Math 550. 2nd Midterm. (Some) Solutions.

Problem 1. Let $F : S^1 \to S^1$ be a mapping of the unit circle $S^1$ into itself.

(a) Prove that if $F(P) = F(-P)$ for all $P$, then the degree of $F$ is even.

(b) If $F(P) \neq F(-P)$ for all $P$, then prove that the degree of $F$ is not zero, and that $F$ is surjective.

Solution. This is similar to Problem 5, Homework 6. \hfill $\square$

Problem 2. Let $X$ be a topological space.

(a) Define the concepts: (i) $X$ has the fixed point property, and (ii) $Y \subset X$ is a retract of $X$.

(b) Prove that if $X$ has the fixed point property and if $Y \subset X$ is a retract of $X$, then $Y$ also has the fixed point property.

Solution. A subspace $Y \subset X$ is a retract of $X$ if there is a continuous map $r : X \to Y$ such that $r(P) = P$ for all $P$ in $Y$.

Suppose that $X$ has the fixed point property, that $Y \subset X$ is a retract of $X$ with retraction map $r : X \to Y$ as above, and that $f : Y \to Y$ is continuous. If $i : Y \to X$ the inclusion mapping (given by $i(P) = P$ for all $P$ in $Y$), then the composite $i \circ f \circ r : X \to X$ is continuous and thus it has a fixed point $P$ in $X$.

We claim that this point $P$ is in $Y$ and that $f(P) = P$. Indeed, the point $f(r(P))$ is in $Y$ because $r(P)$ is in $Y$ and $f$ maps $Y$ into $Y$. It follows that $P$ is in $Y$ because $P = i(f(r(P))) = f(r(P))$. If $P$ is in $Y$, then $r(P) = P$, and so $P = f(P)$. \hfill $\square$

Problem 3. Let $U$ and $V$ be two open subsets of the plane.

(a) Prove that if $U \cap V$ is connected and if $H^1(U) = 0$ and $H^1(V) = 0$, then $H^1(U \cup V) = 0$.

(b) Prove that if $U$ and $V$ are connected, and $H^1(U \cup V) = 0$, then $U \cap V$ is connected.

Solution. (a) We have two show that every closed one form $\omega$ on $U \cup V$ is exact. Since $H^1(U) = H^1(V) = 0$, the restrictions of $\omega$ to $U$ and to $V$ are both exact. Since $U \cap V$ is connected, Lemma 1.14 in the textbook implies that $\omega$ is also exact on $U \cup V$.

(b) To prove that $U \cap V$ is connected is equivalent to proving that the vector space $H^0(U \cap V)$ has dimension 1. Consider the coboundary map

$$\delta : H^0(U \cap V) \to H^1(U \cup V).$$

The kernel of $\delta$ equals $H^0(U \cap V)$ because $H^1(U \cup V) = 0$ (by hypothesis). Proposition 5.7 in the textbook says that a locally constant function $f$ on $U \cap V$ is in the kernel of $\delta$ if and only if there are locally constant functions $f_U$ on $U$ and $f_V$ on $V$ such that $f = f_U - f_V$ on $U \cap V$. But $f_U$ and $f_V$ must be both constant because, by hypothesis, $U$ and $V$ are both connected. Therefore $f = f_U - f_V$ is the difference of two constant functions and is therefore constant. That is, the kernel of $\delta$ consists of the constant functions on $U \cap V$, and thus it has dimension 1. \hfill $\square$

Problem 4. Let $U$ be an open subset of the plane.
(a) Define the concepts: (i) 1-chain on \( U \), (ii) 1-cycle on \( U \), and (iii) 1-boundary on \( U \).

(b) Prove that if \( U \) is convex, then every 1-cycle on \( U \) is a 1-boundary.

**Problem 5.** Let \( X \) be a subset of the plane homeomorphic to the figure 8.

(a) Prove that \( \mathbb{R}^2 \setminus X \) has three connected components.

(b) Prove that two of those components are bounded and the other is unbounded.

**Solution.** The set \( X \) is the union \( X = A \cup B \), where \( A \) and \( B \) are closed subsets homeomorphic to a circle with \( A \cap B = \{ P \} \). Then \((\mathbb{R}^2 \setminus A) \cap (\mathbb{R}^2 \setminus B) = \mathbb{R}^2 \setminus X \) and \((\mathbb{R}^2 \setminus A) \cup (\mathbb{R}^2 \setminus B) = \mathbb{R}^2 \setminus \{ P \} \). Consider the coboundary map

\[
\delta : H^0((\mathbb{R}^2 \setminus A) \cap (\mathbb{R}^2 \setminus B)) = H^0(\mathbb{R}^2 \setminus X) \rightarrow H^1((\mathbb{R}^2 \setminus A) \cup (\mathbb{R}^2 \setminus B)) = H^1(\mathbb{R}^2 \setminus \{ P \}).
\]

