Math 623. Homework 1. Solutions

Problem 1. Let \( z_1, z_2 \) and \( z_3 \) be three distinct complex numbers, with \( |z_1| = |z_2| = |z_3| \). Prove that the following properties are equivalent:

(a) The points are equidistant.

(b) \( z_1 + z_2 + z_3 = 0 \).

(c) There is a number \( a \) such that \( z_1, z_2, z_3 \) are the roots of the equation \( z^3 = a \).

Proof. Let \( z, w \) be two complex numbers such that \( |z| = |w| = 1 \). If the points \( 1, z, w \) are equidistant, then
\[
|1 - z| = |1 - w| = |z - w|,
\]
and so \( z + \overline{z} = w + \overline{w} = wz + \overline{wz} \). Because \( \overline{z} = 1/z \) and \( \overline{w} = 1/w \), it follows that
\[
z + \frac{1}{z} = \frac{w}{z} + \frac{z}{w},
\]
or that \( w(z^2 + 1) = w^2 + z^2 \). Therefore \( z^2 = w \), and similarly \( w^2 = z \). Thus \( z^3 = wz = w^3 \).

Thus \( z^4 = w^2 = z \), so \( z^3 = w^3 = 1 \). From the above \( 1 + z + w = 1 + z^2 + z^2 = \frac{z^3 - 1}{z - 1} = 0 \).

Now we solve the problem. Given \( z_i, i = 1, 2, 3 \), with \( |z_i| = r \), let \( z = z_2/z_1 \) and \( w = z_3/z_1 \).

(a) \( \Rightarrow \) (b) If the points \( z_i \) are equidistant, the the points \( 1, z, w \) are also equidistant. If follows from the calculations above that \( 1 + z + w = 0 \), hence that \( z_1 + z_2 + z_3 = 0 \).

(b) \( \Rightarrow \) (c) If \( z_1 + z_2 + z_3 = 0 \), then \( 1 + z + w = 0 \). Multiply by \( z \) to get \( z + z^2 + zw = 0 \). Take the conjugate to get \( 1 + \frac{1}{z} + \frac{1}{w} = 0 \), so \( wz + w + z = 0 \). It follows that \( z^2 = w \). Thus \( 0 = 1 + z + w = 1 + z + z^2 = \frac{1 - z^3}{1 - z} \),
which implies \( z^3 = 1 \). Similarly \( w^3 = 1 \). It follows that \( z_1^3 = z_2^3 = z_3^3 = a \) with \( a = z_1^3 \).

(c) \( \Rightarrow \) (a) The three roots of \( z^3 = a \) are the vertices of an equilateral triangle. Algebraically, if \( z_i, i = 1, 2, 3 \) are the three roots of \( z^3 = a \), then the numbers \( 1, z, w \) are the three roots of unity. Then \( \overline{z} = w \), \( \overline{w} = z \), \( z^2 = w \) and \( w^2 = z \). It follows that \( |z - w| = |z||1 - w/z| = |1 - z| \) and similarly \( |z - w| = |1 - w| \). This implies that \( |z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1| \).

Problem 2. Let \( s_n \) denote the side length of \( P_n \), the regular \( n \)-sided polygon inscribed in the unit circle.

(a) Prove that
\[
s_{2n} = \sqrt{2 - \sqrt{4 - s_n^2}},
\]
Deduce from this that
\[
s_4 = \sqrt{2}, \quad s_8 = \sqrt{2 - \sqrt{2}}, \quad \cdots \quad s_{2^n + 1} = \sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}}},
\]
(in the last expression there are \( n \) nested square roots).

(b) Prove that
\[
\lim_{n \to \infty} 2^n \sqrt{2 - \sqrt{2 + \sqrt{2 + \cdots + \sqrt{2}}} = \pi}
\]
(in the limit there are \( n \) nested square roots).

Proof. (a) follows easily by induction using elementary trigonometry.

