1. Let $X$ be an infinite set and let $\mathcal{R}$ be the following collection of subsets: $A \in \mathcal{R}$ if and only if $A$ is finite or $A^c$ is finite. Let $\mu$ be the following function on $\mathcal{R}$: $\mu(A) = 1$ if $A$ is finite, and $\mu(A) = 1$ if $A^c$ is finite. Is $\mu$ a measure?

Solution.
You should check that $\mathcal{R}$ is a ring.

There are two cases to consider: (1) $X$ is countable, and (2) $X$ is uncountable.

Case (1) If $X$ is countable, then $\mu$ is not a measure. Indeed, write $X = \{x_0, x_1, x_2, \cdots \}$. Then the sets $A_i = \{x_i\}$, $i = 1, 2, \cdots$ belong to $\mathcal{R}$, are disjoint, and $\mu(A_i) = 0$. Furthermore, $A = \bigcup_{i=1}^{\infty} A_i = \{x_1, \cdots \}$ is such that $A^c = \{x_0\}$ is finite, so also $A \in \mathcal{R}$. But $\mu(A) = 1$. Thus

$$\mu(A) = 1 \neq \sum_{i=1}^{\infty} \mu(A_i) = 0.$$ 

Case (2) If $X$ is uncountable, then $\mu$ is a measure. The key observation is that if $A, B \in \mathcal{R}$, and both are infinite, then they must intersect.

Let $\{A_i\}_{i=1}^{\infty}$ be a disjoint collection of elements of $\mathcal{R}$ such that $A = \bigcup_{i=1}^{\infty} A_i \in \mathcal{R}$.

There are now two cases to consider: (2a) $A$ finite, and (2b) $A^c$ finite.

(2a) If $A$ is finite, then all the $A_i$ are also finite because $A_i \subset A$. Thus $\mu(A) = 0 = \mu(A_i)$.

(2b) If $A^c$ is finite then $A$ is infinite. In this situation, not all the $A_i$ can be finite, for if they were, then $A$ would be a countable union of finite sets, hence countable. But then $X = A \cup A^c$ would be the union of a countable set and a finite set, hence also countable.

Hence at least one of the $A_i$ is infinite. But in fact exactly one of them is infinite because of the key observation above. Thus in this case also

$$\mu(A) = 1 = \sum_{i=1}^{\infty} \mu(A_i) = 1 + 0 + 0 + 0 \cdots = 1.$$