H_{\text{LP}}(0)] = b_1/a_0 - a_1/a_0 \, \text{sec}^{-1} \] (6.3.24)

where \(H_{\text{LP}}\) is given by Eq. 2.0.1. Thus, the sag in the high-pass filter step response is equal to the dc delay of the analogous low-pass filter.

Meyer has found a useful result concerning step response overshoot. All high-pass filters having more than one zero at the origin (i.e., more than +6 dB/oct or +20 dB/dec low-frequency rolloff in its magnitude characteristic) must always have step response overshoot. Thus, no high-pass filter having greater than +20 dB/dec low-frequency rolloff can be constructed that will not exhibit step response overshoot. However, high-pass filters having a single zero at the origin may or may not exhibit overshoot.

In Sec. 4.18, we discussed the pulsed-cosine response of low-pass filters. In like manner, it is of interest to investigate the pulsed-cosine response of high-pass filters. The pulsed-cosine input \(s_1\) to the high-pass filter is given by Eq. 4.18.13, its spectrum \(S_1\) by Eq. 1.9.30, and its energy \(E_1\) by Eq. 1.9.31. The output energy \(E_o\) of the filter is given by Eq. 4.18.14 where \(|H_o|^2\) is the squared-magnitude response of the high-pass filter. After some manipulation, it can be shown that the \(E_o/E_1\) ratio for Butterworth high-pass filters is simply one minus that for Butterworth low-pass filters. Thus, with a simple relabelling of the ordinate in Fig. 4.18.3, the same figure can be used for analyzing the pulsed-cosine response of fourth-order Butterworth high-pass filters.

For example, \(E_o/E_1 + 1\) as \(\omega_o T = 0\) since reducing the pulse duration flattens the sin x/x spectrum which allows more energy to be transmitted through the high-pass filter. For a given \(\omega_o T\), reducing \(B/\omega_o\) increases \(E_o/E_1\) since more bandwidth is available for transmission. For \(B/\omega_o = 3.0\), in continuous wave or CW operation (meaning \(T = \infty\)), less than 0.3% of the energy is transmitted. However, under pulsed operation when \(\omega_o T = 5.3\), then almost 9% of the energy is transmitted. Thus, this different type of response allows us to gain another perspective of high-pass filter response.
Fig. 6.3.14 Impulse and step responses of Chebyshev high-pass filters having ripples of 0.5 and 2 dB and unity ripple bandwidths. (From K. W. Henderson and W. H. Kautz, "Transient responses of conventional filters," IRE Trans. Circuit Theory, vol. CT-5, pp. 333–347, Dec., 1958.)
more highly selective a filter, the smaller was its output noise energy and noise bandwidth. From Eq. 4.18.3, the output energy of a low-pass filter contained in its impulse response, or alternatively, its response to a flat input noise spectrum equaled

$$E = \pi^{-1} \int_{0}^{\infty} |H_{LP}(j\omega)|^2 d\omega = \int_{-\infty}^{\infty} h_{LP}^2(t) dt$$

(6.3.25)

Now let us see how the energy \( E \) is related to the high-pass filter characteristics.\(^6\)

Performing a change of variable \( v = -1/\omega \) in Eq. 6.3.25 and substitution of \( |H_{LP}(-j/\omega)| = |H_{HP}(j\omega)| \) yields

$$E = \pi^{-1} \int_{-\infty}^{0} |H_{LP}(-j/\omega)|^2 \omega \ d(-1/\omega)$$

$$= \pi^{-1} \int_{0}^{\infty} \omega^{-2} |H_{HP}(j\omega)|^2 d\omega = \pi^{-1} \int_{0}^{\infty} \omega^{-2} |H_{HP}(j\omega)|^2 d\omega$$

(6.3.26)

Since the step response of the high-pass filter equals

$$h_{HP}(t) = \mathcal{L}^{-1} [H_{HP}(s)] = \mathcal{L}^{-1} [H_{HP}(s)/s] = \mathcal{F}^{-1} [H_{HP}(j\omega)/j\omega]$$

(6.3.27)

then combining Eqs. 6.3.26 and 6.3.27, we can re-express Eq. 6.3.25 as

$$E = \pi^{-1} \int_{0}^{\infty} \omega^{-2} |H_{HP}(j\omega)|^2 d\omega = \int_{-\infty}^{\infty} h_{HP}^2(t) dt = \int_{-\infty}^{\infty} h_{LP}^2(t) dt$$

(6.3.28)

Thus in a high-pass filter, the output energy \( E \) is equal to the energy contained in its step response. Therefore, the results concerning energy for the low-pass filter can be generalized to the high-pass filter when properly interpreted. The step response energies of various classical high-pass filters are plotted in Fig. 4.18.1.

From a noise viewpoint, integrated white noise is input to the high-pass filter (rather than a step) where \( S_{n}(\omega) = 1/\omega^2 \) for all \( \omega \). The high-pass filter output then equals

$$S_{o}(\omega) = S_{n}(\omega) |H_{HP}(j\omega)|^2 = \omega^{-2} |H_{HP}(j\omega)|^2$$

(6.3.29)

so that using Eq. 4.18.2, the output energy equals the integral of \( S_{o} \) which agrees with Eq. 6.3.28. Since the ideal high-pass filter with unity bandwidth has a magnitude response \( |H_{HP}| = 1 \) for \( |\omega| > 1 \) and zero elsewhere, Eq. 6.3.28 shows that its output energy \( E \) equals \( 1/\pi = 0.318 \) which forms a convenient reference value.

The *noise margin* of a high-pass filter is defined as

$$NM = \int_{0}^{\infty} \omega^{-2} |H_{HP}(j\omega)|^2 d\omega$$

(6.3.30)

by analogy with Eq. 4.18.11 assuming a \( 1/\omega^2 \) input noise spectrum. For convenience, this is called the \( 1/\omega^2 \) noise margin. The *noise bandwidth* \( B_{n} \) of a high-pass filter equals the effective stopband bandwidth of the ideal high-pass filter which passes integrated white noise with the same energy value as the actual filter, or

$$B_{n} = |H_{HP}(j\omega)|_{max}^2 \int_{0}^{\infty} \omega^{-2} |H_{HP}(j\omega)|^2 d\omega$$

(6.3.31)
6.3.5 COMPLEMENTARY TRANSFORMATION

In general, the low-pass to high-pass transformation preserves the magnitude characteristic of a filter but modifies its step response. There is a different transformation which preserves the step response characteristic but modifies the magnitude characteristic. This is desirable in some applications. This transformation is called the complementary or low-transient low-pass to high-pass transformation.

The step response of a Butterworth low-pass filter is shown in Fig. 6.3.15. If we are to maintain the step response characteristics in terms of overshoot (undershoot), rise time (fall time), delay time (storage time), settling time, etc., then the high-pass filter step response must equal

$$r_{HP}(t) = U_{-1}(t) - r_{LP}(t)$$

(6.3.32)

Equivalently, the impulse responses must satisfy

$$h_{HP}(t) = U_0(t) - h_{LP}(t)$$

(6.3.33)

Thus, the time domain responses are complements of one another. Now let us determine the relation between the filter transfer functions.

Taking the Laplace transform of the impulse responses in Eq. 6.3.33 yields

$$H_{HP}(s) = 1 - H_{LP}(s)$$

(6.3.34)

where the region of convergence of $H_{HP}$ coincides with that of $H_{LP}$. Thus, if the low-pass filter has a transfer function

$$H_{LP}(s) = \sum_{i=0}^{m} a_i s^i / \sum_{i=0}^{n} b_i s^i , \quad m \leq n$$

(6.3.35)

then using Eq. 6.3.34, the high-pass filter has transfer function

$$H_{HP}(s) = \sum_{i=0}^{n} (b_i - a_i) s^i / \sum_{i=0}^{n} b_i s^i$$

(6.3.36)

We see that the poles are unchanged but that the original zeros are modified and $(n - m)$ additional zeros are introduced. Thus, a stable low-pass filter remains stable under the transformation, but a minimum phase filter may become nonminimum phase. If we consider a low-pass filter having no finite zeros, then $a_i = 0$ except for $a_0 \neq 0$ and Eq. 6.3.36 becomes

$$H_{HP}(s) = \sum_{i=0}^{n} b_i s^i - b_0 / \sum_{i=0}^{n} b_i s^i = s \sum_{i=1}^{n} b_i s^{i-1} / \sum_{i=0}^{n} b_i s^i$$

(6.3.37)
Thus, all-pole filters have a finite zeros where one of these zeros is located at the origin.

Let us consider the steady-state behavior of filters related by this transformation. Using Eq. 6.3.34, it is easily shown that

$$|H_{HP}| = \left[(1 - \text{Re} H_{LP})^2 + \text{Im} H_{LP}^2\right]^{1/2}, \quad \arg H_{HP} = \tan^{-1} \left[\text{Im} H_{LP}(1 - \text{Re} H_{LP})\right]$$

$$\tau_{HP} = (\text{Re} H_{LP})^2 [\tau_{LP} - d \text{Im} H_{LP}/d\omega] / [(1 - \text{Re} H_{LP})^2 + (\text{Im} H_{LP})^2]$$

(6.3.38)

Thus, the magnitude, phase, and delay characteristics are related in a complicated manner and must be calculated using Eq. 6.3.38. The high-pass filter characteristics cannot be drawn directly from the low-pass filter characteristics.

For example, consider a Butterworth high-pass filter whose step and frequency responses are shown in Fig. 6.3.16. Using the complementary transformation of Eqs. 6.3.32 and 6.3.34, the analogous low-pass filters having the characteristics shown in Fig. 6.3.16 result. We see that the time domain characteristics are preserved. However, the frequency domain characteristics are changed. It is interesting to observe that gain peaking is introduced. We also see that the asymptotic rolloff of the low-pass filter is -20 dB/dec for any order. Thus, the primary benefit of using a higher-order filter is to obtain the desired step response characteristic.

For another example, consider a Bessel low-pass filter having transfer function $H_{LP}(s_n) = P_n(0)/P_n(s_n)$ as discussed in Sec. 4.10. Then the analogous high-pass filter has transfer function

$$H_{HP}(s_n) = 1 - H_{LP}(s_n) = \left[P_n(s_n) - P_n(0)\right]/P_n(s_n)$$

(6.3.39)

The step responses of the two filters are shown in Fig. 6.3.17. Normalizing the bandwidth of the high-pass filter to unity, the frequency responses of the two filters are also shown in Fig. 6.3.18. We see that the frequency domain characteristics are modified. Peaking is introduced in the high-pass filter magnitude characteristic, and the asymptotic rolloff is maintained at +20 dB/dec for all orders. Again we see that out-of-band rejection is not increased by increasing filter order. Thus, the only advantage of increasing the high-pass filter order here is to increase the storage time of the step response.

**EXAMPLE 6.3.8** Determine the frequency responses of a Butterworth low-pass filter and its complementary high-pass filter. Their step responses are shown in Fig. 6.3.15.

**Solution** A Butterworth high-pass filter has the frequency response shown in Fig. 6.3.16. To obtain a Butterworth low-pass filter, we simply apply the standard low-pass to high-pass transformation to this.
frequency response. Inverting frequency, we obtain the desired responses as shown in Fig. 6.3.18.

**EXAMPLE 6.3.9** Design two second-order band-splitting filters having an overall gain of unity and analogous step response characteristics. Design the low-pass filter to have maximally flat magnitude (MFM) response and unity bandwidth.

**Solution** The band-splitting filter consists of a parallel combination of an MFM low-pass filter and a complementary high-pass filter (e.g., see Fig. 2.7.3a). Assume the low-pass filter has transfer function

\[
H_{LP}(s) = \frac{1 + a_1 s}{1 + b_1 s + b_2 s^2}
\]  

(6.3.40)

For \(H_{LP}\) to have an MFM response, we found in Example 4.2.2 that \(b_1^2 - 2b_2 = a_1^2\) and \(a_1^2 = b_2^2 - 1\). Therefore, the high-pass filter must have a transfer function

\[
H_{HP}(s) = 1 - H_{LP}(s) = \frac{s [b_2 s + b_1 - a_1]}{1 + b_1 s + b_2 s^2} = \frac{b_2 s [s + 2/(a_1 + b_1)]}{1 + b_1 s + b_2 s^2}
\]  

(6.3.41)

Fig. 6.3.18 Magnitude responses of...
useful for interpreting transformations.

6.4 LOW-PASS TO BAND-PASS TRANSFORMATIONS

Just as we can transform low-pass filter transfer functions into high-pass filter transfer functions, so can we transform them into other filter types. To transform the low-pass filter gain into a band-pass filter gain, we use the low-pass to band-pass transformation which is

\[ H_{BP}(s) = H_{LP}(p) \left| \begin{array}{l}
\frac{p}{s} = \frac{\omega_0}{B} \left( \frac{s}{\omega_0} + \frac{\omega_0}{s} \right) = \frac{s^2 + \omega_0^2}{B s} \\
(6.4.1)
\end{array} \right. \]

For simplicity, we often use the frequency normalized form where \( s_n = s/\omega_0 \) and \( p = (\omega_0/B) X (s_n + 1/s_n) \). To justify this result, consider Figs. 6.4.1 and 6.4.8. From a mapping viewpoint, we need to map the \( H_{LP}(j\omega) \) magnitude characteristic into the \( H_{BP}(j\omega) \) magnitude characteristic as shown. Thus, we must map the \( j\omega \)-axis of the \( p \)-plane into the \( j\omega \)-axis of the \( s \)-plane with the following constraints: (1) \( p = j0 \) maps into \( s = \pm j\omega_0 \) (and \( -j\omega_0 \)), (2) \( p = j1 \) maps into \( s = \pm j\omega_U \) (and \( -j\omega_L \)), and (3) \( p = -j1 \) maps into \( s = \pm j\omega_L \) (and \( -j\omega_U \)). We also wish to maintain the geometric symmetry of \( H_{LP}(j\omega) \). A transformation which produces this mapping is

\[ p = k(s/\omega_0 + \omega_0/s) \]

(6.4.2)

Setting \( p = jv \) and \( s = j\omega \), then Eq. 6.4.2 becomes

\[ v = k(\omega_0^2 - \omega^2)/\omega_0 \omega \]

(6.4.3)

When \( v = 0 \), then \( \omega = \pm \omega_0 \). When \( v = \pm 1 \) and \( -1 \), then Eq. 6.4.3 requires that

\[ 1 = k(\omega_0^2 - \omega^2)/\omega_0 \omega \]

(6.4.4)

Solving both equations for \( k \) and equating shows that \( \omega_0 \) equals

\[ \omega_0 = \sqrt{\omega_U \omega_L} \]

(6.4.5)

which is the (geometric) center frequency of the band-pass filter. Substituting \( \omega_0 \) into Eq. 6.4.4 gives \( k \) as

\[ k = (\omega_0/\omega_L - \omega_U/\omega_0)^{-1} = [((\omega_L/\omega_U)^{1/2} - (\omega_U/\omega_L)^{1/2})^{-1} \]

\[ = (\omega_U/\omega_L)^{1/2}/(\omega_L - \omega_U) = \omega_0/B = Q \]

(6.4.6)

Thus, \( k \) is equal to the \( Q \) of the band-pass filter. \( 1/k \) must equal the fractional bandwidth of the band-pass filter where the bandwidth equals

![Fig. 6.4.1 Mapping of p-plane into s-plane using the low-pass to band-pass transformation. (The lower-half s-plane is not shown.)](image)
\[ B = \omega_U - \omega_L \]  

(6.4.7)

It is important to note from Fig. 6.4.1 that Q is defined in terms of the \( M_p \) (dB) bandwidth. \( M_p \) can be any value such as 0.01, 0.5, or 3 dB.

