Problem 1. A function $f : U \to \mathbb{C}$ has an analytic $k$th root on $U$ if there is a function $h$ analytic on $U$ such that $h^k = f$ on $U$. Suppose that $f$ is analytic and never 0 on the open set $U$. Prove that $f$ has an analytic logarithm on $U$ if and only if $f$ has an analytic $k$th root for all $k = 2, 3, \cdots$.

Solution. (1) “⇒” If $g$ is an analytic logarithm for $f$ on $U$, then $e^g = f$, and so, $(e^{g/n})^n = f$. Therefore $e^{g/n}$ is an analytic $n$th root for $f$ on $U$.

“⇐” We show that $\int_\gamma f'/f = 0$ for every closed path $\gamma$ in $U$.

If $\gamma$ is a closed path in $U$, then the path $f \circ \gamma$ does not pass through 0 because $f$ is never zero on $U$. Therefore,

$$\text{ind}(f \circ \gamma; 0) = \frac{1}{2\pi i} \int_\gamma \frac{f'(z)}{f(z)} \, dz,$$

by a theorem in class.

Let $h_n$ be an analytic function on $U$ such that $(h_n)^n = f$, $n = 2, 3, \cdots$. Then

$$\frac{f'}{f} = \frac{h_n'}{h_n}.$$ 

Hence

$$\frac{1}{n} \int_\gamma \frac{f'}{f} = \int_\gamma \frac{h_n'}{h_n},$$

or

$$\frac{1}{n} \text{ind}(f \circ \gamma; 0) = \text{ind}(h_n \circ \gamma; 0).$$

Since the index of a point with respect to a closed path is an integer, and since $\text{ind}(f \circ \gamma; 0)$ does not depend on $n$, the only possibility for this identity to hold for all $n$ is that $\text{ind}(h_n \circ \gamma; 0) = 0$ for $n$ sufficiently large. Thus

$$\int_\gamma \frac{f'}{f} = 0$$

and so $f$ has an analytic logarithm on $U$. 

Problem 2. Let $a, b$ be two distinct complex numbers, and let $U$ be the complement of the segment $[a, b]$. Show that $f(z) = (z-a)(z-b)$ has an analytic square root but not an analytic logarithm on $U$.

Problem 3. Let $f, g$ be continuous mappings of a connected set $S \subset \mathbb{C}$ into $\mathbb{C} \setminus \{0\}$.

(1) If $f^n = g^n$ for some $n = 2, 3, \cdots$, then show that $f = e^{2\pi ik/n}g$ on $S$, for some $k = 0, 1, \cdots, n-1$. (Thus if $f$ and $g$ agree at one point, then they agree everywhere.)

(2) Show that (1) does not hold in general if $f$ and $g$ map into $\mathbb{C}$ instead of $\mathbb{C} \setminus \{0\}$. 

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Solution. (1) The function \( h = f/g \) is continuous on \( S \) and \( h^n(z) = 1 \) for each \( z \in S \). Therefore, for each \( z \in S \), \( h(z) \) is one of the \( n \)-th roots of 1, namely, one of the numbers \( e^{2\pi i k/n} \), \( k = 0, 1, \ldots, n-1 \). Since \( h \) is continuous and \( S \) is connected, it follows that there is some \( k = 0, 1, \ldots, n-1 \) such that \( h(z) = e^{2\pi i k/n} \) for all \( z \in S \).

(2) Let \( S \) be the real axis and let \( f(x) = x \) and \( g(x) = |x| \) for \( x \in S \).

Problem 4. Give two Laurent series expansions in powers of \( z \) for the function

\[
 f(z) = \frac{1}{z^2(1-z)}
\]

and specify the regions in which those expressions are valid.

Solution. The function \( f \) has singularities at \( z = 0 \) and \( z = 1 \). There are two Laurent series expansions, one on \( 0 < |z| < 1 \) and another on \( 1 < |z| < \infty \).

First decompose into simple fractions

\[
 \frac{-2}{z^2(z-1)} = \frac{1}{z^2} + \frac{1}{z} + \frac{1}{1-z}
\]

If \( |z| < 1 \), then

\[
 \frac{1}{1-z} = \sum_{n=0}^{\infty} z^n
\]

Therefore,

\[
 f(z) = \frac{-1/2}{z^2} + \frac{-1/2}{z} + \sum_{n=0}^{\infty} z^n, \quad 0 < |z| < 1.
\]

If \( 1 < |z| \), then \( |1/z| < 1 \), and

\[
 \frac{1}{1-z} = \frac{-1/z}{1-(1/z)} = \frac{-1}{z} \sum_{n=0}^{\infty} \frac{1}{z^n} = \sum_{n=1}^{\infty} (-1) \frac{1}{z^n}.
\]

Therefore,

\[
 f(z) = \frac{1}{2} z^{-2} + \frac{1}{2} z^{-1} + \sum_{n=-\infty}^{3} (-1) z^n, \quad 1 < |z| < \infty.
\]

Problem 5. Let \( f \) be analytic and never 0 on the open set \( U \subset \mathbb{C} \), and let \( g \) be a continuous logarithm of \( f \) on \( U \). Prove that \( g \) is analytic on \( U \).

Solution. Analyticity is a local property, thus to show that \( f \) is analytic on \( U \) it suffices to show that \( f \) is analytic a disk about each point in \( U \). Let \( D \) be a disk contained in \( U \). Then \( D \) is convex and \( f \) is analytic and nowhere 0 on \( D \), so it has an analytic logarithm \( h \) on \( D \). Since \( g \) is a continuous logarithm for \( f \) on \( D \) and \( D \) is connected, \( g = h + 2\pi k \), for some integer \( k \). Thus \( g \) is analytic on \( D \), because it is the sum of two analytic functions on \( D \).