Problem 3 Given the power series

\[ f(z) = \sum_{n=0}^{\infty} a_n z^n \]

with radius of convergence \( R \), let \( z_0 \) be the zero of \( f \) which lies closest to the origin. Prove that

\[ \frac{r|a_0|}{M(r) + |a_0|} \leq |z_0| \]

for any \( r < R \), where \( M(r) = \max \{|f(z)| \} \).

Solution Assume that \( a_0 \neq 0 \), otherwise \( z_0 = 0 \) and there is nothing to do. Then, if \( |f(z) - a_0| < |a_0| \) on \( |z| = \rho \), the function \( f \) has no zeros on \( |z| \leq \rho \) because of Rouche’s theorem, since the constant function \( a_0 \) has no zeros. Thus the problem reduces to showing that if \( \rho < \frac{r|a_0|}{M(r) + |a_0|} \) and \( |z| = \rho \), then \( |f(z) - a_0| < |a_0| \).

If \( r < R \), then \( \frac{r|a_0|}{M(r) + |a_0|} < R \). Therefore, if \( |z| = \rho < \frac{r|a_0|}{M(r) + |a_0|} \), then the following manipulations of power series are permissible.

\[
|f(z) - a_0| \leq \sum_{n=1}^{\infty} |a_n||z|^n
= \sum_{n=1}^{\infty} |a_n|\rho^n
< \sum_{n=1}^{\infty} |a_n| \left( \frac{r|a_0|}{M(r) + |a_0|} \right)^n \quad \text{(series of positive coefficients)}
= \sum_{n=1}^{\infty} |a_n| \frac{r^n}{M(r) + |a_0|} \left( \frac{|a_0|}{M(r) + |a_0|} \right)^n
\leq \sum_{n=1}^{\infty} M(r) \left( \frac{|a_0|}{M(r) + |a_0|} \right)^n \quad \text{(Cauchy’s estimate)}
= \left( \frac{M(r)|a_0|}{M(r) + |a_0|} \right) \sum_{n=0}^{\infty} \frac{|a_0|}{M(r) + |a_0|} \cdot \frac{1}{M(r) + |a_0|}
= \frac{|a_0|}{M(r) + |a_0|} \cdot \frac{1 - \frac{|a_0|}{M(r) + |a_0|}}{1 - \frac{|a_0|}{M(r) + |a_0|}} \quad \text{(geometric series)}
= |a_0|
\]

The strict inequality appears in the third line of this string.