### 6.4.1 EFFECT ON THE S-PLANE

Now let us investigate the effect this transformation has upon the poles and zeros of the low-pass filter. Rewriting Eq. 6.4.1, we see that \( s \) must satisfy

\[ s^2 - pBs + \omega_o^2 = 0 \]  

(6.4.8)

Solving for \( s \) using the biquadratic equation gives

\[ s = \left( pB/2 \right) \pm \left[ \left( pB/2 \right)^2 - \omega_o^2 \right]^{1/2} \]  

(6.4.9)

so normalized \( s \), denoted as \( s_n \), equals

\[ s_n = s/\omega_o = \frac{p/2Q}{\left[ (p/2Q)^2 - 1 \right]^{1/2}} \]  

(6.4.10)

Every pole and zero \( p \) of the low-pass filter transforms into a pair of poles and zeros in the band-pass filter. The location of these critical frequencies depends upon \( p \) and the Q of the transformation. When \( |p/2| \ll Q \), then \( s \) equals

\[ s = \left( pB/2 \right) \pm j\omega_o \left[ 1 - \left( pB/2\omega_o \right)^2 \right]^{1/2} \approx \left( pB/2 \right) \pm j\omega_o \left[ 1 - \left( pB/2\omega_o \right)^2 / 2 \right] \approx \frac{pB}{2} \pm j\omega_o \]  

(6.4.11)

so \( s_n \) equals

\[ s_n = s/\omega_o \approx p/2Q \pm j1 \]  

(6.4.12)

When \( |p/2| \gg Q \), then \( s \) equals

\[ s = \left( pB/2 \right) \left[ 1 \pm \left[ 1 - (2\omega_o/pB)^2 \right]^{1/2} \right] \approx \left( pB/2 \right) \left[ 1 \pm \left[ 1 - \frac{1}{2}(2\omega_o/pB)^2 \right] \right] \approx pB, \quad Q\omega_o/p \]  

(6.4.13)

so \( s_n \) equals

\[ s_n = s/\omega_o \approx p/Q, \quad Q/p \]  

(6.4.14)

Therefore, we see that when \( |p/2| \ll Q \), the band-pass filter poles and zeros are obtained from the low-pass filter poles and zeros by scaling \( p \) by half the bandwidth and then translating vertically to \( \pm j\omega_o \). However, when \( |p/2| \gg Q \), then one band-pass pole-zero is obtained by scaling \( p \) by \( B \). The second pole-zero is obtained by inverting the first pole-zero and scaling by \( \omega_o^2 \). For intermediate values of \( p \), the pole-zero locations must be calculated using Eqs. 6.4.9 or 6.4.10. These results show that real low-pass poles-zeros remain real under low-Q transformations but become complex under high-Q transformations. Alternatively, the values may be obtained directly from Fig. 6.4.2. Here we scale \( p \) as

\[ p/Q = x + jy \]  

(6.4.15)

and simply read off the \( s_n \) values as

\[ s_n = s/\omega_o = \sigma + j\omega \]  

(6.4.16)
\[ s_{n1, 2, 3, 4} = (-0.707 \pm j0.707)/2Q \pm j1 \quad (6.4.17) \]

from Eq. 6.4.12. For the low-Q case, the band-pass filter poles equal

\[ s_{n1, 2} = Q^{-1} \exp \left[ j(180^0 \pm 45^0) \right], \quad s_{n3, 4} = Q \exp \left[ j(180^0 + 45^0) \right] \quad (6.4.18) \]

from Eq. 6.4.14. For the intermediate-Q case, we use Fig. 6.4.2. For example, when Q = 1, then \( p/Q = -0.707 \pm j0.707 \) and the band-pass filter poles equal

\[ s_{n1, 2} = -0.23 \pm j0.65 = 0.688 \exp \left[ j(180^0 \pm 70.5^0) \right] \]
\[ s_{n3, 4} = -0.47 \pm j1.36 = 1.69 \exp \left[ j(180^0 \pm 70.5^0) \right] \quad (6.4.19) \]

It is important to observe that the two sets of band-pass filter poles obtained from one pair of low-pass filter poles always have the same Q and damping factor as we shall soon discuss.

Now let us consider the mapping of low-pass filter zeros at infinity. Since these are equivalently low-Q zeros, each such low-pass filter zero maps into two band-pass filter zeros; one is located at the origin and the other at infinity. Thus, if the low-pass filter is the all-pole type of order \( n \), then its associated band-pass filter will have \( n \) zeros at both the origin and infinity. If the
low-pass filter has \( n \) finite poles and \( m \) finite zeros, then its associated band-pass filter will have \( (n - m) \) zeros at both the origin and infinity.

**EXAMPLE 6.4.1**  In Example 2.7.3, we designed a second-order band-pass filter using trial-and-error methods. Now design the filter to have a Butterworth characteristic using transformations.

**Solution**  A second-order band-pass filter is obtained from a second-order low-pass filter. The Butterworth filter having unity 3 dB bandwidth has a transfer function

\[
H_{LP}(p) = \frac{10}{p^2 + 1.414p + 1}
\]  

(6.4.20)

where \( p = -0.707 \pm j0.707 = \exp[j(180^\circ \pm 45^\circ)] \). To determine the poles of the band-pass filter, we must know its \( Q \). The center frequency \( \omega_0, 3 \) dB bandwidth \( B \), and \( Q \) of the band-pass filter equal

\[
f_o = [600(3000)]^{1/2} = 1340 \text{ Hz}, \quad B = 3000 - 600 = 2400 \text{ Hz}, \quad Q = 1340/2400 = 0.558
\]  

(6.4.21)

Since \( |p/2| \) is the same order of magnitude as \( Q \), this is an intermediate-\( Q \) situation. Therefore, the scaled low-pass filter pole equals

\[
\frac{p}{Q} = \frac{-0.707 \pm j0.707}{0.558} = -1.267 \pm j1.267 = \alpha + j\beta
\]  

(6.4.22)

From Fig. 6.4.2, we find that the normalized band-pass filter poles equal

\[
s_n = s/\omega_0 = \sigma + j\omega = -0.26 \pm j0.44 = 0.51 \exp[j(180^\circ \pm 59^\circ)]
\]

\[
= -1.02 \pm j1.70 = 1.98 \exp[j(180^\circ \pm 59^\circ)]
\]  

(6.4.23)

Therefore, the band-pass filter transfer function equals

\[
H_{BP}(s_n) = \frac{10 s_n^2/0.558^2}{[(s_n + 0.26)^2 + 0.44^2] [(s_n + 1.02)^2 + 1.70^2]}
\]  

(6.4.24)

where \( s_n = s/2\pi(1340 \text{ Hz}) \). The block diagram of the band-pass filter is shown in Fig. 6.4.4.

**EXAMPLE 6.4.2**  Design a fourth-order Butterworth band-pass filter in block diagram form having a center frequency of 1 KHz, a 3 dB bandwidth of 1 KHz, and a midband gain of 10.
Solution  

The fourth-order Butterworth low-pass filter has a transfer function

\[
H(p) = \frac{10}{(p^2 + 0.765p + 1)(p^2 + 1.848p + 1)} \tag{6.4.25}
\]

Since \( Q = 1 \) KHz/1 KHz = 1 and \( s_n = s/2\pi(1000 \text{ Hz}) \), substituting \( p = (s_n^2 + 1)/s_n \) into the transfer function yields

\[
H(s) = \frac{10s_n^4}{[(s_n^2 + 1)^2 + 0.765s_n(s_n^2 + 1) + s_n^2] [(s_n^2 + 1)^2 + 1.848s_n(s_n^2 + 1) + s_n^2]} \tag{6.4.26}
\]

The band-pass poles are easily found from Fig. 6.4.2. For the low-pass filter pole pair \( p_1, p_1^* = -0.383 \pm j0.924 \), the band-pass filter poles equal

\[
s_{n1, 2} = -0.11 + j0.61 = 0.62 \exp [(180^\circ \pm 80^\circ)]
\]

\[
s_{n3, 4} = -0.28 + j1.58 = 1.61 \exp [(180^\circ \pm 80^\circ)] \tag{6.4.27}
\]

For the low-pass filter pole pair \( p_2, p_2^* = -0.24 + j0.383 \), then the band-pass filter poles are

\[
s_{n5, 6} = -0.36 + j0.71 = 0.80 \exp [(180^\circ \pm 63^\circ)]
\]

\[
s_{n7, 8} = -0.57 + j1.11 = 1.25 \exp [(180^\circ \pm 63^\circ)] \tag{6.4.28}
\]

The block diagram of the band-pass filter is easily drawn in Fig. 6.4.5.

The band-pass filter poles and zeros can be obtained from the low-pass filter poles and zeros in polar form (rather than rectangular form as in Fig. 6.4.2) using a different approach. Jones has shown\(^9\) that under the low-pass to band-pass transformation, a pair of low-pass filter roots having magnitude \( \omega_n \) and a \( Q \) of \( Q_{BP} \) map into a pair of band-pass filter roots which always have identical \( Q_{BP} \)'s. Defining a constant, \( \delta \), to equal \( \delta = Q/\omega_n \), then \( Q_{BP} \) is equal to

\[
Q_{BP} = (Q_{LP}/2^{1/2}) \left[ 1 + 4\delta^2 + [(1 + 4\delta^2)^2 - 4\delta^2/(Q_{LP}^2)]^{1/2} \right]^{1/2} \tag{6.4.29}
\]

We already observed in Figs. 6.4.4 and 6.4.5 that the band-pass filter poles always have the same damping factor. The normalized critical frequencies equal

\[
\omega_{o1}/\omega_o, \omega_{o2}/\omega_o = 2^{1/2} \left[ (Q_{BP}/\delta Q_{LP}) \pm [(Q_{BP}/\delta Q_{LP})^2 - 4]^{1/2} \right] \tag{6.4.30}
\]

Under the limiting condition that \( \delta \to \infty \) or 0, then

\[
Q_{BP} \approx 25 Q_{LP}, \quad \omega_o/\omega_o \approx 1 \pm (1 - 1/8Q_{LP}^2)]/2\delta \approx 1 \pm 1/2\delta \quad \text{for} \quad \delta \to \infty
\]

\[
Q_{BP} \approx (1 + 25\delta^2)Q_{LP} \approx Q_{LP}, \quad \omega_o/\omega_o \approx (1 - 25\delta^2)\delta \approx \delta \quad \text{for} \quad \delta \to 0
\]

\[
\approx (1 + 25\delta^2)/\delta \approx 1/\delta
\]

---

EXAM!!

Solution had \( \omega_o \), given by
Fig. 6.4.6 Relation between $Q$ and $\omega_0$ of critical frequencies of low-pass and band-pass filters.

The normalized critical frequencies and the $Q_{BP}$ of the roots are plotted in Fig. 6.4.6. These figures are useful primarily for the narrowband case.

**EXAMPLE 6.4.3** Determine the band-pass filter poles of Example 6.4.1 in polar form.

**Solution** In Example 6.4.1, we found that $Q = 0.558$ and that the Butterworth low-pass filter poles had $\omega_n = 1$ and $Q_{LP} = 0.707$. Therefore, since $\delta = Q/\omega_n = 0.558$, the band-pass filter poles have a $Q_{BP}$ given by Eq. 6.4.29 of

$$Q_{BP} = (0.707/2^{1/2}) \left( 1 + 4(0.558^2) + (1 + 4(0.558^2))^2 - 4(0.558^2)/0.707^2 \right)^{1/2}$$

$$= 0.5 \left[ 1 + 1.25 + (2.25^2 - 2.49)^{1/2} \right] = 0.5(2.25 + 1.60)^{1/2} = 0.98 \quad (6.4.32)$$

The normalized band-pass filter pole frequencies equal...
\[ t_0 = 10.0\, \text{KHz} \]
\[ Q_p = 22 \]
\[ Q_p = 45 \]
\[ f_{oz} = 7.60\, \text{KHz} \]
\[ f_{oz} = 13.2\, \text{KHz} \]

Fig. 6.4.7 Block diagram realization of third-order elliptic band-pass filter of Example 6.4.4.

\[
\omega_{a1}/\omega_o, \omega_{a2}/\omega_o = \frac{1}{2i} \left\{ \frac{0.98}{(0.558)(0.707)} + \left( \frac{(0.98)(0.558)(0.707)}{2} - 4i \right)^{1/2} \right\} \\
= 0.5(2.48 \pm 2i1.7) = 0.51, \quad 1.98
\]  

(6.4.33)

from Eq. 6.4.30. We determined these same poles in Example 6.4.1 using Fig. 6.4.2.

**EXAMPLE 6.4.4** Determine the transfer function and block diagram for a third-order elliptic band-pass filter having a center frequency of 10 KHz, a bandwidth of 1 KHz, and a maximum gain of 10. The in-band ripple equals 1.25 dB and the minimum stopband rejection equals 60 dB. Use the narrowband approximation and then verify the results using Fig. 6.4.6.

