These are notes to supplement the lectures on elliptic equations. Our model problem will be the Dirichlet boundary value problem

\[-\Delta u = f,\]
on a bounded open set \( \Omega \subset \mathbb{R}^2 \) with \( u = 0 \) on \( \partial \Omega \). For simplicity we will assume that \( \partial \Omega \) is a twice differentiable curve, and that \( \Omega \) is convex.

**Preliminaries** we will introduce two Hilbert spaces on \( \Omega \), first

\[ L^2(\Omega) \]
is the space of all real valued functions or limits of sequences of real valued functions such that

\[ \int_{\Omega} u^2 \, dx < \infty. \]
The limit is here taken in the following sense. We say \( u_n \) converges to \( u \) in \( L^2(\Omega) \) if

\[ \int_{\Omega} (u_n - u)^2 \, dx \to 0. \]
The second Hilbert space is a subspace of the first one and consists of all differentiable real valued functions on \( \Omega \) such that

\[ \int_{\Omega} (u^2 + |\nabla u|^2) \, dx < \infty, \quad u|_{\partial \Omega} = 0 \]
and the limits of sequences of such functions. We denote this space by \( H^1_0(\Omega) \). Limits in this space are taken in a slightly different sense, since they also involve the derivatives of these functions. But we will not need any specifics here.

To continue we will introduce the notion of a bilinear form: Given a Hilbert Space \( V \) a symmetric bilinear form \( \phi \) is a function

\[ \phi : V \times V \to \mathbb{R}, \]
such that:

\[
\phi(u, v) = \phi(v, u) \\
\phi(u, av + bw) = a\phi(u, v) + b\phi(u, w) \\
\phi(av + bw, u) = a\phi(v, u) + b\phi(w, u)
\]

for all \(u, v, w \in V\) and all \(a, b \in \mathbb{R}\). The simplest example is if \(V\) is a real \(n\)-dimensional vector space and \(A\) is a symmetric \(n \times n\) matrix, then

\[\phi(u, v) = u^T Av\]

is a symmetric bilinear form.

To continue we will construct a symmetric bilinear form connected with our model problem. To do this let \(u\) and \(v\) be two elements of \(H^{1}_0(\Omega)\), then we can consider

\[-u\Delta v.\]

integrating this expression over \(\Omega\) and applying Green’s Formula we get:

\[-\int_{\Omega} u\Delta v \, dx = -\int_{\partial\Omega} v \frac{\partial u}{\partial n} \, ds + \int_{\Omega} \nabla u \nabla v \, dx.\]

The boundary term vanishes, and it is easily checked that

\[\Phi(u, v) = \int_{\Omega} \nabla u \nabla v \, dx\]

is indeed a symmetric bilinear form on \(H^{1}_0(\Omega)\). Next we introduce the notion of a week solution to our model problem:

\(u\) is a week solution to the model problem, if and only if

\[\Phi(u, v) - (f, v) = 0\]

for all \(v \in H^{1}_0(\Omega)\). Here, the inner product

\[(f, v) = \int_{\Omega} fv \, dx\]

It is easy to see that this inner product satisfies all properties of an innerproduct on
$L^2(\Omega)$. In particular it satisfies the Cauchy-Schwartz Inequality:

$$|(f,v)| \leq |f||v|,$$

where

$$|f| = \left(\int_{\Omega} f^2 \, dx\right)^{\frac{1}{2}}.$$

The next step is to show that finding a weak solution is equivalent to finding a minimum of a function. This is important, since the weak solution condition must be satisfied for all $v \in H^1_0(\Omega)$, a condition which is almost impossible to check.

**Theorem.** Let $V$ be a Hilbert Space and $H$ be a subspace. Let $f \in V$, and let $\phi$ be a symmetric bilinear form on $H$. Then $u$ is a weak solution to

$$\phi(u,v) - (f,v) = 0 \quad \forall v \in H,$$

if and only if the functional

$$F(w) = \frac{1}{2} \phi(w,w) - (f,w)$$

has a minimum at $w = u$.

