1. Let $X$ be a set.
   (a) Carefully define the concepts $\sigma$-ring and $\sigma$-field of subsets of $X$.
   (b) Let $\mathcal{R}$ be the collection of countable subsets of $X$. Show that $\mathcal{R}$ is a $\sigma$-ring, and determine when $\mathcal{R}$ is a $\sigma$-field.

2. Let $X$ be a countable infinite set, and let $\mathcal{F}$ be the collection of its finite subsets and their complements. For $A \in \mathcal{F}$, let $\mu(A) = 0$ if $A$ is finite and $\mu(A) = 1$ if the complement of $A$ is finite.
   (a) Show that $\mathcal{F}$ is a field.
   (b) Show that $\mu$ is finitely additive. Is it countably additive?

3. Let $(X, \mathcal{F}, \mu)$ be a probability space (with $\mathcal{F}$ a $\sigma$-field) and let $B_1, B_2, \ldots$ be a countable collection of sets from $\mathcal{F}$ such that $\sum_{n=1}^{\infty} \mu(B_n) < \infty$. Show that $\mu(\limsup B_n) = 0$.

4. Let $X$ be a set and $\mathcal{F}$ a $\sigma$-field of subsets of $X$.
   (a) Define the concept of measurable function $f : X \to [-\infty, +\infty]$.
   (b) For $X = \mathbb{R}$ and $\mathcal{F}$ the Lebesgue measurable sets, show that the function $f : \mathbb{R} \to [-\infty, +\infty]$ given by
      \[
      f(x) = \begin{cases} 
      -x & \text{if } x < 0, \\
      +\infty & \text{if } x \geq 0,
      \end{cases}
      \]

5. Let $(X, \mathcal{F}, \mu)$ be a probability space (with $\mathcal{F}$ a $\sigma$-field).
   (a) Define the concepts: Two sets $A$ and $B$ from $\mathcal{F}$ are independent; a countable collection $A_n, n = 1, 2, \ldots$, of sets from $\mathcal{F}$ is independent.
   (b) Let $X$ be the unit interval, $\mathcal{F}$ the Lebesgue measurable sets and $\mu$ the Lebesgue measure. Let $B_n$ be the subset of the unit interval corresponding to the event "HTH at the $n$th, $n+1$st and $n+2$st trial" in a Bernoulli sequence. Show that the sets $B_1, B_4, B_7, B_{10}, \ldots$ are independent.

6. Let $X$ be a set, $\mathcal{R}$ a $\sigma$-field of subsets of $X$, and $\mu_1$ and $\mu_2$ measures on $\mathcal{R}$. Assume that $\mu_1(X) = \mu_2(X) < \infty$. Let $\mathcal{L}$ be the family of those subsets $A \in \mathcal{R}$ for which $\mu_1(A) = \mu_2(A)$. Show that $\mathcal{L}$ has the following properties:
   (a) If $A, B \in \mathcal{L}$ and $B \subset A$, then $A \setminus B \in \mathcal{L}$.
   (b) If $A_n \in \mathcal{L}$, $n = 1, 2, \ldots$, are mutually disjoint, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{L}$.

7. Let $(X, \mathcal{F}, \mu)$ be a probability space (with $\mathcal{F}$ a $\sigma$-field), and let $\{B_n\}_{n=1}^{\infty}$ be a countable collection of subsets from $\mathcal{F}$.
   (a) Prove that $\mu(\limsup B_n) \geq \limsup \mu(B_n)$.
   (b) Suppose that $X = [0, 1]$, $\mathcal{F}$ is the field of Lebesgue measurable subsets of $[0, 1]$, and $\mu$ is Lebesgue measure. Show that if there exists a $\delta > 0$ such that $\mu(B_n) \geq \delta$ for every $n$, then there is at least one point which belongs to infinitely many $B_n$’s.