**Solution** The third-order elliptic low-pass filter has a transfer function

\[
H(p) = 0.143 \frac{p^2 + 5.54^2}{(p + 0.459)(p + 0.222)^2 + 0.953^2} 
\]

(6.4.34)

from Table 4.8.2 where \(|H(0)| = 10\). Since the band-pass filter has \(f_o = 10\) KHz and \(B = 1\) KHz, then \(Q = 10\) KHz/1 KHz = 10. Using the narrowband approximation \(s_n = p/2Q = \pm j = 0.05p \pm j1\), then the individual low-pass filter poles \(p_1\) and their corresponding band-pass filter poles \(s_n1\) must equal

\[ p_1 = -0.459 = 0.459 \exp(\pm j180^\circ) \quad \text{so} \quad s_{n1,2} = \pm 0.023 \pm j1 \]

\[ p_2, p_2^* = -0.222 \pm j0.953 = 0.98 \exp(\pm j(180^\circ \pm 76.9^\circ)) \quad \text{so} \quad s_{n3,4,5,6} = \pm 0.011 \pm j1 \]

\[ s_{n7,8} = \pm j1 \]

(6.4.35)

Likewise for the zeros \(z_n\),

\[ z_1, z_1^* = 0.5 \pm j5.54 = 5.54 \exp(\pm j90^\circ) \quad \text{so} \quad s_{n7,8} = \pm j1 \]

(6.4.36)

Now let us use Fig. 6.4.6 to find the exact poles and zeros. Treating the real low-pass pole as complex with \(Q_{LP} = 0.5\), since \(\delta = 10/0.459 = 21.8\), then from Fig. 6.4.6,

\[ s_{n1,2} = 1.0; \quad Q_{BP} = 22 \]

(6.4.37)

Since the complex poles \(p_2\) and \(p_2^*\) have a \(Q_{LP} = 1/(2 \cos 76.9^\circ) = 2.21\), and \(\delta = 10/0.98 = 10.2\), then from Fig. 6.4.6,

\[ s_{n3,4,5,6} = 0.95, 1.05; \quad Q_{BP} = 45 \]

(6.4.38)

The approximate results of Eq. 6.4.35 are in close agreement with the exact results of Eqs. 6.4.37 and 6.4.38. The complex zeros \(z_1\) and \(z_1^*\) have a \(Q_{LP} = 1/(2 \cos 90^\circ) = \infty\). Since \(\delta = 10/5.54 = 1.8\), then from Fig. 6.4.6,

\[ s_{n7,8} = 0.76, 1.32; \quad Q_{BP} = \infty \]

(6.4.39)

We see from Eq. 6.4.39 that the approximate results of Eq. 6.4.36 are slightly in error. We also see that either approach requires some algebra. The transfer function equals

\[ H_{BP}(s) = H_{LP}(p) \bigg|_p = 10(s_n + 1/s_n) \]
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\[ H_1(p) = \frac{H_0}{p+1} \rightarrow H_1(s) = \frac{H_0 s_n/Q}{s_n^2 + s_n/Q + 1} \]  

(6.4.41)

for \( s_n = s/2\pi(10 \text{ KHz}) \) and \(|H_{BP}(j1)| = 10\). The block diagram of the filter is shown in Fig. 6.4.7.

We should make an important observation concerning the midband gain \( H_0 \) of the band-pass filter and the gains of the individual blocks in its block diagram realization. First-order band-pass filters have transfer functions

\[ H_1(p) = \frac{H_0}{p+1} \rightarrow H_1(s) = \frac{H_0 s_n/Q}{s_n^2 + s_n/Q + 1} \]  

(6.4.42)

so their midband gains equal \( H_0 \). Therefore, the low-pass and band-pass filter blocks have identical gains. However, the transfer function of a second-order band-pass filter equals

\[ H_2(p) = \frac{H_0}{p^2 + p/Q_{LP} + 1} \rightarrow \]

\[ H_2(s) = \frac{H_0 s_n^2/Q^2}{s_n^4 + s_n^3/Q_{LP} + (2 + 1/Q_{LP})s_n^2 + s_n/Q_{LP} + 1} \]  

(6.4.43)

which can be factored as

\[ H_2(s) = \frac{(H_0^{1/2}Q_{BP}/Q)k_{BP}/Q_{BP} (H_0^{1/2}Q_{BP}/Q)s_n/kQ_{BP}}{s_n^2 + k_{BP}/Q_{BP} + k^2 s_n^2 + s_n/kQ_{BP} + 1/k^2} \]  

(6.4.44)

where \( k \) is the order of unity. Here the midband gain of each second-order band-pass filter stage equals \( \sqrt{H_0} Q_{BP}/Q \) which is a factor \( Q_{BP}/Q \sqrt{H_0} \) higher than that in the first-order case.

This result is useful for determining the proper gains. For example in the second-order Butterworth filter of Fig. 6.4.4, the gains must equal \([10(0.98/0.558)^2]^{1/2} = 5.6\). In the fourth-order Butterworth filter of Fig. 6.4.5, the gains must equal \([10(1.11)^2(2.89^2)]^{1/4} = 3.19\). This is obtained by factoring each half of Eq. 6.4.26 as shown in Eq. 6.4.43.

A simpler approach for determining the gain constant \( H_0 \) of a higher order band-pass filter is to use Eq. 6.4.1 directly. Here we see that at high-frequencies when \( p \to \infty \), then \( s \to \infty \) and the low-pass and band-pass filter gains equal

\[ H_{BP}(s_n) = H_{LP}(p) \bigg|_{p = Qs_n} \bigg|_{s = Qs_n} \to \infty \]  

(6.4.45)

Hence, when the low-pass filter has an asymptotic gain given by \( H_{LP} \sim H_0/p^n \), then the band-pass filter has an asymptotic gain \( H_{BP} \sim H_0/(Qs_n)^n = (H_0/Q^n)/s_n^n \). Thus, the multiplying coefficient of \( H_{BP} \) is \( H_0/Q^n \). For example, in the third-order elliptic filter of Fig. 6.4.7, \( H_{BP}(p) \sim 0.143/p \) at high-frequencies using Eq. 6.4.34. Therefore, \( H_0 \) must equal 0.143/10 = 0.0143 in Eq. 6.4.40; when the high-frequency gain of the two BSF stages is unity, then the midband gain of the BPF stage must equal 22(0.0143) = 0.315 using Eq. 6.4.41.

We should also make one comment about the form of our low-pass to band-pass transformation. We have used \( p = (\omega_o/Q)(s/\omega_o + \omega_o/s) \). This transformation maps a unity-bandwidth
6.4.2 MAGNITUDE, PHASE, AND DELAY RESPONSES

The magnitude characteristic of a band-pass filter has geometric symmetry about its center frequency \( \omega_0 \) using the low-pass to band-pass transformation. Thus, its gain at frequency \( \omega \) is identical to that at frequency \( \omega_0/\omega \). Therefore, the shape of the characteristic is preserved for transformations of any \( Q \) (i.e., low, intermediate, or high). A magnitude characteristic is shown in Fig. 6.4.8a. A more general plot is drawn in Fig. 6.4.9 to show the entire \( s \)-plane. Note that \( H_{LP}(0) \) maps into \( H_{BP}(\pm j\omega_0) \) and \( H_{LP}(\infty) \) maps into \( H_{BP}(\infty) \).

The phase characteristic also has geometric symmetry about \( \omega_0 \). It is identical to the phase of the low-pass filter after it is recentered from \( 0 \) to \( \omega_0 \). This is shown in Fig. 6.4.8b.

The delay of the band-pass filter can be determined from the low-pass filter delay. The band-pass filter delays equals

\[
\tau_{BP}(j\omega) = -\frac{d \arg H_{BP}(j\omega)}{d\omega} = -\frac{d \arg H_{LP}(j\nu)}{d\nu} \frac{dv}{d\omega} = \tau_{LP}(j\nu) \frac{dv}{d\omega} \tag{6.4.45}
\]

where \( \nu \) is given by Eq. 6.4.3 as

\[
\nu = (\omega_0/B)(\omega/\omega_0 - \omega_0/\omega) = (\omega - \omega_0^2/\omega)/B \tag{6.4.46}
\]

Therefore, since

\[
dv/d\omega = [1 + (\omega_0/\omega)^2]/B \tag{6.4.47}
\]

then substituting Eq. 6.4.47 into Eq. 6.4.45, the delay of the band-pass filter can be expressed as

\[
\tau_{BP}(j\omega) = B^{-1} [1 + (\omega_0/\omega)^2] \tau_{LP}(j\nu) \bigg|_{\nu = Q(\omega/\omega_0 - \omega_0/\omega)} \tag{6.4.48}
\]

We can write the passband delay of a narrowband filter. If we express the frequency variation from center frequency \( \omega_0 \) as \( \Delta \omega = \omega - \omega_0 \), then we can express \( \nu \) given by Eq. 6.4.46 as

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![Fig. 6.4.8 Effect of low-pass to band-pass transformation upon (a) magnitude and (b) phase characteristics.](image-url)
Fig. 6.4.9 Magnitudes of third-order elliptic (a) low-pass filter gain $H_{LP}(p)$ and (b) band-pass filter gain $H_{BP}(s)$ in the $p$- and $s$-planes, respectively.

\[ v = (\omega^2 - \omega_0^2)/B\omega = \Delta\omega(\omega + \omega_0)/B\omega \]  

(6.4.49)

When the frequency derivation is small so that $\Delta\omega \ll 0$, then $\omega \approx \omega_0$ and $v \approx 2\Delta\omega/B$. Thus, the passband delay of a band-pass filter given by Eq. 6.4.48 equals

\[ \tau_{BP}(i(\omega_0 + \Delta\omega)) \cong (B/2)^{-1}\tau_{LP}(i\Delta\omega/(B/2)) \]  

(6.4.50)

Since $\tau_{LP}$ is even, then $\tau_{BP}$ is symmetrical about the center of the passband. However the $dV/d\omega$ term in the $\tau_{BP}$ expression introduces asymmetry; the delay becomes more asymmetrical with increasing deviation from the center frequency and decreasing $Q^{10}$. We will illustrate this in a moment. The delay at the center frequency of the band-pass filter equals

\[ \tau_{BP}(i\omega_0) = (B/2)^{-1}\tau_{LP}(i0) = (B/2)^{-1}(b_1/b_0 - a_1/a_0) \]  

(6.4.51)

It is $2/B$ times the dc delay of the low-pass filter whose gain is given by Eq. 2.0.1. Thus, the midband delay is inversely proportional to half the filter bandwidth. The low-frequency delay of the band-pass filter equals

\[ \tau_{BP}(i\omega) = (Q\omega_0)^{-1}d\tau_{LP}(\infty)/d(1/\omega^2) = (Q\omega_0)^{-1}(b_{n-1}/b_n - a_{m-1}/a_m), \quad \omega \ll \omega_0 \]  

(6.4.52)

while the high-frequency delay equals

\[ \tau_{BP}(i\omega) = (B/\omega^2) d\tau_{LP}(\infty)/d(1/\omega^2) = (B/\omega^2)(b_{n-1}/b_n - a_{m-1}/a_m), \quad \omega \gg \omega_0 \]
Fig. 6.4.10  Conversion of delay under (a) low-pass and (b) high-Q transformations in Example 6.4.5.

**EXAMPLE 6.4.5** Consider a second-order low-pass filter having \( \omega_o = 1 \) and \( \zeta = 0.5 \). Its delay characteristics are shown in Fig. 2.10.2. Determine and sketch the delay characteristic of the analogous second-order band-pass filter.

**Solution** The transfer function of the low-pass filter equals \( H_{LP}(p) = 1/(p^2 + p + 1) \). The transfer function of the band-pass filter is given by Eq. 6.4.1 where \( \omega_o \) and \( B \) must be specified. The delay of the band-pass filter is given by Eq. 6.4.45. Since the low-pass filter delay equals

\[
\tau_{LP}(v) = \frac{1 + v^2}{1 - v^2 + v^4} \tag{6.4.54}
\]

from Eq. 2.10.5, the dc value equals one second while \( \tau_{LP}(v) \approx 1/v^2 \) for \( |v| \gg 1 \). Thus, the midband delay of the band-pass filter equals \( 2/B \) from Eq. 6.4.51. The low-frequency delay equals \( 1/Q \omega_o \) for \( \omega \ll \omega_o \) from Eq. 6.4.52, while the high-frequency delay equals \( B/\omega^2 \) for \( \omega \gg \omega_o \) from Eq. 6.4.53. The delay of a low-Q and high-Q band-pass filter is sketched in Fig. 6.4.10. We see that the low-Q filter has asymmetrical passband delay but that the high-Q filter has almost symmetrical passband delay.

Band-pass filters are specified in the frequency domain, the time domain, or in both domains simultaneously. Just as with low-pass filters, we can utilize the frequency domain results and nomographs of Chaps. 4 and 5. We shall soon see that we can also utilize the time domain results for narrowband band-pass filters. In general, the frequency domain specifications of band-pass filters have the form shown in Fig. 6.4.11. \( f_L \) and \( f_U \) are the required corner frequencies for less than \( M_p \) in-band gain variation and \( f_1, f_2, f_3, f_4, \) etc., are the frequencies for stopband rejections of \( M_s1, M_s2, \) etc. In many applications, the gain characteristic will have geometric symmetry about the center frequency \( f_0 \) where \( f_0^2 = f_L f_U = f_1 f_3 = f_2 f_4 = \ldots \). To utilize the design data of the previous chapters, we convert this into equivalent information for a low-pass filter using

\[
\Omega = Q \left| \frac{f}{f_o} - \frac{f_0}{f_1} \right| \tag{6.4.55}
\]

We normalize each stopband frequency \( f_s \) by \( f_o \), form its reciprocal, take the absolute value of the difference, and multiply by \( Q \). This gives us the normalized stopband frequencies of the equivalent low-pass filter. Since the upper and lower frequencies are assumed to have geometric symmetry, their normalized \( \Omega \)'s will be equal to one another (i.e., \( \Omega_1 = \Omega_2 \), etc.). We then draw
the equivalent low-pass filter specification as shown in Fig. 6.4.11b where $f_0$ corresponds to $\Omega = 0$, the upper and lower band-edge frequencies correspond to $\Omega = 1$, and the various cutoff frequencies are as shown. Using this specification, we now determine the characteristics for the low-pass filter. The filter type is selected, the order determined, and the normalized low-pass filter pole-zero data obtained from the tables. Then using the low-pass to band-pass transformation, the required transfer function of the band-pass filter can be written.

**EXAMPLE 6.4.6** Determine the minimum order Butterworth, Chebyshev, and elliptic band-pass filters required to meet the magnitude characteristics shown in Fig. 6.4.12a.