**Proof:** Assume that $F$ has a minimum at $u$, and let $v$ be an arbitrary element of $H$. For $t \in \mathbb{R}$ let

$$G(t) = F(u + tv).$$

$G$ is a real valued function on $\mathbb{R}$, which has a minimum at $t = 0$ (since $G(0) = F(u)$ which was said to be a minimum.) From Calculus we know that $G'(0) = 0$ if $G$ is differentiable. But

$$G(t) = \frac{1}{2} \phi(u + tv,u + tv) - (f,u + tv) = \frac{1}{2} \phi(u,u) + \phi(u,v)t + \frac{1}{2} \phi(v,v)t^2 - (f,u) - (f,v)t,$$

is just a second degree polynomial in $t$ and therefore differentiable.

$$G'(t) = \phi(u,v) + \phi(v,v)t - (f,v),$$

and

$$G'(0) = \phi(u,v) - (f,v).$$
So, \( u \) is a minimum if and only if \( \phi(u, v) - (f, v) \) for an arbitrary \( v \in H \), i.e. \( u \) is a weak solution, and the proof is complete.

The advantage of the minimum problem is, that it only involves one unknown \( u \), not two. The next step is to show that the given minimum problem actually has a solution. This is done in The next theorem.

**Theorem.** Let \( v \) be a Hilbert space and \( H \) a closed subspace of \( V \). If the symmetric bilinear form

\[
\phi : H \times H \to \mathbb{R},
\]

satisfies

\[
\phi(u, u) \geq c|u|^2,
\]

for all \( u \in H \), then the function

\[
F(u) = \frac{1}{2}\phi(u, u) - (f, u)
\]

has a minimum in \( H \) for all \( f \in V \).

**Proof:** Define

\[
M = \{ a \in \mathbb{R} : a = F(u), u \in H \}.
\]

Since

\[
|(f, u)| \leq |f||u| \leq \frac{1}{2\epsilon}|f|^2 + \frac{\epsilon}{2}|u|^2;
\]

we have

\[
\frac{1}{2}\phi(u, u) - (f, u) \geq \frac{1}{2}\phi(u, u) - |(f, u)| \geq \frac{1}{2}\phi(u, u) - \frac{1}{2\epsilon}|f|^2 - \frac{\epsilon}{2}|u|^2 \geq \left( c - \frac{\epsilon}{2} \right)|u|^2 - \frac{1}{2\epsilon}|f|^2.
\]

Since \( \epsilon \) is arbitrary, we may choose \( \epsilon = 2c \) and get

\[
F(u) \geq \frac{1}{4c}|f|^2.
\]

Since \( f \) is given, this implies that the set \( M \) is bounded below. Any set of real numbers that is bounded below has a greatest lower bound. So let

\[
m = \inf M.
\]
Now for any $n > 0$ there is a $u_n \in H$ such that

$$F(u_n) \leq m + \frac{1}{n},$$

for if there is no such $u_n$ for a given value of $n$, then $m + \frac{1}{n}$ would be a lower bound on $M$ and $m$ could not be the greatest lower bound. We have thus constructed a sequence of functions $u_n$ such that $F(u_n) \to m$ for $n \to \infty$. However, we still need to show that the sequence itself has a limit in $H$. First of all, observe that for this sequence the expression,

$$a_n = \frac{1}{2} \phi(u_n, u_n)$$

is a bounded sequence of numbers, and therefore it has a convergent subsequence. Let $(u_n)$ denote this subsequence. Then

$$\frac{1}{2} \phi(u_n, u_n) \to a$$

some $a \in \mathbb{R}$, and $(f, u_n) = \frac{1}{2} \phi(u_n, u_n) - F(u_n) \to a - m$ since $F(u_n)$ converges to $m$. Since, we could have done this process with any $f \in V$ from the beginning, we conclude that

$$(f, u_n)$$

converges for every $f \in V$. Next, for any $f \in V$ consider the function

$$\Psi(f) = \lim_{n \to \infty} (f, u_n)$$

It is easy to show that

$$\Psi(af + bg) = a\Psi(f) + b\Psi(g),$$

i.e. that $\Psi$ is a linear function from $V$ to $\mathbb{R}$. The Riesz Representation Theorem is a Theorem which goes beyond the scope of this class which states that for any such linear function $\Psi: V \to \mathbb{R}$ there exists a $v \in V$ such that

$$\Psi(f) = (f, v)$$

Let $u^*$ be that value for our given $\Psi$, i.e.

