Problem 1 Let \( a \in \mathbb{C} \). Show that the function \( f(z) = 1/(z-a) \) is representable by a power series on \( \mathbb{C} \setminus \{a\} \).

Can you generalize this? (If \( f : U \to \mathbb{C} \) is representable by a power series on \( U \) and \( f(z) \neq 0 \) for every \( z \) in \( U \), then \( 1/f \) is representable by a power series in \( U \).)

Problem 2 Compute the integral
\[
\int_{\gamma} \frac{1}{z^2 - 1} \, dz
\]
where \( \gamma \) is the circle \(|z| = 2\), oriented counterclockwise.

Problem 3 Let \( f \) be analytic on \( D(z_0; r) \). Let \( R_n(z) \) be the remainder after the term of degree \( n \) in the Taylor series expansion for \( f \) about \( z_0 \).

(a) Show that
\[
R_n(z) = \frac{(z - z_0)^{n+1}}{2\pi i} \int_{\Gamma} \frac{f(w)}{(w - z)(w - z_0)^{n+1}} \, dw,
\]
where \( \Gamma \) is the circle \(|z - z_0| = r_1\) (oriented counterclockwise).

(b) If \(|z - z_0| \leq s < r_1\), show that
\[
|R_n(z)| \leq \max_{z \in \Gamma} |f(z)| \frac{r_1}{r_1 - s} \left( \frac{s}{r_1} \right)^{n+1}.
\]

Problem 4 Show that the following series all have radius of convergence equal to 1:
\[
\sum_{n=0}^{\infty} \frac{z^n}{n^2}, \quad \sum_{n=0}^{\infty} \frac{z^n}{n}, \quad \sum_{n=1}^{\infty} z^n.
\]
Show that first series converges everywhere on the unit circle; that the third series converges nowhere on the unit circle; and that the second series converges for at least one point on the unit circle and diverges for at least one point on the unit circle.

Problem 5 Compute the integrals
\[
\int_{\gamma} \frac{e^z}{z} \, dz \quad \text{and} \quad \int_{\gamma} \frac{1}{1 + z^2} \, dz
\]
where \( \gamma \) is the counterclockwise oriented circle \(|z| = 2\).