**Solution** To enable us to determine the order of the band-pass filters, we need to first determine the $Q$ and $f_0$ for the filter. Since the center frequency and bandwidth equal

$$f_0 = \frac{500(2000)}{35} = 1000 \text{ Hz}, \quad B = 2000 - 500 = 1500 \text{ Hz}$$

using Eqs. 6.4.5 and 6.4.7, then from Eq. 6.4.6, $Q$ equals

$$Q = \frac{1000}{1500} = 0.667$$

(6.4.57)

Therefore, using Eq. 6.4.55, the normalized stopband frequencies are

$$\Omega_s = 0.667 [4000/1000 - 1000/4000] = 0.667 [250/1000 - 1000/250] = 2.5$$

(6.4.58)

which are equal since they have geometric symmetry about $f_0$. We now express the band-pass filter requirements in terms of the equivalent low-pass filter requirements. This is shown in Fig. 6.4.12b. From the nomographs of Chap. 4, we find

Butterworth: $n \geq 6$, Chebyshev: $n \geq 4$, Elliptic: $n \geq 3$

Therefore, a third-order elliptic band-pass filter will meet this gain requirement.
Another useful result which simplifies determining filter order is use of the $M_1 - M_2$ dB shaping factor $S = \frac{B}{W_1/BW_2}$ where $M_1$ and $M_2$ are arbitrary rejections. It is easily shown that $S$ is independent of $Q$ and is invariant under the low-pass to band-pass transformation. Thus, the shaping factor can be calculated directly from the bandwidth ratio for the band-pass filter. Then using this as the normalized stopband frequency $\Omega_s$ for the low-pass filter, the order is determined directly. For instance, in Example 6.4.6, since $BW_{1.25$ dB} = 1500 Hz and $BW_{40$ dB} = 3750 Hz, then the 1.25-40 dB shaping factor $S = \frac{3750}{1500} = 2.5$. Thus, $\Omega_s = 2.5$ directly in Fig. 6.4.12b without the need of Eqs. 6.4.56-6.4.58. The filter order is then determined as usual. This approach considerably reduces the work required to find the orders of filters having geometric symmetry.

In some filter applications, asymmetrical magnitude specifications may be given. We shall discuss this in connection with filter magnitude responses which have arithmetic symmetry rather than geometric symmetry. In such asymmetrical gain situations, we can still utilize the filter nomographs, assuming of course that filters having geometric symmetry are acceptable. The technique merely requires selection of the most stringent (i.e., minimum) stopband frequency $\Omega_s$ at a given attenuation as we show in the following example.

**EXAMPLE 6.4.7** Determine the minimum order Butterworth, Chebyshev, and elliptic band-pass filters required to meet the magnitude characteristics shown in Fig. 6.4.13a.

**Solution** In this case, we have

$$ f_0 = \left[ \frac{500(1000)}{12} \right] = 707 \text{ Hz, } B = 1000 - 500 = 500 \text{ Hz, } Q = \frac{707}{500} = 1.41 \quad (6.4.59) $$

The upper and lower stopband frequencies of the equivalent low-pass filter are found to be $\Omega_1 = 1.70$, $\Omega_2 = 2.02$, $\Omega_3 = 3.50$, and $\Omega_4 = 7.75$ using

$$ \Omega_1 = 1.41 \frac{f_0}{707 - 707/\Omega_1} \quad (6.4.60) $$

The equivalent low-pass filter gain is drawn in Fig. 6.14.13b. The required stopband frequencies are unequal and so the smallest frequency is chosen. This gives the most stringent gain requirement. From Chap. 4, then

- Butterworth: $n_1 \geq 10$ and $n_2 \geq 11$ (use $n \geq 11$)
- Chebyshev: $n_1 \geq 6$ and $n_2 \geq 6$ (use $n \geq 6$)
- Elliptic: $n_1 \geq 4$ and $n_2 \geq 5$ (use $n \geq 5$)

### 6.4.3 IMPULSE AND COSINE STEP RESPONSES

Low-pass filters are characterized by their step responses. Band-pass filters are characterized by their responses to a unit cosine step input $\cos \omega_c t U_1(t)$ where $\omega_c$ is the center frequency of the filter. Note that this band-pass filter input is analogous to the low-pass filter step input, since

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\[ \mathcal{L}^{-1} \left[ \frac{1}{p} \mid p = (s^2 + \omega_0^2)^2/(Bs^2) \right] = \mathcal{L}^{-1} \left[ \frac{B s/(s^2 + \omega_0^2)}{} \right] = B \cos \omega_0 t \ U_{1}(t) \]  

(6.4.61)

from Table 1.8.1. When the impulse response of the analogous low-pass filter is known, then the unit cosine step response of the band-pass filter can be found as we will now show.

The impulse response of the band-pass filter equals

\[ h_{BP}(t) = \mathcal{L}^{-1} \left[ H_{BP}(s) \right] = \mathcal{L}^{-1} \left[ H_{LP}(p) \mid p = Q(s/\omega_0 + \omega_0/s) \right] \]  

(6.4.62)

It can be shown that if \( f(t) \) has a Laplace transform \( F(s) \), then \(^2\)

\[ \mathcal{L}^{-1} \left[ (s + 1)/s \right] = \int_0^t J_0(2 \sqrt{(t - \tau)\tau}) f(\tau) \ d\tau \]  

(6.4.63)

Applying the time differentiation theorem from Table 1.8.2 to Eq. 6.4.63, then

\[ \mathcal{L}^{-1} \left[ (s + 1)/s \right] = \frac{d}{dt} \int_0^t J_0(2 \sqrt{(t - \tau)\tau}) f(\tau) \ d\tau \ U_{1}(t) + (0) U_{0}(t) \]  

(6.4.64)

Since the derivative of an integral equals

\[ \frac{d}{dt} \int_0^t f(t, \tau) \ d\tau \ U_{1}(t) = [f(t, t) + \int_0^t \frac{df(t, \tau)}{dt} \ d\tau] \ U_{1}(t) \]  

(6.4.65)

then we can re-express Eq. 6.4.64 as

\[ \mathcal{L}^{-1} \left[ (s + 1)/s \right] = [f(t) - \int_0^t \sqrt{(t - \tau)J_1(2 \sqrt{(t - \tau)\tau}) f(\tau)} \ d\tau] \ U_{1}(t) \]  

(6.4.66)

Applying this result to a low-pass filter where \( \omega_0 = 1 \), the impulse response of the band-pass filter equals

\[ h_{BP}(t) = Q[H_{LP}(t) - \int_0^t \sqrt{(t - \tau)J_1(2 \sqrt{(t - \tau)\tau}) h_{LP}(\tau)} \ d\tau] \ U_{1}(t) \]  

(6.4.67)

from Eq. 6.4.62 (note \( t' = t/Q \)). Therefore, the unit cosine step response equals

\[ r_{BP}(t) = h_{BP}(t) \ast \{ \cos t \ U_{1}(t) \} \]  

(6.4.68)

using the convolution theorem of Table 1.8.2. These are very useful results. They show that the impulse response of a band-pass filter is proportional to the integral of the product of the low-pass filter impulse response times \( j_0 \). An analogous result was found for the high-pass filter.

Now let us find the responses for the two limiting cases of narrowband and wideband band-pass filters. In the narrowband case, we found that \( p = (s \pm j\omega_0)/(B/2) \) from Eq. 6.4.11. The approximate transfer function of the band-pass filter can be expressed as \(^{11}\)

\[ H_{BP}(s) \approx H_{LP}(p) \mid p = 2(s \pm j\omega_0)/B + H_{LP}(p) \mid p = 2(s - j\omega_0)/B \]  

(6.4.69)

Therefore, the impulse response of the band-pass filter must equal

\[ h_{BP}(t) = \mathcal{L}^{-1} \left[ H_{RP}(s) \right] \]
using the time scaling and frequency shifting theorems of Table 1.8.2 and the Laplace transforms of Table 1.8.1. The response is a cosine step whose envelope equals \( B h_{LP}(Bt/2) \) where \( h_{LP} \) is the impulse response of the low-pass filter. In like manner, the unit cosine step response equals:

\[
I_{BP}(t) = \mathcal{L}^{-1}[R_{BP}(s)] \\
= B^{-1} \mathcal{L}^{-1}[H_{LP}(p)|p = 2(s + j\omega_0)/B] + B^{-1} \mathcal{L}^{-1}[H_{LP}(p)|p = 2(s - j\omega_0)/B] \\
= r_{LP}(Bt/2) \cos \omega_0 t \ U_1(t)
\]  

(6.4.71)

Therefore, narrowband band-pass filters have impulse and cosine step responses which can be drawn by inspection using the impulse and step responses of their analogous low-pass filters.

In wideband band-pass filters, this is not true. In this case, the poles and zeros are determined using Eq. 6.4.13 where \( p = s/B \) and \( Q\omega_0/s \). The transfer function of the band-pass filter can be expressed as:

\[
H_{BP}(s) = H_{LP}(p)|p = s/B \times H_{LP}(p)|p = Q\omega_0/s
\]

(6.4.72)

Therefore, the filter impulse response must equal:

\[
h_{BP}(t) = \mathcal{L}^{-1}[H_{BP}(s)] = \mathcal{L}^{-1}[H_{LP}(p)|p = s/B] \star \mathcal{L}^{-1}[H_{HP}(p)|p = s/Q\omega_0] \\
= B h_{LP}(Bt) \star Q\omega_0 h_{HP}(Q\omega_0 t) = \omega_0^2 h_{LP}(Bt) \star h_{HP}(Q\omega_0 t)
\]

(6.4.73)

using the high-pass filter result of Eq. 6.3.17. Thus, we convolve the impulse responses of low-pass and high-pass filters to determine the impulse response of a wideband band-pass filter. The cosine step response is then obtained using \( h_{BP} \) and Eq. 6.4.68.

**EXAMPLE 6.4.8** A third-order elliptic band-pass filter having a center frequency \( f_c \) of 10 KHz, a bandwidth \( B \) of 1 KHz, and a \( Q \) of 10 was analyzed in Example 6.4.4. Sketch its cosine step response assuming unity passband gain.

**Solution** Treating this as a high-Q filter, its cosine step response is easily sketched using its analogous low-pass filter step response which is shown in Fig. 4.8.5. Using \( M_p = 1.25 \pm 1 \text{ dB} \) and \( \theta = 12^\circ \pm 10^\circ \) since \( M_p = 60 \text{ dB} \) (see Table 4.8.2), we first sketch the envelope of the band-pass filter response \( r_{LP}(t_n) \) as shown in Fig. 6.4.14. We then denormalize the time scale as

\[
t = \frac{t_n}{B/2} = \frac{t_n}{2\pi(1 \text{ KHz})/2} = 0.318 t_n \text{ (msec)}
\]

(6.4.74)
Next, we "fill in" this envelope with the cosine signal \( \cos [2\pi(10 \text{ KHz})t] \). Because \( f_0 \gg B \), this signal simply "darkens" the envelope in Fig. 6.4.14.

**EXAMPLE 6.4.9** Determine the transfer function form and sketch the impulse response of a wideband Butterworth band-pass filter having \( Q = \frac{1}{2} \) so \( B/\omega_0 = 2 \). As discussed in Prob. 2.21, such filters are called *octave-bandwidth* filters because their fractional bandwidths equal two.

**Solution** The \( n \)-th-order Butterworth band-pass filter having a wide bandwidth has a transfer function given by Eq. 6.4.72. The octave-bandwidth filter having unity center frequency has a transfer function

\[
H_{\text{BP}}(n)(s) = H_{\text{LP}}(n)(s/2) H_{\text{HP}}(n)(2s)
\]  

(6.4.75)

where the \( n \)-th-order Butterworth polynomials for \( H_{\text{LP}} \) are listed in Table 4.3.1. \( H_{\text{HP}} \) is obtained using the low-pass to high-pass transformation. The impulse response of the octave-bandwidth filter equals

\[
h_{\text{BP}}(n)(t) = h_{\text{LP}}(n)(2t) * h_{\text{HP}}(n)(t/2)
\]  

(6.4.76)

from Eq. 6.4.73. The result of this convolution for an eighth-order Butterworth octave-bandwidth band-pass filter is shown in Fig. 6.4.15.

Sometimes it is necessary to calculate filter response to inputs other than a cosine step. One such example arises in video processing applications in which a band-pass filter is excited by a Gaussian cosine pulse having a value

\[
f(t) = e^{-(2 \ln 2)(t/T)^2} \cos \omega_0 t U_{-1}(t)
\]  

(6.4.77)

The 3 dB pulse duration equals \( T \). In such instances, Laplace or Fourier transforms are used to calculate responses. Although the method is straightforward, the results become complicated.

Another closely related area is that of the pulsed-cosine response of band-pass filters. This is important in radar and other similar applications. In this situation, the filter input becomes

\[
s(t) = \cos \omega_0 t \left[ U_{-1}(t + T/2) - U_{-1}(t - T/2) \right]
\]  

(6.4.78)

where \( \omega_0 \) is the center frequency of the filter and \( T \) is the duration of the cosine pulse. Schafer and Wang have investigated the behavior of Butterworth band-pass filters under pulsed-cosine operation. The output energy of the filter is given by Eq. 4.18.14 where \( |H_{\text{BP}}|^2 \) is the squared-magnitude response of the band-pass filter. This is a complicated expression even for a Butterworth filter.\(^{13}\) \( E_0 \) is plotted in Fig. 6.4.16 for a fourth-order Butterworth band-pass filter, where \( E_0 \) has been normalized to the input energy \( E \) given by Eq. 1.9.31. The fractional 3 dB bandwidths (1/Q) equal 0.001, 0.004, 0.02, 0.1, 0.2, 0.3, and 0.6. The curves show that for a fixed fractional bandwidth \( B/\omega_0 \), there is a minimum normalized time \( \omega_0 T \) required to pass a reasonable amount of energy through the filter. For example, for a \( Q = 10 \), then \( \omega_0 T \gg 35 \) to transfer at
least 50% of the input energy through the filter. If \( f_o = 10 \) KHz, then the pulse duration \( T \geq 35/(2\pi \times 10^4) = 0.56 \) msec. For a discussion of pulsed-cosine signal distortion, see Ref. 14.

