1. Prove that \((P \land (P \rightarrow Q)) \rightarrow Q\) is a tautology. Hint: Use the truth table.

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<tr>
<th>P</th>
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<th>P → Q</th>
<th>P \land (P → Q)</th>
<th>(P \land (P → Q)) → Q</th>
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Since the last column is all true for every possible truth value of \(P\) and \(Q\), then \((P \land (P \rightarrow Q)) \rightarrow Q\) is a tautology.

2. Write the contrapositive of the following statement:

If \(n\) is a perfect square and \(n\) is even then 4 divides \(n\).

If 4 does not divide \(n\) then \(n\) is not a perfect square or \(n\) is odd (\(n\) not even is also acceptable)

3. Let \(A = \{a, b, c\}\). Write at least 5 different elements of \(\mathcal{P}(\mathcal{P}(A))\). Recall that \(\mathcal{P}(A)\) stands for the power set of \(A\).

\(\emptyset, \{a\}, \{b\}, \{a, b\}, \{b\}, \emptyset, \) etc.

We must have a set whose elements are subsets of \(A\), for example \(\{a\}\) and \(\{b, c\}\) are subsets of \(A\). Thus \(\{\{a\}, \{b, c\}\} \in \mathcal{P}(\mathcal{P}(A))\).

4. The universe is the set of positive integers \(\mathbb{N}\). Let \(P(x)\) be the following propositional function:

\(P(x) : (\forall n)(n \text{ divides } x \rightarrow (3n \leq x \text{ or } n = x))\)

(a) Write a useful negation of \(P(x)\).

\(\neg P(x) : (\exists n)(n \text{ divides } x \text{ and } (3n > x \text{ and } n \neq x))\)

(b) Decide if \(P(5)\) is true or not, explain.

\(P(5) : (\forall n)(n \text{ divides } 5 \rightarrow (3n \leq 5 \text{ or } n = 5))\).

If \(n\) divides 5 and \(n \in \mathbb{N}\) then \(n = 1\) or \(n = 5\) (since 5 is prime). If \(n = 1\) then \(3 = 3n \leq 5\).

If \(n = 5\) then \(n = 5\). In both cases the consequence is true, then \(P(5)\) is true.

(c) Decide if \(P(8)\) is true or not, explain.

We show \(P(8)\) is false by proving that \(\neg P(8)\) is true.

\(\neg P(8) : (\exists n)(n \text{ divides } 8 \text{ and } (3n > 8 \text{ and } n \neq 8))\)

Let \(n = 4\), observe that 4 divides 8, \(3n = 12 > 8\), and \(4 \neq 8\). Thus \(n = 4\) is the number that exists in the previous proposition. Thus \(\neg P(8)\) is true.

5. Prove that if \(x\) is a positive real number then \(x + \frac{1}{x} \geq 2\). For which \(x\) does equality occur?

**Direct proof:** Since \(x\) is real then \((x - 1)^2 \geq 0\), this is equivalent to \(x^2 - 2x + 1 \geq 0\).

Which is equivalent to \(x^2 + 1 \geq 2x\). Since \(x > 0\) then when we divide by \(x\) the inequality is maintained, thus

\[
\frac{x + \frac{1}{x}}{x} = \frac{x^2 + 1}{x} \geq \frac{2x}{x} = 2,
\]

and the proof is finished.
Contradiction proof: Suppose $x$ is a positive real number and $x + \frac{1}{x} < 2$. Since $x > 0$ we can multiply by $x$ and the inequality will be maintained. Then

$$x^2 + 1 = x \left( x + \frac{1}{x} \right) < x(2) = 2x,$$

which is equivalent to

$$(x-1)^2 = x^2 - 2x + 1 < 0.$$  

This is a contradiction since the square of every real number is positive.

In both proofs, equality occurs if $x + \frac{1}{x} = 2$, and solving for $x$ one obtains $x = 1$.

6. Let $a$ and $b$ be integers. Prove that if $a$ divides $b$, and $b$ divides $(105 - a)$; then $a$ divides 105.

**Direct proof:** If $a$ divides $b$ and $b$ divides $105 - a$ then by definition there are integers $k_1$ and $k_2$ such that $b = ak_1$ and $105 - a = bk_2$.

Substituting the first equation into the second we have

$$105 - a = ak_1k_2.$$  

Then

$$105 = a(1 + k_1k_2),$$

and since $1 + k_1k_2$ is an integer then $a$ divides 105.

7. Let $A$ and $B$ be sets. Prove that if $A \cap B = A$ then $A \subseteq B$.

**Proof.** (By chasing the element) Assume $A \cap B = A$. Let $x \in A$, since $A \cap B = A$ then $x \in A \cap B$, in other words $x \in A$ (which we already knew), and $x \in B$. Therefore all elements in $A$ are also in $B$, i.e., $A \subseteq B$.

8. Let $A$ and $B$ be sets. Prove that if $A \cap B = \emptyset$ then $(A - B) \cup (B - A) = A \cup B$.

**Proof.** Assume $A \cap B = \emptyset$. We will prove that $A - B = A$, then by symmetry $B - A = B$ and the result will follow.

We will prove $A - B = A$ by ‘chasing the element method’. Let $x \in A - B$, then by definition $x \in A$ and $x \notin B$. So in particular $x \in A$. Thus all elements in $A - B$ are in $A$, i.e. $A - B \subseteq A$.

Now, let $x \in A$. If $x \in B$ then $x \in A \cap B$ (it is in both $A$ and $B$), but by assumption $A \cap B = \emptyset$, so this is not possible. Thus $x \notin B$, summarizing, we have that $x \in A$ and $x \notin B$, i.e., $x \in A - B$. Therefore $A \subseteq A - B$. Putting together $A - B \subseteq A$ and $A \subseteq A - B$ we conclude that $A - B = A$.

9. Prove that the average of two consecutive primes is never a prime. For example $6 = (5+7)/2$, $26 = (23+29)/2$.

**Proof.** Let $x$ and $y$ be two consecutive prime numbers, $x < y$. This means that there are no primes $p$ between $a$ and $b$. Suppose by contradiction that the average of $x$ and $y$ is prime, i.e. $\frac{x+y}{2}$ is prime. Observe that

$$2x < x + y < 2y,$$

then

$$x < \frac{x+y}{2} < y.$$  

Which means $\frac{x+y}{2}$ is between $x$ and $y$, this is a contradiction since there are no primes between $a$ and $b$. Therefore $\frac{x+y}{2}$ is always composite.