(b) A regular \( m \)-gon \( P_m \) inscribed in the unit circle circumscribes a disk of radius \( \sin(\pi/m) \). Therefore, the area of \( P_m \) satisfies \( \pi \sin^2(\pi/m) < \text{Area}(P_m) < \pi \), hence \( \lim_{m \to \infty} \text{Area}(P_m) = \pi \). Now use (a) with \( m = 2^n \). \( \square \)
Problem 3. (a) Prove that a line in \( \mathbb{C} \) satisfies an equation of the form 
\[
\alpha z + \bar{\alpha} \bar{z} + b = 0
\]
(a complex and \( b \) real), and conversely.

(b) Prove that a line in \( \mathbb{C} \) can be described as the set of points in \( \mathbb{C} \) equidistant from two fixed points, and use this fact to show that isometries take lines to lines.

(c) Prove that if \( \ell \) is the line equidistant from the points \( a \) and \( b \), then reflection on \( \ell \) takes \( a \) to \( b \).

Proof. (a) In Cartesian coordinates \((x, y)\), a line is given by a linear equation of the form 
\[
2Ax + 2By + C = 0
\]
for some real numbers \( A, B, C \). Use that \( 2x = z + \bar{z} \) and \( 2iy = z - \bar{z} \) to write that equation in the form 
\[
A(z + \bar{z}) - iB(z - \bar{z}) + C = 0.
\]
This simplifies to \((A - iB)z + (A + iB)\bar{z} + c = 0\), as desired.

Problem 4. (a) Prove that the map \( z \mapsto e^{i\alpha}z + v \) is a translation if \( e^{i\alpha} = 1 \) and otherwise a rotation about \( v/(1 - e^{i\alpha}) \).

(b) Prove that \( z \mapsto e^{i\alpha} \bar{z} + v \) is a glide reflection. Find its axis and the length of translation.

Proof. (a) The map \( f(z) = e^{i\alpha}z + v \) is an isometry because 
\[
|f(z) - f(w)| = |e^{i\alpha}z - e^{i\alpha}w| = |z - w|.
\]
If \( e^{i\alpha} = 1 \), then \( f \) is a translation. If \( e^{i\alpha} \neq 1 \), then \( f(1/1 - e^{i\alpha}) = 1/1 - e^{i\alpha} \). Thus \( f \) has exactly one fixed point, and so it must be a rotation.

(b) The map \( g(z) = e^{i\alpha} \bar{z} + v \) is the composite \( g = f \circ r \), where \( r(z) = \bar{z} \). Because of (a) and the classification of isometries, \( f \) is the product of two reflections. Thus \( g \) is the product of three reflections, and so it must be a glide reflection. It could be a trivial glide reflection, that is, just a reflection.

The map \( g^2 \) is a translation (it could be the identity if \( g \) was just a reflection) whose direction is that of the axis of \( g \) and whose length is twice the length of \( g \). Since \( g^2(z) = z + e^{i\alpha} \bar{v} + v \), the length of \( g \) is \(|e^{i\alpha} \bar{v} + v|/2\). The axis is the line that passes through \( v/2 \) and has direction vector \( e^{i\alpha} \bar{v} + v \) (it passes through \( v/2 \) because the axis bisects the line segment joining \( 0 \) and \( g(0) \)).

Problem 5. (a) Prove that two rotations by the same angle \( \rho_1 \) and \( \rho_2 \) are conjugate by an isometry, that is, there is an isometry \( f \) such that \( \rho_2 = f \circ \rho_1 \circ f^{-1} \).

(b) When are two translations conjugate?

Proof. (a) If \( P \) and \( Q \) are the centers of rotation of \( \rho_1 \) and \( \rho_2 \), let \( \tau \) be the translation by \( P - Q \).

(b) Let \( \tau_1(z) = z + u_1 \) and \( \tau_2(z) = z + u_2 \). They are conjugate by \( f(z) = e^{i\alpha}z + v \), that is, \( \tau_2 = f \tau_1 f^{-1} \), if and only if \( u_2 = e^{i\alpha}u_1 \). They are conjugate by \( g(z) = e^{i\alpha} \bar{z} + v \) if and only if \( u_1 = u_2 \).