### 6.4.4 NOISE RESPONSE

Now let us consider the noise energy \( E \) of a band-pass filter. Making the low-pass to band-pass transformation in Eq. 4.18.3 gives

\[
E = (2\pi)^{-1} \int_{-\infty}^{\infty} |H_{LP}(j\omega)|^2 d\omega = (2\pi)^{-1} \int_{0}^{\infty} |H_{BP}(j\omega)|^2 \left[ 1 + (\omega_o/\omega)^2 \right] d\omega
\]

\[
= (2\pi)^{-1} \int_{0}^{\infty} |H_{BP}(j\omega)|^2 d\omega + (Q\omega_o/2\pi) \int_{0}^{\infty} \omega^{-2} |H_{BP}(j\omega)| d\omega \tag{6.4.79}
\]

\( E \) equals the weighted sum of the energies in the white noise response and the \((1/f^2)\) noise response of the band-pass filter. Trofimenkov shows that these noise energies are equal so that \( E \) equals

\[
E = (B\pi)^{-1} \int_{0}^{\infty} |H_{BP}(j\omega)|^2 d\omega = (Q\omega_o/\pi) \int_{0}^{\infty} \omega^{-2} |H_{BP}(j\omega)|^2 d\omega \tag{6.4.80}
\]

Thus, the same performance indices determined for the low-pass filter apply to the band-pass filter when scaled by the bandwidth \( B \) (e.g., see Fig. 4.18.1). In terms of the impulse of the band-pass filter, \( E \) equals

\[
E = (B/2)^{-1} \int_{-\infty}^{\infty} h_{BP}(t)^2 dt \tag{6.4.81}
\]

using Parseval’s theorem of Eq. 4.18.2.

### 6.4.5 ARITHMETICALLY-SYMMETRICAL TRANSFORMATION

In some applications, it is desirable for filters to have magnitude and delay characteristics that exhibit arithmetic symmetry about \( \omega_o \) rather than geometric symmetry. These include radar,
television, telephone, and other communication systems which have signal frequencies distributed arithmetically about their carrier frequencies. In order for band-pass filters to have arithmetic symmetry, they must have a periodic magnitude response. Therefore, they have an infinite number of passbands and stopbands. The additional passbands introduce spurious responses which are undesirable. Thus, we must use a transformation which produces approximate arithmetic symmetry in the magnitude and delay characteristics, or use the optimization techniques of Chap. 5. A transformation which produces almost symmetrical delay and fairly symmetrical magnitude characteristics in the passband (for large Q) is the narrowband or arithmetically-symmetrical transformation where \( s = (B/2)p + j\omega_0 \) \( (6.4.82) \)

Letting \( s = j\omega \) and \( p = jv \) for ac steady-state, then

\[ v = (\omega \pm \omega_0)/(B/2) \] \( (6.4.83) \)

where

\[ \omega_0 = (\omega_U + \omega_L)/2, \quad B = \omega_U - \omega_L, \quad Q = \omega_0/B \] \( (6.4.84) \)

Here we see that the bandwidth remains constant but that the center frequency is the arithmetical mean of the corner frequencies rather than the geometric mean (of Eqs. 6.4.5–6.4.7).

The band-pass filter transfer function is defined to equal

\[ H_{BP}(s) = (s/\omega_0)^n H_{LP}(p) | \quad p = 2(s - j\omega_0)/B \quad H_{LP}(p) | \quad p = 2(s + j\omega_0)/B \] \( (6.4.85) \)

This transformation scales the entire pole/zero pattern of the low-pass filter by \( B/2 \) and translates it to frequencies \( \pm j\omega_0 \). It also inserts \( n \) zeros at the origin. This produces the narrowband pole-zero distribution shown in Fig. 6.4.3b. Expressing the frequency deviation from center frequency \( \omega_o \) as \( \Delta \omega = \omega - \omega_o \), then the gain equals

\[ H_{BP}(j(\omega_o + \Delta \omega)) = (1 + \Delta \omega/\omega_0)^n H_{LP}(j[4Q + \Delta \omega/(B/2)]) \] \( (6.4.86) \)

When \( \Delta \omega \ll \omega_o \), the first and third terms are constant with values \( 1 \) and \( H_{LP}(j4Q) \), respectively, so that \( H_{BP}(j(\omega_o + \Delta \omega)) \approx H_{LP}(j2\Delta \omega/B) \). Therefore, \( H_{BP} \) exhibits arithmetic symmetry in the immediate vicinity of \( \omega_o \). The gains at the band-edges where \( \Delta \omega = \pm B/2 \) equal

\[ H_{BP}(j(\omega_o \pm B/2)) = (1 \pm 1/2Q^2) H_{LP}(\pm j1) H_{LP}(j(4Q \pm 1)) \] \( (6.4.87) \)

The difference between these gains, or the gain unbalance, becomes negligible for large \( Q \). Calculating the gains at \( \omega_0 \) and \( 2\omega_o \) shows that

\[ H_{BP}(0) = 0, \quad H_{BP}(j\omega_o) = j^n H_{LP}(0) H_{LP}(j4Q), \quad H_{BP}(j2\omega_o) = (j2)^n H_{LP}(j2Q) H_{LP}(j6Q) \] \( (6.4.88) \)

Although the dc gain is zero, the gain at frequency \( 2\omega_0 \) is not quite zero, although it approaches zero for increasing \( Q \) and filter order. We also note from Eq. 6.4.85 that the asymptotic gain slopes equal \( \pm 20n \text{ dB/dec} \).

The phase of the band-pass filter equals

\[ \arg H_{BP}(j(\omega_o + \Delta \omega)) = \arg H_{LP}(j[\Delta \omega/(B/2)]) + \arg H_{LP}(j[4Q + \Delta \omega/(B/2)]) + 90^0 \] \( (6.4.89) \)
\[ \tau_{BP}(j\omega) = \tau_1(j\omega) + \tau_2(j\omega) \]  
where \( \tau_1 \) and \( \tau_2 \) are the delays of the individual low-pass filter gain terms in Eq. 6.4.85. These delay terms equal

\[ \tau_1, 2(j\omega) = -\frac{d \arg H_{BP}(j\omega)}{d\omega} = -\frac{d \arg H_{LP}(jv)}{dv} \frac{dv}{d\omega} = \tau_{LP}(jv) \frac{dv}{d\omega} \]  

(6.4.91)

where \( \tau_{LP} \) is the delay of one of the low-pass filters and \( v \) is given by Eq. 6.4.83. Therefore, since

\[ \frac{dv}{d\omega} = 2/B \]  

(6.4.92)

then combining Eqs. 6.4.90 and 6.4.91 gives

\[ \tau_{BP}(j\omega) = (B/2)^{-1} \left[ \tau_{LP}(jv) \left| v = (\omega - \omega_0)/(B/2) \right. + \tau_{LP}(jv) \left| v = (\omega + \omega_0)/(B/2) \right. \right] \]  

(6.4.93)

Here we note that \( dv/d\omega \) is constant rather than being frequency dependent as in the regular low-pass to band-pass transformation (see Eq. 6.4.47). In terms of the frequency deviation from center frequency \( \omega_0 \), then Eq. 6.4.93 can be rewritten as\(^8\)

\[ \tau_{BP}(j(\omega_0 + \Delta\omega)) = (B/2)^{-1} \left[ \tau_{LP}(j(\Delta\omega)/(B/2)) + \tau_{LP}(j[4Q + \Delta\omega/(B/2)]) \right] \]  

(6.4.94)

This equation shows us why the linear phase transformation well-preserves the delay characteristics. The first term is the transformed delay characteristic of the low-pass filter. The second term represents a delay error term. It is small since \( 4Q \) significantly increases the value of the argument for \( \Delta\omega = 0 \) and since \( \tau_{LP} \to 0 \) as \( \omega \to \infty \). It is interesting to note that the first term is the delay of a filter using the “half-transformation”\(^9\) where \( s = pB/2 + j\omega_0 \). This transformation scales and translates the entire pole-zero cluster of the low-pass filter to frequency \( \omega_0 \). Thus, a low-pass filter having constant delay in its passband becomes a band-pass filter also having constant passband delay. Of course, it is nonrealizable since the poles and zeros do not have complex conjugate values. These conjugate poles and zeros must be added for realizability, and this introduces delay distortion.

**EXAMPLE 6.4.10** Determine the transfer function for a fourth-order Butterworth band-pass filter which is arithmetically symmetrical about center frequency \( \omega_0 = 1 \). Design for a bandwidth of \( \frac{1}{4} \) where \( Q = 2 \).

**Solution** The fourth-order Butterworth low-pass filter has a transfer function

\[ H_{LP}(p) = \frac{1}{(p + 0.924)^2 + 0.382^2} \]  

(6.4.95)

The proper transformation to obtain approximate arithmetic symmetry is \( p = 4(s \pm j) \) from Eq. 6.4.82. The pole-zero pattern of the resulting band-pass filter is shown in Fig. 6.4.17. The pattern is scaled by \( B/2 = 1/4 \) and then translated to \( j/\omega_0 = j1 \). The gain of the band-pass filter, given by Eq. 6.4.85, is

![Diagram showing pole transformation](Fig. 6.4.17 Pole transformation of low-pass filter into a quasi-band-pass filter having arithmetically-symmetrical magnitude and delay responses. (The conjugate poles are not shown.))
The frequency domain specifications of band-pass filters having ideal arithmetic symmetry are shown in Fig. 6.4.11a where \( f_u - f_o = f_o - f_l, \) \( f_1 - f_o = f_3 - f_o, \) etc. We convert this data into equivalent information for a low-pass filter by using
EXAMPLE 6.4.11 A band-pass filter having an arithmetically-symmetrical magnitude characteristic was considered in Example 6.4.7. However, it was realized using the standard low-pass to band-pass transformation which resulted in a filter having geometrically-symmetrical magnitude characteristics. Now realize the filter using the transformation to preserve arithmetic symmetry.

Solution We first determine the equivalent low-pass filter requirement. In this case, we have

$$f_0 = \frac{(1000 + 500)}{2} = 750 \text{ Hz}, \quad B = 1000 - 500 = 500 \text{ Hz}, \quad Q = \frac{750}{500} = 1.5$$

(6.4.101)

from Eq. 6.4.84. Now we can determine the normalized stopband frequencies of the equivalent low-pass filter as

$$\Omega_{s1} = \frac{750 - 250}{500/2} = 2, \quad \Omega_{s2} = \frac{750 - 125}{500/2} = 2.5$$

(6.4.102)

which correspond to stopband rejections of 40 and 60 dB, respectively. Thus, the equivalent low-pass filter has the magnitude requirement shown in Fig. 6.4.13b where the $\Omega$-axis is relabeled as (1, 2, 2.5).

From the nomographs of Chap. 4, we find that

- Butterworth: $n_1 > 7$ and $n_2 > 9$ (use $n = 9$)
- Chebyshev: $n_1 > 5$ and $n_2 > 6$ (use $n = 6$)
- Elliptic: $n_1 > 4$ and $n_2 > 4$ (use $n = 4$)

Due to the low $Q$ of 1.5, we do not anticipate good arithmetic symmetry in this situation. One way to improve magnitude symmetry is to select the number of zeros $m$ (letting $n = m$) in Eq. 6.4.85 to minimize the gain imbalance. Taking $m = 3$ reduces this gain imbalance to about 1 dB (see Prob. 6.40).

It is useful to again note that the $M_1-M_2$ dB shaping factor $S = BW_2/BW_1$ is invariant under this transformation and is independent of $Q$. We can again calculate bandwidth ratios to determine stopband frequencies and reduce our work. For instance in Example 6.4.11, $BW_{1.25dB} = 500 \text{ Hz}, BW_{40dB} = 1000 \text{ Hz},$ and $BW_{60dB} = 1250 \text{ Hz}$ using Fig. 6.4.13a. Thus, the $1.25-40 \text{ dB}$ $S = 1000/500 = 2$ and the $1.25-60 \text{ dB}$ $S = 2.5$ which agree with the results of Eq. 6.4.102. The filter order is then determined as usual.

The impulse and cosine step responses of band-pass filters having transfer functions given by Eq. 6.4.85 are given by Eqs. 6.4.70 and 6.4.71. Thus, we simply use the narrowband filter results discussed previously. It is important to re-emphasize that the transformation of Eq. 6.4.82 will yield magnitude characteristics having arithmetic symmetry only for large $Q$, but that delay will remain arithmetically-symmetrical even for small $Q$'s.

Before proceeding further, it should be pointed out that little quantitative design data has been tabulated for classical and optimum band-pass filters. In theory, it can be derived directly, so little effort has been expended in this area. However, there are some limited results and actual design data which the engineer may find useful in designing Butterworth, 20 Bessel, 21 Gaussian and raised-cosine (ISE gain/phase), 22 and linear phase (ISE delay) 23 band-pass filters.

6.4.6 FREQUENCY DISCRIMINATOR TRANSFORMATIONS

Another useful transformation is used to design frequency discriminators for FM applications. These are band-pass filters whose gain is proportional to input frequency (i.e., they are band-limited differentiators). A low-pass frequency discriminator has a linear magnitude response from
Fig. 6.4.19 (a) Low-pass and (b) band-pass (low-Q) frequency discriminators.

\[ H_{DLP}(s) = sH_{LP}(s) \]  
\[ H_{DBP}(s) = sH_{LP}(p) \mid p = (s/\omega_o + \omega_o/s)/B = sH_{BP}(s) \]

This transformation yields gain linearity with respect to log \( \omega \). If we require linearity with respect to \( \omega \) instead, then we use the narrowband transformation given by Eq. 6.4.82 in Eq. 6.4.104. These transformations are suitable for wide-bandwidth or low-Q discriminators. However, for narrow bandwidths, they are not too useful because of the small change in gain \( K \). In these cases, a high-Q band-pass discriminator is required, and we use the transformation

\[ H_{DBP}(s) = pH_{LP}(p) \mid p = (s/\omega_o + \omega_o/s)/B = \frac{1}{B\omega_o} s^2 + \omega_o^2 \]

Its response is shown in Fig. 6.4.20a. By modifying the parameters used in only the \( H_{LP} \) portion of Eq. 6.4.105 so that \( \omega'_o = \sqrt{\omega_o \omega_2} \) and \( B' = \omega_2 - \omega_o \), a band-pass filter results which eliminates the lower side of the gain characteristic as shown in Fig. 6.4.20b. Again, the gain exhibits linearity with respect to \( (\omega - 1/\omega) \). The narrowband transformation may be used to obtain
\[ \arg H_{\text{DLF}}(j\omega) = \arg H_{\text{LP}}(j\omega) + (\pi/2)[2U_{1}(\omega) - 1] \]

\[ \arg H_{\text{DBP}}(j\omega) = \arg H_{\text{BP}}(j\omega) + (\pi/2)[2U_{1}(\omega) - 1] \quad \text{(low-Q)} \] (6.4.106)

\[ \arg H_{\text{DBP}}(j\omega) = \arg H_{\text{BP}}(j\omega) + (\pi/2)[2U_{1}(\omega + \omega_{0}) - 1] + (\pi/2)[2U_{1}(\omega - \omega_{0}) - 1] \quad \text{(high-Q)} \]

from Eqs. 6.4.103, 6.4.104, and 6.4.105, respectively. The delays, therefore, equal

\[ \tau_{\text{DLF}}(j\omega) = \tau_{\text{LP}}(j\omega) + \pi U_{0}(\omega) \]

\[ \tau_{\text{DBP}}(j\omega) = \tau_{\text{BP}}(j\omega) + \pi U_{0}(\omega) \quad \text{(low-Q)} \] (6.4.107)

\[ \tau_{\text{DBP}}(j\omega) = \tau_{\text{BP}}(j\omega) + \pi U_{0}(\omega + \omega_{0}) + \pi U_{0}(\omega - \omega_{0}) \quad \text{(high-Q)} \]

In the time domain, the unit step response of low-pass discriminators equal

\[ t_{\text{DLF}}(t) = h_{\text{LP}}(t) \] (6.4.108)

while the unit step response of the low-Q band-pass discriminator and the unit cosine step response of the high-Q band-pass discriminator equal

\[ t_{\text{DBP}}(t) = h_{\text{BP}}(t) \] (6.4.109)

Thus, these various discriminator responses can be drawn directly from those of their analogous low-pass and band-pass filters.

