Exercise 3.1.1. Let $I$ be the unit interval $0 \leq x \leq 1$, and let $I_{k,n}$ be the subinterval 
\[
\frac{k}{n} \leq x \leq \frac{k+1}{n} \quad 0 \leq k \leq n.
\]
Let $f_1$ be the characteristic function of $I_{0,1}$, $f_2$ and $f_3$ the characteristic functions of $I_{0,2}$ and $I_{1,2}$, and so on. Show that the sequence $f_n$ converges to 0 in $L^1(I)$ but it does not converge pointwise anywhere.

Solution. A figure will help you to understand what is going on.

Given an integer $n = 1, 2, \cdots$ there is a unique integer $m = 1, 2, \cdots$ such that 
\[
\frac{m(m-1)}{2} + 1 < n < \frac{m(m+1)}{2} + 1
\]
and so the interval $I_{k,n}$, $(0 \leq k < n)$ has length $1/m$. Thus the integral 
\[
\int_I |f_n| \cdot \mu_L = \frac{1}{m} \leq \frac{2}{\sqrt{n}}
\]
which converges to 0 as $n \to \infty$.

For $x \in I$, the sequence of values $f_n(x)$ contains infinitely many 0’s and infinitely many 1’s and thus it cannot converge. □

Exercise 3.1.3. In Exercise 3.1.1 above, extract a subsequence of $f_n$ which converges to 0 almost everywhere.

Solution. Take $g_n = f_{(n(n-1)/2)+1}$.

□

Exercise 3.1.2. Let $f_n$ be the function on $(0, 1]$ that is equal to 0 in $[1/n, 1]$ and equal to $n$ in $(0, 1/n)$. Show that $f_n$ converges pointwise to 0 everywhere in $(0, 1]$ as $n \to \infty$, but it does not converge in $L^1$.

Solution. The function $f_n$ can be expressed as 
\[
f_n = n\chi_{(0,1/n)}.
\]
Therefore, given $x \in (0, 1]$, if $n > 1/x$, then $f_n(x) = 0$. It follows that 
\[
\lim_{n \to \infty} f_n(x) = 0 \text{ for all } x \in (0, 1].
\]
To show that $f_n$ does not converge in $L^1$ we will show that it is not a Cauchy sequence there. Let $n, m$ be integers with $n < m$. Then the difference 
\[
f_m - f_n = (m - n)\chi_{(0,1/m)} - n\chi_{(1/m, 1/n)}
\]

Therefore, given $x \in (0, 1]$, if $n > 1/x$, then $f_n(x) = 0$. It follows that 
\[
\lim_{n \to \infty} f_n(x) = 0 \text{ for all } x \in (0, 1].
\]
To show that $f_n$ does not converge in $L^1$ we will show that it is not a Cauchy sequence there. Let $n, m$ be integers with $n < m$. Then the difference 
\[
f_m - f_n = (m - n)\chi_{(0,1/m)} - n\chi_{(1/m, 1/n)}
\]
and thus
\[ |f_m - f_n| = (m - n)\chi_{(0,1/m)} + n\chi_{[1/m,1/n)}. \]
Therefore, the \( L^1 \)-norm
\[
\|f_m - f_n\|_1 = \int_{(0,1]} |f_m - f_n| \cdot \mu_L
\]
\[
= 2\frac{m - n}{m}
\]
which does not converge to 0 as \( n \to \infty \) and \( m \to \infty \) (take \( m = 2n \to \infty \)). □

Exercise 3.2.2. Let \( V \) be an inner product space. Show that
\[
\|v + w\|^2 + \|v - w\|^2 = 2\|v\|^2 + 2\|w\|^2.
\]

Solution. This is a calculation. Compute each term on the left side:
\[
\|v + w\|^2 + = \langle v + w, v + w \rangle
\]
\[
= \langle v, v \rangle + \langle w, w \rangle + \langle v, w \rangle + \langle w, v \rangle
\]
\[
= \|v\|^2 + \|w\|^2 + \langle v, w \rangle + \langle w, v \rangle
\]
and
\[
\|v - w\|^2 + = \langle v - w, v - w \rangle
\]
\[
= \langle v, v \rangle + \langle w, w \rangle - \langle v, w \rangle - \langle w, v \rangle
\]
\[
= \|v\|^2 + \|w\|^2 - \langle v, w \rangle - \langle w, v \rangle
\]
and add them up. □

Exercise 3.2.4. Let \( X = (0,1] \) equipped with Lebesgue measure \( \mu \). Show that the function \( f(x) = x^{-3/4} \) is in \( \text{calL}^1(X, \mu) \) but not in \( L^2(X, \mu) \).

Solution. Use Exercise 2.4.1. The functions \( f \) and \( f^2 \) are nonnegative and measurable on \( J = (0,1] \), and Riemann integrable on \( [a,1] \) for every \( 0 < a < 1 \). The Riemann integrals are
\[
\int_a^1 f(x) \cdot dx = \int_a^1 x^{-3/4} \cdot dx = 4 - 4a^{1/4}
\]
and
\[
\int_a^1 f(x) \cdot dx = \int_a^1 x^{-3/4} \cdot dx = -2 + \frac{2}{a^{1/2}}
\]
Therefore, by Exercise 2.4.1,
\[
\int f \cdot \mu = \lim_{a \to 0} \left( 4 - 4a^{1/4} \right) = 4 < \infty
\]
and
\[
\int f \cdot \mu = \lim_{a \to 0} \left( -2 + \frac{2}{a^{1/2}} \right) = \infty.
\]