$$\lim_{n \to \infty} (f, u_n) = (f, u^*).$$
(This is almost as good as saying \( u^* = \lim_{n \to \infty} u_n \), and in finite dimensional Hilbert spaces it is actually the same, however, we are in an infinite dimensional space, so we need a little more). To continue, consider

\[
\frac{1}{2} \phi(u_n, u^*) - (f, u^*)
\]

this is again a bounded sequence of real numbers, which has a convergent subsequence (which continue to denote by \( u_n \)). To continue, observe that the sequence

\[ b_n = |u_n|^2 \]

satisfies

\[
\frac{1}{c} \phi(u_n, u_n) \geq |u_n|^2 \geq 0
\]

so it must also have a convergent subsequence. Again we denote this sequence by \( (u_n) \).

Let \( b^2 \) denote the limit of this sequence. Since \( (u_n, u^*) \) converges to \( |u^*|^2 \) it is clear that \( b^2 = |u^*|^2 \). Finally, consider

\[
|u_n - u^*|^2 = (u_n - u^*, u_n - u^*) = (u_n, u_n) - 2(u_n, u^*) + |u^*|^2
\]

taking the limit as \( n \to \infty \) we see that the right hand side converges to 0 and therefore

\[
\lim_{n \to \infty} u_n = u^*.
\]

However, we still need to show that \( u^* \) is actually in the subspace \( H \). However, this follows from the fact that \( H \) is closed. This completes the proof of this Theorem.

The final objective is to show that our model problem satisfies the hypothesis of our two theorems. To do this we need Poincaré’s Inequality. We prove this in the next theorem for the special case of a convex set.

**Theorem.** Let \( \Omega \) be a convex subset of \( \mathbb{R}^2 \) and \( u \) a differentiable function on \( \Omega \), which vanishes on the origin. The there exists a positive constant \( C \) such that.

\[
\int_{\Omega} |u|^2\, dx\, dy \leq C \int_{\Omega} |\nabla u|^2\, dx\, dy.
\]

**Proof:** Let \((x, y)\) be any point in \( \Omega \). Since \( \Omega \) is convex we can connect this point with a point \((x, y_0)\) on the boundary of \( \Omega \) by a line segment that is parallel to the \( y \)-axis.
Then from the fundamental theorem of calculus we have:

\[ u(x, y) - u(x, y_0) = \int_{y_0}^{y} \frac{\partial u(x, s)}{\partial y} \, ds \]

or since \( u(x, y_0) = 0 \)

\[ u(x, y) = \int_{y_0}^{y} \frac{\partial u(x, s)}{\partial y} \, ds. \]

By the same argument, now in \( x \)-direction we get

\[ u(x, y) = \int_{x_0}^{x} \frac{\partial u(t, y)}{\partial x} \, dt. \]

Now we combine both expressions to get

\[
|u(x, y)|^2 = \left| \int_{x_0}^{x} \int_{y_0}^{y} \frac{\partial u(x, s)}{\partial y} \frac{\partial u(t, y)}{\partial x} \, ds \, dt \right|
\]

\[
\leq \int_{x_0}^{x} \int_{y_0}^{y} \left| \frac{\partial u(x, s)}{\partial y} \right| \left| \frac{\partial u(t, y)}{\partial x} \right| \, ds \, dt
\]

\[
\leq \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dxdy.
\]

Integrating over \( \Omega \) again gives the desired result.

From this it follows that our model problem fits the hypothesis of the above theorems.