Another useful result in band-pass discriminator design is to note that the band-stop transfer function given by Eq. 2.8.3 can be used in place of \( p \) in Eq. 6.4.105. This is convenient when we wish to use an existing band-filter as part of the band-pass discriminator, because band-stop filters are easier to implement than filters having inverse band-pass responses (i.e., \( s^{2} + \omega_{0}^{2}/s \) in Eq. 6.4.105). Parameter \( \gamma \) in Eq. 2.8.3 is determined from the discriminator bandwidth.22

### 6.5 LOW-PASS TO BAND-STOP TRANSFORMATIONS

Low-pass filter transfer functions are transformed into band-stop filter transfer functions using the *low-pass to band-stop transformation* which is

\[ H_{\text{BS}}(s) = H_{\text{LP}}(p) \left| \frac{1/p - \omega_{0}}{B} \left( \frac{s}{\omega_{0}} + \frac{\omega_{0}}{s} \right) \right| = H_{\text{BP}}(p) \left| \frac{p - \omega_{0}}{B} \left( \frac{s}{\omega_{0}} + \frac{\omega_{0}}{s} \right) \right| \] (6.5.1)

For simplicity, we often use the frequency normalized form where \( s_{n} = s/\omega_{0} \) so that \( 1/p = (\omega_{0}/B) \) \((s_{n} + 1/s_{n})\). Comparing this transformation with the low-pass to band-pass transformation of the previous section, we see they are reciprocals. Thus, we can make the interesting and very useful observation that band-stop filters can also be obtained by using the low-pass to band-pass transformation on *high pass* filter transfer functions.

To justify the low-pass to band-stop transformation, this latter observation is very helpful. We need to map the \( H_{\text{LP}}(j\omega) \) magnitude characteristic into the \( H_{\text{BS}}(j\omega) \) magnitude characteristic as shown in Fig. 6.5.1. Thus, we must map the \( j\omega \)-axis of the \( p \)-plane into the \( j\omega \)-axis of the \( s \)-plane with the following constraints: (1) \( p = \mp \infty \) maps into \( s = \mp j\omega_{0} \) (and \( -j\omega_{0} \)), (2) \( p = j1 \) maps into \( s = \pm j\omega_{L} \) (and \( -j\omega_{L} \)), and (3) \( p = -j1 \) maps into \( s = \pm j\omega_{U} \) (and \( -j\omega_{U} \)). We also wish to maintain...
the geometric symmetry of $H_{LP}(j\nu)$. A transformation which produces this mapping is

$$\frac{1}{p} = k(\frac{\omega_0}{s + s/\omega_0})$$  \hspace{1cm} (6.5.2)

Setting $p = j\nu$ and $s = j\omega$, then Eq. 6.5.2 becomes

$$\frac{1}{\nu} = k(\omega_0^2 - \omega^2)/\omega_0\omega_0$$  \hspace{1cm} (6.5.3)

When $\nu = \infty$, $1/\nu = 0$, and then $\omega = \pm \omega_0$. When $\nu = +1$ and $-1$, then Eq. 6.5.3 requires that

$$1 = k(\omega_0^2 - \omega_L^2)/\omega_0\omega_L, \quad -1 = k(\omega^2 - \omega_U^2)/\omega_0\omega_U$$  \hspace{1cm} (6.5.4)

Eq. 6.5.4 is identical to Eq. 6.4.4 of the band-pass filter. Therefore, the (geometric) center frequency $\omega_0$, bandwidth $B$, and $Q$ of the band-stop filter equal

$$\omega_0 = \sqrt{\omega_U\omega_L}, \quad B = \omega_U - \omega_L, \quad Q = \frac{\omega_0}{B} = k$$  \hspace{1cm} (6.5.5)

from Eqs. 6.4.5–6.4.7. It is important to remember that the bandwidth of the notch filter is the width of its stopband.

### 6.5.1 EFFECT ON THE S-PLANE

The poles and zeros of the band-stop filter can be easily related to those of the low-pass filter. Rewriting Eq. 6.5.1, we find that $s$ satisfies

$$j\omega = s$$

**Fig. 6.5.2** Mapping of p-plane into s-plane using the low-pass to band-stop transformation. (The lower-half s-plane is not shown.)
which is identical to Eq. 6.5.6. Solving the low-pass filter results by making the \((p, 1/p)\) interchange. Solving Eq. 6.5.6 for \(s\) using the biquadratic equation gives
\[
s = \frac{B}{2p} \pm \left[ \left( \frac{B}{2p} \right)^2 - \omega_o^2 \right]^{1/2}
\] (6.5.7)
so \(s_n\) equals
\[
s_n = \frac{s}{\omega_o} = \frac{1}{2Qp} \pm \left[ \left( \frac{1}{2Qp} \right)^2 - 1 \right]^{1/2}
\] (6.5.8)
As before, every pole and zero of the low-pass filter transforms into a pair of poles and zeros in the band-stop filter. The location of the critical frequencies depends upon the \(p\) and \(Q\) of the transformation. When \(|1/2p| \ll Q\), then
\[
s \approx \frac{B}{2p} \pm j\omega_o
\] (6.5.9)
so \(s_n\) equals
\[
s_n = \frac{s}{\omega_o} \approx \frac{1}{2Qp} \pm j1
\] (6.5.10)
from Eqs. 6.4.11 and 6.4.12. When \(|1/2p| \gg Q\), then \(s\) equals
\[
s = \frac{B}{p}, \quad pQ\omega_o
\] (6.5.11)
so \(s_n\) equals
\[
s_n = \frac{s}{\omega_o} \approx \frac{1}{pQ}, \quad pQ
\] (6.5.12)
from Eqs. 6.4.13 and 6.4.14. Therefore, when \(|1/2p| \ll Q\), the band-stop filter poles and zeros are obtained from the low-pass filter poles and zeros by inverting, scaling \(1/p\) by half the bandwidth, and then translating vertically to \(\pm j\omega_o\). However, when \(|1/2p| \gg Q\), then one band-stop filter pole-zero is obtained by inverting \(p\) and scaling by \(B\). The second pole-zero is obtained by inverting the first pole-zero and scaling by \(\omega_o^2\). For intermediate values of \(p\), the pole-zero locations must be calculated using Eq. 6.5.7. Again we see that real low-pass filter poles-zeros remain real under a low-Q transformation, but become complex under a high-Q transformation.

Alternatively, the value may be obtained directly from Fig. 6.4.2 where we let
\[
1/pQ = x + jy
\] (6.5.13)
and simply read off
\[
s_n = s/\omega_o = a + j\omega
\] (6.5.14)
We may also use the polar form equations, Eqs. 6.4.29 and 6.4.30, and Fig. 6.4.6 in determining band-stop filter roots. However, we must replace \(\omega_n\) by \(1/\omega_n\), so that \(\delta = Q\omega_n\). Conceptually, the idea of first transforming a low-pass filter to a high-pass filter and then applying the low-pass to band-pass transformation, allows us to utilize all of the previous band-pass filter results.

This is illustrated in Fig. 6.5.3 for a second-order Butterworth band-stop filter. We first transform the low-pass filter pole-zero pattern into that for a high-pass filter. By simple frequency inversion, the two zeros at infinity map into zeros at the origin. Then we apply a low-pass to band-pass transformation to obtain the pole-zero pattern for the band-stop filter shown. Depending upon \(Q\), the three general patterns of Fig. 6.4.3 also apply to the band-stop filter, if we move the zeros from the origin to \(\pm j1\).

**EXAMPLE 6.5.1** Determine the transfer function for a second-order Chebyshev band-stop filter having 3 dB in-band ripple. Sketch the gain characteristic of the filter.
SEC. 6.5.1

Solution  

The associated low-pass filter has a transfer function

\[
H_{LP}(p) = \frac{0.707(0.841)^2}{(p + 0.322)^2 + 0.777^2} = \frac{0.707(0.841)^2}{p^2 + 0.644p + 0.841^2} \quad (6.5.15)
\]

Substituting the low-pass to band-stop transformation given by Eq. 6.5.1 into Eq. 6.5.15, then the band-stop filter transfer function equals

\[
H_{BS}(s) = 0.707(0.841)^2 \left[ \frac{1}{Q^2} \left( \frac{s_n}{s_n^2 + 1} \right)^2 + \frac{0.644}{Q} \left( \frac{s_n}{s_n^2 + 1} \right) + 0.841^2 \right]^{-1} \quad (6.5.16)
\]

which can be re-expressed as

\[
H_{BS}(s) = 0.707 \frac{(s_n^2 + 1)^2}{s_n^4 + (0.911/Q)s_n^3 + (1 + 1/0.841Q^2)s_n^2 + (0.911/Q)s_n + 1} \quad (6.5.17)
\]

EXAMPLE 6.5.2  

Determine the poles and zeros of the band-stop filter of Example 6.5.1.  

Solution  

From Eq. 6.5.15, the low-pass filter has poles located at

\[
p_1, 2 = -0.322 \pm j0.777 = 0.841 \exp [(180^0 \pm 67.5^0)] \quad (6.5.18)
\]

and two zeros at infinity. To find the poles of the band-stop filter, we begin by inverting the low-pass filter poles as

\[
1/p_1, 2 = 0.841^{-1} \exp [-j(180^0 \pm 67.5^0)] = 1.19 \exp [(180^0 \pm 67.5^0)] = -0.455 \pm j1.10 \quad (6.5.19)
\]

In the high-Q case, the transformed poles equal

\[
s_n1, 2, 3, 4 = -0.455/2Q \pm j(1 \pm 1.10/2Q) \quad (6.5.20)
\]

from Eq. 6.5.10. For the low-Q case, the transformed poles equal

\[
s_n1, 2 = 1.19Q^{-1} \exp [(180^0 \pm 67.5^0)], \quad s_n3, 4 = 0.841Q \exp [(180^0 \pm 67.5^0)] \quad (6.5.21)
\]

from Eq. 6.5.12. For the intermediate-Q case, we use Figs. 6.4.2 or 6.4.6. For example, when Q = 1, then 1/pQ = -0.455 \pm j1.10 = x + jy and using Fig. 6.4.2,

\[
s_n1, 2 = -0.12 \pm j0.58 = 0.59 \exp [(180^0 \pm 78.5^0)] \quad (6.5.22)
\]

\[
s_n3, 4 = -0.34 \pm j1.68 = 1.71 \exp [(180^0 \pm 78.5^0)]
\]

We again note that the two pairs of band-stop filter poles are obtained from one pair of low-pass filter poles. They will always have the same Q and damping factor. The zeros are easily determined since each of the two zeros of the low-pass filter at infinity map into two zeros of transmission at s = \pm j\omega_0.

EXAMPLE 6.5.3  

A second-order Butterworth band-stop filter having a center frequency \( f_c \) of 1 kHz, a bandwidth B of 100 Hz, and a maximum gain of 10 is required. Determine its transfer function and
sketch its pole-zero pattern. Draw the block diagram realization of the filter.

Solution The band-stop filter has a \( Q = 1 \) KHz/100 Hz = 10. The second-order Butterworth low-pass filter has poles located at

\[
p = -0.707 \pm j0.707 = \exp \left[ j(180^\circ \pm 45^\circ) \right]
\]

Using Fig. 6.4.6 where \( \zeta = Q_\omega \eta = 10 \) and \( Q_{LP} = 0.707 \), then the normalized poles of the band-stop filter equal

\[
|s_n| = 0.962, \quad 1.038; \quad Q_{BS} = 14.1
\]

(6.5.24)

Since the low-pass filter has two zeros at \( p = \infty \), the band-stop filter has two zeros at \( s_n = \zeta j1 \). This is verified from Fig. 6.4.6 using \( \zeta = \infty \) and \( Q_{LP} = 1.0 \) where

\[
|s_n| = 1.0; \quad Q_{BS} = \infty
\]

(6.5.25)

Thus, the transfer function of the band-stop filter equals

\[
H(s) = \frac{10(s_n^2 + 1)^2}{(s^2 + 0.962 s_n^2 + 1.038 s_n^2 + 1.962^2 s_n^2 + 1.038^2 s_n^2)}
\]

(6.5.26)

where \( s_n = s/2\pi(1 \text{ KHz}) \). Of course, we may apply the low-pass to band-stop transformation directly to the low-pass transfer function as

\[
H(s) = H(p) \mid 1/p = Q(s_n + 1/s_n) = \frac{10}{p^2 + 1.414p + 1} \quad p = \frac{Q_{LP} s_n}{s_n^2 + 1}
\]

(6.5.27)

Factoring Eq. 6.5.27 into the product of second-order terms yields Eq. 6.5.26. The block diagram realization of the filter is shown in Fig. 6.5.4.

