Problem 1. Let \((X, \mathcal{F}, \mu)\) be a probability space. Let \(A_1, A_2, \ldots\) be a sequence of subsets of \(X\) belonging to \(\mathcal{F}\).

(a) Show that if \(A_1 \supset A_2 \supset A_3 \supset \cdots\), then

\[
\mu\left(\bigcap_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n).
\]

(b) Show that if \(A_1 \subset A_2 \subset A_3 \subset \cdots\), then

\[
\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to \infty} \mu(A_n).
\]

Solution (a) The sets \(B_1 = A_1 \setminus A_2, B_2 = A_2 \setminus A_3, \ldots\) are disjoint, belong to \(\mathcal{F}\), and

\[
\bigcup_{n=1}^{\infty} B_n = A_1 \setminus \bigcap_{n=1}^{\infty} A_n.
\]

To see this identity, let \(x \in \bigcup_{n=1}^{\infty} B_n\). Then \(x \in B_n\) for at least one index \(n\). Thus \(x \in A_n \subset A_1\) and \(x \notin A_{n+1} \supset \bigcap_{n=1}^{\infty} A_n\). The reverse containment: if \(x \in A_1 \setminus \bigcap_{n=1}^{\infty} A_n\), then \(x \notin \bigcup_{n=1}^{\infty} A_n\), and thus there is a smallest index \(n \geq 2\) (because \(x \in A_1\)) such that \(x \notin A_n\). So \(x \in A_{n-1} \setminus A_n = B_{n-1}\). Thus it follows that

\[
\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \mu(A_1) - \lim_{n \to \infty} \mu(A_n).
\]

On the other hand

\[
\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n) \quad \text{(because the } B_n \text{ are disjoint and } \mu \text{ countably additive)}
\]

\[
= \sum_{n=1}^{\infty} (\mu(A_n) - \mu(A_{n+1})) \quad \text{(because } B_n = A_n \setminus A_{n+1} \text{ and } \mu \text{ is additive)}
\]

\[
= \mu(A_1) - \lim_{n \to \infty} \mu(A_n) \quad \text{(by calculus of series),}
\]

which together with Equation (1) and the fact that \(\mu\) is a probability measure (why?) gives (a).

(b) Let \(B_n = X \setminus A_n\). Then \(B_1 \supset B_2 \supset \cdots\), as in (a). Thus

\[
\mu\left(\bigcap_{n=1}^{\infty} B_n\right) = \lim_{n \to \infty} \mu(B_n).
\]

But \(\bigcap_{n=1}^{\infty} B_n = X \setminus \bigcup_{n=1}^{\infty} A_n\) and \(\mu(B_n) = \mu(X \setminus A_n) = \mu(X) - \mu(A_n) = 1 - \mu(A_n)\), and (b) obtains.

Problem 2. Let \(X\) be a probability space and \(A_n\) measurable sets. Show that the probability of \(\liminf_n A_n^c\) is 0 if and only if the probability of \(\limsup_n A_n\) is 1.
Solution By definition
\[
\liminf_{n} A_n = \bigcup_{k=1}^{\infty} \bigcap_{n>k} A_n
\]
and
\[
\limsup_{n} A_n = \bigcap_{k=1}^{\infty} \bigcup_{n>k} A_n
\]
Thus taking complements
\[
(\limsup_{n} A_n)^c = X \setminus \left( \bigcap_{k=1}^{\infty} \bigcup_{n>k} A_n \right) = \bigcup_{k=1}^{\infty} (X \setminus A_n) = \liminf_{n} A_n^c
\]
and since \(m(X) = 1\) it follows that
\[
1 - \mu(\limsup_{n} A_n) = \liminf_{n} A_n^c.
\]

Problem 3. Let \((X, \mathcal{F}, \mu)\) be a probability space, and let \(A_1, A_2, \ldots\) be in \(\mathcal{F}\). Show that
\[
\mu(\liminf_{n} A_n) \leq \liminf_{n} \mu(A_n) \leq \limsup_{n} \mu(A_n) \leq \mu(\limsup_{n} A_n).
\]
Solution If \(x_n\) is a sequence of real numbers then
\[
\limsup_{n} x_n = \lim \left( \sup_{k \geq n} x_n \right)
\]
and
\[
\liminf_{n} x_n = \lim \left( \inf_{k \geq n} x_n \right)
\]
so \(\limsup_{n} x_n \geq \liminf_{n} x_n\) and the middle inequality follows.

To prove the first inequality, let \(B_k = \bigcap_{n \geq k} A_n.\) Then \(B_1 \subset B_2 \subset \cdots\), and
\[
\bigcup_{k=1}^{\infty} B_k = \liminf_{n} A_n.
\]
Since \(X\) is a probability space, Problem 1 applies:
\[
\mu \left( \bigcup_{k=1}^{\infty} B_k \right) = \lim_{k} \mu(B_k)
\]
Moreover, \(B_k \subset A_n\) for every \(n > k\), so \(\mu(B_k) \leq \liminf_{n \geq k} \mu(A_n)\). Hence, by definition of \(\liminf\), it obtains \(\liminf_{k} \mu(B_k) \leq \liminf_{n} \mu(A_n)\).

The third inequality is proved similarly.

Problem 4. Let \((X, \mathcal{F}, \mu)\) be a probability space. Show that if \(A_1, A_2, \ldots, A_n\) are independent sets from \(\mathcal{F}\), then the sets \(A_1^c, A_2^c, \ldots, A_n^c\) are also independent.
Solution It suffices to show that for any sequence of integers \(2 \leq i_1 < i_2 \cdots < i_k \leq n\), we have
\[
\mu(A_{i_1}^c \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \mu(A_{i_1}^c) \mu(A_{i_2}) \cdots \mu(A_{i_k})
\]
Now, since \(A_1 \cup A_2^c = X\),
\[
A_{i_1} \cap \cdots \cap A_{i_k} = (A_{i_1}^c \cup A_1) \cap A_{i_1} \cap \cdots \cap A_{i_k} = (A_{i_1}^c \cap A_{i_1} \cap \cdots \cap A_{i_k}) \bigcup (A_1 \cap A_{i_1} \cap \cdots \cap A_{i_k})
\]
is a disjoint union, so that taking measures
\[
\mu(A_{i_1} \cap \cdots \cap A_{i_k}) = \mu(A_{i_1}^c \cap A_{i_1} \cap \cdots \cap A_{i_k}) + \mu(A_1 \cap A_{i_1} \cap \cdots \cap A_{i_k})
\]
Since the sets \(A_k\) are independent, this can be written
\[
\mu(A_{i_1}) \cdots \mu(A_{i_k}) = \mu(A_{i_1}^c \cap A_{i_1} \cap \cdots \cap A_{i_k}) + \mu(A_1) \mu(A_{i_1}) \cdots \mu(A_{i_k})
\]
and so
\[
\mu(A_1^c \cap A_{i_1} \cap \cdots \cap A_{i_k})
= (\mu(A_{i_1}) \cdots \mu(A_{i_k})) - (\mu(A_1) \mu(A_{i_1}) \cdots \mu(A_{i_k}))
= (1 - \mu(A_1)) \mu(A_{i_1}) \cdots \mu(A_{i_k})
= \mu(A_1^c) \mu(A_{i_1}) \cdots \mu(A_{i_k})
\]