Math 655. Homework 3. Due 2/26/03

Problem 1 Let $f$ be an analytic function on a connected open set $U \subset \mathbb{C}$.

(1) Show that if $f$ is real valued, then $f$ is constant on $U$.

(2) Show that if $f$ has constant absolute value, then $f$ is constant on $U$.

Problem 2 Let $f$ be analytic on $\mathbb{C}$ and real valued on $|z| = 1$. Show that $f$ is constant.

Problem 3 Let

$$f(z) = \int_{[1,z]} \frac{1}{w} \, dw$$

where $[1, z]$ is the line segment from 1 to $z$ in $\mathbb{C}$. Show that $f$ is a well defined analytic function on $\mathbb{C} \setminus \{z = x + iy \mid x \leq 0\}$, and compute its power series expansion centered at the point $z_0 = 1$.

Problem 4 Show that if $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ is a polynomial of degree $n \geq 1$, then $|P(z)| \to \infty$ as $|z| \to \infty$. In fact, show that if $|z| \geq \max\{1, 2n|a_{n-1}|, \cdots, 2n|a_0|\}$, then $|P(z)| \geq |z|^n/2$.

Problem 5 Let $f$ be an entire function such that

$$|f(z)| \leq A|z|^k$$

for all $z \in \mathbb{C}$, for some constant $A$ and integer $k$. Show that $f$ is a polynomial of degree $\max\{0, k\}$. 