We should make an important observation concerning the midband gain \( H_\omega \) of the band-stop filter and the gains of the individual blocks in its block diagram realization. From Eq. 6.5.1, we see that at low-frequencies when \( p \to 0 \), then \( s_n \to 0 \) or \( \infty \) and the low-pass and band-stop filter gains equal

\[
H_{BS}(s_n) = H_{LP}(p) \mid 1/p = Q(s_n + 1/s_n) \quad p \to 0 \quad \text{and} \quad s_n \to 0 \text{ or } \infty
\]

(6.5.28)

Hence, when the low-pass filter has a dc gain given by \( H_\omega \), then the band-stop filter has an identical dc and high-frequency gain \( H_\omega \). This is much more convenient than the band-pass filter case where calculations are needed. For example, \( H_{LP}(0) \) and \( H_{BS}(\infty) \) are identical in Example 6.5.1 (Eqs. 6.5.15 and 6.5.17), and Example 6.5.3 (Eq. 6.5.27) as required.

We shall make the same comment about the form of our low-pass to band-stop transformation as we made for the band-pass case. We have used \( 1/p = (\omega_\omega/Q)(s/\omega_\omega + \omega_\omega/s) \). This transformation maps a unity-bandwidth low-pass filter into a band-stop filter having (stopband) bandwidth \( B \). This is especially convenient since most of our tabulated low-pass filters have unity bandwidths. Other authors have chosen to use the transformation \( 1/p = \omega_\omega (s/\omega_\omega + \omega_\omega/s) \), or in terms of normalized frequency when \( \omega_\omega = 1 \), \( p = s + 1/s \). Here the bandwidth of the band-stop filter equals
that of the low-pass filter. We used this form in Chap. 2 when we introduced the various transformations, and indeed, we saw that bandwidth was preserved in the examples discussed there. However, we feel that our first form is more advantageous for present purposes.

### 6.5.2 Magnitude, Phase, and Delay Responses

The magnitude characteristic of a band-stop filter has geometric symmetry about its center frequency \( \omega_0 \) using the low-pass to band-stop transformation. Thus, its gain at frequency \( \omega \) is identical to that at frequency \( \omega_0 / \omega \). The shape of the characteristic is preserved for transformations of any \( Q \). A magnitude characteristic is shown in Fig. 6.5.5a. A more general s-plane plot is shown in Fig. 6.5.6. Note that \( H_{LP}(0) \) maps into \( H_{BS}(0) \) and \( H_{BS}(\infty) \) to form the drumhead. \( H_{LP}(\infty) \) maps into \( H_{BS}(\pm j \omega_0) \).

The phase characteristic also has geometric symmetry about \( \omega_0 \). It is identical to the phase of the analogous high-pass filter after it is recentered from 0 to \( \omega_0 \). This is shown in Fig. 6.5.5b, where it is important to note the phase change of \( 180(n - m) \)° at \( \omega_0 \).

The delay of the band-stop filter can be determined from the low-pass filter delay. The band-stop filter delay equals

\[
\tau_{BS}(\omega) = -\frac{d \arg H_{BS}(j \omega)}{d \omega} = -\frac{d \arg H_{LP}(jv)}{dv} \frac{dv}{d \omega} = \tau_{LP}(jv) \frac{dv}{d \omega}
\]

(6.5.29)

where \( v \) is given by Eq. 6.5.3 as

\[
-1/v = (\omega_0/B)(\omega/\omega_0 - \omega_0/\omega) = (\omega - \omega_0^2/\omega)/B
\]

(6.5.30)

Therefore, since

\[
dv/d\omega = [1 + (\omega_0/\omega)^2] v^2/B = (B/\omega)^2 [1 + (\omega_0/\omega)^2] / B [1 - (\omega_0/\omega)^2]^2
\]

(6.5.31)

then the band-stop filter delay given by Eq. 6.5.29 can be expressed as

\[
\tau_{BS}(\omega) = (B/\omega)^2 [1 + (\omega_0/\omega)^2] / B [1 - (\omega_0/\omega)^2]^2 \tau_{LP}(jv) \left| \frac{1}{v} = Q(\omega_0/\omega - \omega/\omega_0) \right.
\]

(6.5.32)

![Fig. 6.5.5 Effect of low-pass to band-stop transformation upon (a) magnitude and (b) phase characteristics.](image)
The delay expression is more complicated than that for band-pass filters (cf. Eq. 6.4.48).

We see that the $dv/d\omega$ term in Eq. 6.5.29 introduces delay asymmetry as in the band-pass filter case. The delay becomes more asymmetrical with increasing deviation from center frequency. We can find the approximate stopband delay of a filter. If we express the frequency variation from center frequency $\omega_0$ as $\Delta \omega = \omega - \omega_0$, then we can express Eq. 6.5.30 as

$$-1/v = (\omega^2 - \omega_0^2)/B\omega = \Delta \omega (\omega + \omega_0)/B\omega$$  \hspace{1cm} (6.5.33)

Substituting this result into Eq. 6.5.32, the stopband delay of a band-stop filter equals

$$\tau_{BS}(j(\omega_0 + \Delta \omega)) \approx -B/2 \tau_{LP}(B/2\Delta \omega)$$  \hspace{1cm} (6.5.34)

Due to the instantaneous phase change at $\omega = \omega_0$, the delay has impulses of area $\pi(n-m)$ located at $\pm \omega_0$. The delay at the center frequency of the band-stop filter equals

$$\tau_{BS}(j\omega_0) = (B/2)^{-1} \tau_{LP}(1/\omega_0^2) = (B/2)^{-1} (b_{-1}/b_n - a_{m-1}/a_m)$$  \hspace{1cm} (6.5.35)

It is $2/B$ times the high-frequency slope of the low-pass filter whose gain is given by Eq. 2.0.1. Thus, the midband delay is inversely proportional to half the filter bandwidth. The low-frequency delay of the band-stop filter equals

$$\tau_{BS}(j\omega) = (Q\omega_0)^{-1} \tau_{LP}(0) = (Q\omega_0)^{-1} (b_1/b_0 - a_1/a_0), \hspace{1cm} \omega \ll \omega_0$$  \hspace{1cm} (6.5.36)

while the high-frequency delay equals

$$\tau_{BS}(j\omega) = (B/\omega_0^2) \tau_{LP}(0) = (B/\omega_0^2) (b_1/b_0 - a_1/a_0), \hspace{1cm} \omega \gg \omega_0$$  \hspace{1cm} (6.5.37)

This is easily proved using Eqs. 2.10.34, 2.10.36, and 6.5.32. The low-frequency delay is constant
and the high-frequency delay rolls off as $1/\omega^2$.

**EXAMPLE 6.5.4** A second-order low-pass filter having $\omega_n = 1$ and $\zeta = 0.5$ was considered in Example 6.4.5. Determine and sketch the delay characteristic of the analogous second-order band-stop filter.

**Solution** The low-pass filter gain equals $H_{LP}(p) = 1/(p^2 + p + 1)$. The band-stop filter gain is found from Eq. 6.5.1 where $\omega_o$ and $B$ must be specified. The low-pass filter has a delay given by Eq. 6.4.54. Since it has a dc delay of one second, the band-stop filter has a low-frequency delay of $1/(Q\omega_o)$ for $\omega \ll \omega_o$ from Eq. 6.5.36, and a high-frequency delay of $B/\omega_o^2$ for $\omega \gg \omega_o$ from Eq. 6.5.37. From Eq. 6.5.35, the midband delay equals $\tau_{BS}(j\omega_o) = 2/B$ with delay impulses of area $2\pi$ at $\omega = \pm \omega_o$. These delay characteristics are shown in Fig. 6.5.7. The general shape of $\tau_{BS}$ is obtained by taking the product of the $\tau_{LP}$ and $dv/d\omega$ curves.

Band-stop filters are specified in an analogous fashion to band-pass filters. The specifications may be in the frequency domain, the time domain, or in both domains simultaneously. We can utilize the frequency domain results and nomographs of Chaps. 4 and 5. We can also utilize the time domain results for narrowband notch filters. In general, the frequency domain specifications of band-stop filters have the form shown in Fig. 6.5.8. $f_1$ and $f_0$ are the required corner frequencies for less than $M_p$ in-band gain variation and frequencies $f_1, f_2, f_3, f_4$, etc., are the frequencies for various stopband rejections of $M_{s1}, M_{s2}$, etc. In many applications, the gain characteristic will have geometric symmetry about the center frequency $f_0$ where $f_0^2 = f_2f_4 = f_1f_3 = f_2f_4 = \ldots$. We convert this data to that for the equivalent low-pass filter using Eq. 6.5.3 where

$$\Omega_i = \frac{1}{Q} \left| \frac{f_i}{f_0} - \frac{f_0}{f_i} \right|$$  (6.5.38)

We normalize each stopband frequency $f_i$ by $f_0$, form its reciprocal, take the absolute value of the difference, multiply by $Q$, and take the reciprocal. Alternatively, in filters having geometric symmetry, we can simply determine the $M_1 - M_0$ dB shaping factor $S = BW_1/BW_0$ (note the inverted
**EXAMPLE 6.5.5** Determine the minimum order Butterworth, Chebyshev, and elliptic band-stop filters required to meet the magnitude characteristics shown in Fig. 6.5.9a.

**Solution**

We first find the $Q$ and $f_0$ for the filter. Since

\[ f_0 = \left(\frac{250}{4000}\right)^{1/2} = 1000 \text{ Hz}, \quad B = 4000 - 250 = 3750 \text{ Hz} \]

(6.5.39)

using Eq. 6.5.5, then $Q$ equals

\[ Q = \frac{1000}{3750} = 0.267 \]

(6.5.40)

Therefore, the normalized stopband frequencies equal

\[ \Omega_5 = \left[0.267 \left(\frac{2000}{1000} - 1 \right) \right]^{-1} = \left[0.267 \left(\frac{500}{1000} - 1 \right) \right]^{-1} = 1/0.4 = 2.5 \]

(6.5.41)

using Eq. 6.5.38. Alternatively, since we have geometric symmetry and $B_{1.25dB} = 3750$ Hz and $B_{40dB} = 1500$ Hz, then $\Omega_5 = 3750/1500 = 2.5$. The equivalent low-pass filter gain is drawn in Fig. 6.5.9b. From the nomographs of Chap. 4, we find

Butterworth: $n \geq 6$, Chebyshev: $n \geq 4$, Elliptic: $n \geq 3$

Thus, a third-order elliptic band-stop filter will meet this gain requirement.

In some filter applications, asymmetrical magnitude specifications may be given. This occurs in filters requiring arithmetic symmetry. In these situations, we can still utilize the nomographs, assuming of course that filters having geometric symmetry are acceptable. We select the most stringent:

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Fig. 6.5.10 (a) Band-stop filter specification and its (b) equivalent low-pass filter specification of Example 6.5.6.

EXAMPLE 6.5.6 Determine the minimum order Butterworth, Chebyshev, and elliptic band-stop filters required to meet the magnitude requirements shown in Fig. 6.5.10a.

Solution In this case, we have

\[ f_0 = \left[125(1375)\right]^{1/2} = 415 \text{ Hz}, \quad B = 1375 - 125 = 1250 \text{ Hz}, \quad Q = 415/1250 = 0.332 \]

which is a low-Q requirement. The upper and lower stopband frequencies of the equivalent low-pass filter are found to be \( \Omega_1 = 1.12, \Omega_2 = 1.51, \Omega_3 = 2.85, \) and \( \Omega_4 = 8.04 \) using

\[ \Omega_q = \left[0.332 \frac{|f_0/415 - 415/f_0|}{f_0} \right]^{-1} \]

The equivalent low-pass filter is drawn in Fig. 6.5.10b. The required stopband frequencies are unequal and so the smallest frequency is chosen at any given attenuation level. This gives the most stringent gain requirement. From the nomographs of Chap. 4,

Butterworth: Impractical, Chebyshev: \( n \geq 12, 9 \) (use \( n \geq 12 \)), Elliptic: \( n \geq 6, 6 \) (use \( n \geq 6 \))

Therefore, a sixth-order elliptic band-stop filter can be used.

6.5.3 IMPULSE AND COSINE STEP RESPONSES

Band-stop filters are characterized in the time domain by their responses to a unit cosine step input \( \cos \omega_o t \) at \( \tau = 1 \) where \( \omega_o \) is the center frequency of the filter. When the impulse response of the analogous high-pass filter is known, then the impulse response of the band-stop filter can be easily written. The band-stop filter has an impulse response

\[ h_{BS}(t) = L^{-1}[H_{BS}(s)] = L^{-1}[H_{HP}(p) |p = Q(s/\omega_o + \omega_o/s)] \]

Therefore, by analogy with Eqs. 6.4.62 and 6.4.67,

\[ h_{BS}(t) = Q[h_{HP}(t') - \int_0^{t'} \sqrt{\tau(t'-\tau)} J_1(2\sqrt{(t'-\tau)\tau}) h_{HP}(\tau) d\tau] U_{-1}(t) \]

when \( \omega_o = 1 \) and \( t' = t/Q \). Therefore, the unit cosine step response equals
\[ H_{BS}(s) = H_{HP}(p) \bigg|_{p = 2(s + j\omega_0)/B} + H_{HP}(p) \bigg|_{p = 2(s - j\omega_0)/B} \]  

(6.5.47)

Therefore, following the same procedure as that in Eq. 6.4.70, the impulse response of the band-stop filter must equal

\[ h_{BS}(t) = B_s H_{HP}(Bt/2) \cos \omega_0 t \ U_1(t) \]  

(6.5.48)

The unit cosine step response equals

\[ r_{BS}(t) = r_{HP}(Bt/2) \cos \omega_0 t \ U_1(t) \]  

(6.5.49)

by analogy with Eq. 6.4.71. Therefore, narrowband band-stop filters have impulse and cosine step responses which can be drawn by inspection using the impulse and step responses of their analogous high-pass filters.

In wideband band-stop filters, this is not true. In this case, the poles and zeros are given by 1/p = s/B and Q\omega_0/s. The transfer function of the band-stop filter can be expressed as

\[ H_{BS}(s) = H_{HP}(p) \bigg|_{p = s/B} \times H_{HP}(p) \bigg|_{p = Q\omega_0/s} \]  

(6.5.50)

Therefore, the filter impulse response must equal

\[ h_{BS}(t) = \mathcal{L}^{-1}[H_{BS}(s)] = \omega_0^2 h_{HP}(Bt) \ast h_{LP}(Q\omega_0 t) \]  

(6.5.51)

by analogy with Eq. 6.4.73. Thus, we convolve the impulse responses of the low-pass and high-pass filters to determine the impulse response of a wideband band-stop filter. The cosine step response is then obtained using \( h_{BS} \) and Eq. 6.5.46.

It is important to remember that when low-transient or complementary band-stop filters are required, they can be obtained by utilizing the low-transient high-pass filters discussed earlier, and the low-pass to band-pass transformation. This gives us considerable latitude in choosing band-stop filter responses.

**EXAMPLE 6.5.7** A fourth-order Butterworth band-stop filter has a center frequency \( f_0 \) of 10 KHz, a bandwidth \( B \) of 1 KHz, and unity gain. Sketch its cosine step response.

**Solution** Since \( Q = 10 \text{ KHz}/1 \text{ KHz} = 10 \), we treat this as a high-Q filter. Its cosine step response is easily sketched using its analogous high-pass filter step response. This step response is shown in Figs. 6.3.13 and 6.3.16. Using Fig. 6.3.16, we first sketch the envelope response \( t_{HP}(t_n) \) of the band-stop filter as shown in Fig. 6.5.11. We then de-normalize the time scale as

\[ t = \frac{t_n}{B/2} = \frac{1}{1 \text{ KHz}/2} = 2t_n \text{ (msec)} \]  

(6.5.52)

Next, we “fill in” this envelope with the cosine signal \( \cos [2\pi(10 \text{ KHz})t] \). Because \( f_0 \gg B \), this signal

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Fig. 6.5.12  Cosine step response envelope of low-transient fourth-order Butterworth band-stop filter of Example 6.5.8.

simply “darkens” the envelope in Fig. 6.5.11.

EXAMPLE 6.5.8  A low-transient fourth-order Butterworth band-stop filter has a center frequency $f_o$ of 10 KHz, a bandwidth B of 1 KHz, and unity gain. Sketch its cosine step response.

Solution  Since $Q = 10$ as in Example 6.5.7, we treat this as a high-Q filter. Its cosine step response is readily drawn using the analogous high-pass filter step response in Fig. 6.3.15. This response gives the envelope response $t_{HP}(t_n)$ of the band-stop filter as shown in Fig. 6.5.12. We then denormalize the time scale as

$$t = \frac{t_n}{0.33B/2} = 6.06t_n \text{ (msec)}$$  \hspace{1cm} (6.5.53)

Remember that the low-transient fourth-order Butterworth high-pass filter had a 3 dB bandwidth of $f_{3dB} \approx 0.33$ from Fig. 6.3.18, we we must renormalize $f_{3dB}$ accordingly. Next we fill in the envelope with the cosine signal $\cos [2\pi (10 \text{ kHz}) t]$. Comparing this result with that of Fig. 6.5.11, we see the reduced overshoot and increased storage time of the response. Of course, the penalty paid is the decreased rate of rejection in the stopband magnitude characteristics.

We considered the pulsed-cosine response of band-pass filters in the last section. Now let us consider the pulsed-cosine response of band-stop filters. The pulsed-cosine input $s_i$ to the band-stop filter is given by Eq. 6.4.78, its spectrum $S_i$ by Eq. 1.9.30, and its energy $E_i$ by Eq. 1.9.31. The output energy $E_o$ of the filter is given by Eq. 6.4.79 where $|H_n|^2$ is the squared-magnitude response of the band-stop filter. It can be shown that the $E_o/E_i$ ratio for Butterworth band-stop filters is simply one minus that for Butterworth band-pass filters. Thus, with a simple relabelling of the ordinate in Fig. 6.4.16, the same figure can be used for analyzing the pulsed-cosine response of fourth-order Butterworth band-stop filters.

We see from this figure that there is a maximum normalized time $\omega_o T$ required to pass a reasonable amount of energy through the filter. For example, for a $Q = 10$, then $\omega_o T \leq 35$ to transfer at least 50% of the input energy through the filter. If $f_o = 10$ KHz, then the pulse duration $T \leq 35/2\pi \times 10^{-4} = 0.56$ msec.

6.5.4 ARITHMETICALLY SYMMETRICAL TRANSFORMATION

In applications involving signal frequencies which are arithmetically distributed about their carrier frequencies, band-stop filters having arithmetic symmetry in magnitude and/or delay are required.
\[ \omega_0 = (\omega_U + \omega_L)/2, \quad B = \omega_U - \omega_L, \quad Q = \omega_0/B \] 

The band-stop filter transfer function is defined to equal
\[
H_{BS}(s) = H_{LP}(p)|_{1/p = 2(s + j\omega_0)/B} \quad H_{LP}(p)|_{1/p = 2(s - j\omega_0)/B}
\]
\[
= H_{HP}(p)|_{p = 2(s + j\omega_0)/B} \quad H_{HP}(p)|_{p = 2(s - j\omega_0)/B}
\] (6.5.56)

This transformation scales the entire pole-zero pattern of the high-pass filter by B/2 and translates it to frequencies \( \pm j\omega_0 \). This produces the narrowband pole-zero distribution shown in Fig. 6.5.3. Expressing the frequency deviation from center frequency \( \omega_0 \) as \( \Delta \omega = \omega - \omega_0 \), then \( H_{BS} \) equals
\[
H_{BS}(j(\omega_0 + \Delta \omega)) = H_{HP}(j\Delta \omega/(B/2)) \quad H_{HP}(j[4Q + \Delta \omega/(B/2)])
\] (6.5.57)

When \( \Delta \omega \ll \omega_0 \), the second term is constant with value \( H_{LP}(j4Q) \) so that \( H_{BS}(j(\omega_0 + \Delta \omega)) \sim H_{HP}(j2\Delta \omega/B) \). Therefore, \( H_{BS} \) exhibits arithmetic symmetry in the immediate vicinity of \( \omega_0 \). The gains at the band-edges where \( \Delta \omega = \pm B/2 \) equal
\[
H_{BS}(j(\omega_0 \pm B/2)) = H_{HP}(\pm j1) \quad H_{HP}(j(4Q \pm 1))
\] (6.5.58)

The gain unbalance becomes negligible for large \( Q \). Calculating the gains at \( \omega_0 \), dc, and \( 2\omega_0 \) shows that
\[
H_{BS}(j\omega_0) = 0, \quad H_{BS}(0) = H_{HP}(-j2Q)H_{HP}(j2Q), \quad H_{BS}(j2\omega_0) = H_{HP}(j2Q)H_{HP}(j6Q)
\] (6.5.59)

Although the gain at center frequency \( \omega_0 \) is zero, the gains at frequencies of dc and \( 2\omega_0 \) are slightly unbalanced by the ratio \( H_{HP}(-j2Q)/H_{HP}(j6Q) \). This gain unbalance approaches zero for increasing \( Q \).

The phase of the band-stop filter equals
\[
\arg H_{BS}(j(\omega_0 + \Delta \omega)) = \arg H_{HP}(j\Delta \omega/(B/2)) + \arg H_{HP}(j[4Q + \Delta \omega/(B/2)])
\] (6.5.60)

The second term is constant with a value \( \arg H_{HP}(j4Q) \) for small \( \Delta \omega \ll 2\omega_0 \). Thus, the phase in the vicinity of the stopband is equal to the phase of the high-pass filter phase (around dc) augmented by \( \arg H_{HP}(j4Q) \).

By analogy with Eq. 6.4.94, the delay of the band-stop filter equals
\[
\tau_{BS}(j(\omega_0 + \Delta \omega)) = (B/2)^{-1}[\tau_{HP}(j\Delta \omega/(B/2)) + \tau_{HP}(j[4Q + \Delta \omega/(B/2)])]
\] (6.5.61)

The delay characteristics of the high-pass filter are well-preserved under this transformation. This is due to the second term which represents delay error \( \Delta \omega \). It is small for large \( Q \) and small \( \Delta \omega \).

The frequency domain specifications of band-stop filters having ideal arithmetic symmetry are shown in Fig. 6.5.8a where \( f_U - f_o = f_o - f_L, f_1 - f_o = f_3 - f_o \), etc. We convert this data into equivalent information for a low-pass filter by using
\[
\Omega_1 = \left(\frac{|f_1 - f_o|}{B/2}\right)^{-1}
\] (6.5.62)

from Eq. 6.5.54. Here we simply determine the deviation of the stopband frequencies from the center frequency, normalize by half the bandwidth, and take the reciprocal. Alternatively, we can simply calculate bandwidth ratios and shaping factors. Either approach gives us the normalized stopband frequencies of the equivalent low-pass filter. We can now utilize the various magnitude

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Characteristics for the low-pass filters of the earlier chapters, select the desired type, determine the required orders, and obtain the normalized poles and zeros from the tables. Then using the arithmetically-symmetrical low-pass to band-stop transformation, we can write the gain of the required band-stop filter. It is important to always compare the gain unbalance at the band-edges using Eq. 6.5.58, and at dc and $2\omega_o$ using Eq. 6.5.59, to insure that the actual filter response will be close to the ideal response on which Eq. 6.5.62 is based.

**EXAMPLE 6.5.9** The telephone system uses holding tones and test tones in measuring impulse noise, gain hits, phase hits, phase jitter, and crosstalk in transmission channels. Generally, the tone frequencies are 1010 and 2805 Hz. Notch filters are often used to eliminate these tones when telephone lines are being monitored for other purposes. Assume that the 1010 Hz notch filter has a passband gain of 0 dB and that the other specifications are: (1) 0.5 dB frequencies $< 400$, $> 1620$ Hz; (2) $3$ dB frequencies $= 838$, 1182 Hz; and (3) 50 dB frequencies $= 995$, 1025 Hz as shown in Fig. 6.5.13a. What order Butterworth, Chebyshev, and elliptic filter is required? Which filter is most appropriate?

**Solution** The center frequency, bandwidth, and $Q$ of the band-stop filter equal

$$f_o = (400 + 1620)/2 = 1010 \text{ Hz}, \quad \beta = 1620 - 400 = 1220 \text{ Hz}, \quad Q = 1010/1220 = 0.828$$

(6.5.63)

We normalize the stopband frequencies as

$$\frac{1}{\Omega_{s1}} = \frac{1010 - 838}{1220/2} = 1182 - 1010 = \frac{1}{3.55}, \quad \frac{1}{\Omega_{s2}} = \frac{1010 - 995}{1220/2} = 1025 - 1010 = \frac{1}{40.7}$$

(6.5.64)

Alternatively, since $BW_{dB} = 344$ Hz and $BW_{6dB} = 30$ Hz, the normalized stopband frequencies are $\Omega_{s1} = 1220/344 = 3.55$ and $\Omega_{s2} = 1220/30 = 40.7$. Thus, the equivalent low-pass filter has the magnitude requirements shown in Fig. 6.5.13b. From the nomographs of Chap. 4, we find

- Butterworth and Chebyshev: $n_1 \geq 1$ and $n_2 \geq 2$ (use $n \geq 2$)
- Elliptic: $n_1 \geq 3$ and $n_2 \geq 3$ (use $n \geq 3$)

For minimum complexity and the best delay and step response characteristics, we shall use a second-order Butterworth notch filter.

We should verify that the resulting filter has approximate arithmetic symmetry. The gain unbalance from $|H_{HP}(\pm 1)| = -0.5$ dB at the edge of the passband equals

$$|H_{HP}(4Q + 1)| = |H_{HP}(0.31)| = |H_{LP}(0.232)| = -0.01 \text{ dB}$$

(6.5.65)
from Eq. 6.5.59 which is also small. Thus, we assume the band-stop filter will have close to arithmetic symmetry.

The impulse and cosine step responses of band-stop filters having transfer functions given by Eq. 6.5.56 are given by Eqs. 6.5.49 and 6.5.49. Thus, we simply use the narrowband filter results discussed previously. It is important to re-emphasize that the transformation of Eq. 6.5.44 will yield magnitude characteristics having arithmetic symmetry only for large Q, but that delay will remain arithmetically-symmetrical even at small Q's.

6.6 LOW-PASS TO ALL-PASS TRANSFORMATION

Low-pass filters are transformed into all-pass filters using the low-pass to all-pass transformation where

\[ H_{AP}(s) = \frac{H_{LP}(s)}{H_{LP}(-s)} = H_{LP}(s)H_{LP}^{-1}(-s) \]  

(6.6.1)

This transformation produces zeros which are images of the poles, and visa versa. All-pass filters were introduced in Sec. 2.11. Stable, causal all-pass filters (the usual case of interest) have only left-half-plane poles with their right-half-plane zero images. They have no j\( \omega \)-axis poles or zeros. Thus, only low-pass filters having no finite zeros (except perhaps on the j\( \omega \)-axis) are used in Eq. 6.6.1. All-pass filters have gains, phase, and delay responses which satisfy

\[ |H_{AP}(j\omega)| = 1, \quad \arg H_{AP}(j\omega) = 2\arg H_{LP}(j\omega), \quad \tau_{AP}(j\omega) = 2\tau_{LP}(j\omega) \]  

(6.6.2)

The phase characteristic of the all-pass filter is twice that of the low-pass filter (excluding any steps due to j\( \omega \)-axis zeros) and must always be monotonically nonincreasing. All-pass filters can only introduce positive delay (never negative). The delay is twice that of its analogous low-pass filter. The use of Eq. 6.6.2 is illustrated in Examples 2.10.1–2.10.8, 2.11.1, and 2.11.2.

In most cases, the low-pass filter selected is an all-pole filter where \( H_{LP}(s) = D_n(0)/D_n(s) \). Then the analogous all-pass filter has a transfer function

\[ H_{AP}(s) = \frac{D_n(-s)}{D_n(s)} \]  

(6.6.3)

Usually, the low-pass filter is selected which has the required delay characteristic. For example, if \( H_{AP} \) is a delay element, then a Bessel or equiripple delay low-pass filter may be used. In situations where \( H_{AP} \) is a delay equalizer, then we select a low-pass filter having a delay characteristic \( \tau_e \) which complements that of the delay characteristic to be equalized. In other words, \( \tau_{LP} \) is chosen so that

\[ \tau_e + 2\tau_{LP} = \text{constant} \]  

(6.6.4)

This was illustrated in Examples 2.10.6, 2.11.4, and 2.11.5. In general, the optimization methods of Chap. 5 are used to determine the parameters of the required delay equalizer.

Although the magnitude, phase, and delay responses of all-pass filters are easy to determine from their transfer functions, their step responses are not. Unfortunately, there are no convenient transformation theorems known which directly relate low-pass and all-pass filter step responses. They must be calculated directly from their transfer functions using Laplace transform methods. This was discussed in Sec. 3.12.