A class \([\omega]\) in \( H^1(\mathbb{R}^2 \setminus \{ P \}) \) can be represented as \([\omega] = [\lambda \omega_P]\), for some scalar \( \lambda \), because the vector space \( H^1(\mathbb{R}^2 \setminus \{ P \}) \) has dimension 1 with basis \([\omega_P]\). The image of \( \delta \) consists of the classes \([\omega]\) of closed 1-forms on \( \omega \) on \( \mathbb{R}^2 \setminus \{ P \} \) such that \( \omega \) is exact on \( \mathbb{R}^2 \setminus A \) and on \( \mathbb{R}^2 \setminus B \). Since \( A \) is a bounded, connected closed set and \( P \) is a point in \( A \), the class \([\omega_P]\) is not zero in \( H^1(\mathbb{R}^2 \setminus A) \) (cf. Problem 5, Homework 7). Therefore, the 1-form \( \omega_P \) is not exact on \( \mathbb{R}^2 \setminus A \). Similarly, the 1-form \( \omega_P \) is not exact on \( \mathbb{R}^2 \setminus B \). Therefore, \([\omega] = [\lambda \omega_P]\) is in the image of \( \delta \) if and only if \( \lambda = 0 \). That is, the image of \( \delta \) is the trivial subspace of \( H^1(\mathbb{R}^2 \setminus \{ P \}) \), and so the kernel of \( \delta \) equals the vector space \( H^0(\mathbb{R}^2 \setminus X) \).

There are several methods of showing that the kernel of \( \delta \) has dimension 3. I will explain two such methods. One is direct and the other is more algebraic. (You can probably simplify the explanation below, but I preferred to spell out all the details.)

**Method 1.** Because of the Jordan curve theorem we know that the open set \( \mathbb{R}^2 \setminus A \) has two connected components, one bounded, say \( U_0 \), and the other unbounded, say \( U_\infty \), and \( A \) is the common boundary of both. Similarly, let \( V_0 \) and \( V_\infty \) be the bounded and unbounded components, respectively, of \( \mathbb{R}^2 \setminus B \). Then

\[
\mathbb{R}^2 \setminus X = (\mathbb{R}^2 \setminus A) \cap (\mathbb{R}^2 \setminus B) = (U_0 \cup U_\infty) \cap (V_0 \cup V_\infty) = (U_0 \cap V_0) \cup (U_0 \cap V_\infty) \cup (U_\infty \cap V_0) \cup (U_\infty \cap V_\infty),
\]
a disjoint union of four open sets.

If \( f_A \) is a locally constant function on \( \mathbb{R}^2 \setminus A \), then we can represent \( f \) as a pair of numbers \( f_A = (u_0, u_\infty) \), where \( u_0 \) is the (constant) value of \( f_A \) on the component \( U_0 \) and \( u_\infty \) is the value of \( f_A \) on \( U_\infty \). A locally constant function \( f_B \) on \( \mathbb{R}^2 \setminus B \) is similarly represented by a pair of numbers \( (v_0, v_\infty) \). If \( P \) is in \( \mathbb{R}^2 \setminus X \), then \( P \) is exactly in one of the 4 sets \((U_0 \cap V_0), (U_0 \cap V_\infty), (U_\infty \cap V_0), \) or \((U_\infty \cap V_\infty)\), and therefore

\[
f(P) = f_A(P) - f_B(P) = \begin{cases} 
    u_0 - v_0, & \text{if } P \text{ is in } U_0 \cap V_0; \\
    u_0 - v_\infty, & \text{if } P \text{ is in } U_0 \cap V_\infty; \\
    u_\infty - v_0, & \text{if } P \text{ is in } U_\infty \cap V_0; \\
    u_\infty - v_\infty, & \text{if } P \text{ is in } U_\infty \cap V_\infty.
\end{cases}
\]

Thus it seems that we need 4 numbers, \( u_0, u_\infty, v_0, v_\infty \), to determine the locally constant function \( f = f_A - f_B \) on \((\mathbb{R}^2 \setminus A) \cap (\mathbb{R}^2 \setminus B) \). However, one of these numbers is redundant. Indeed, I claim that \( f \) can also be represented as \( f = g_A - g_B \), where \( g_A \) and \( g_B \) are locally constant on \( \mathbb{R}^2 \setminus A \) and \( \mathbb{R}^2 \setminus B \), respectively, and \( g_A = (x, y) \), \( g_B = (0, z) \), which will show that only three variables are required to determine \( f \). To see this, suppose that \( f \) is given as \( f = f_A - f_B \) with \( f_A = (u_0, u_\infty) \) and \( f_B = (v_0, v_\infty) \). Let \( x = u_0 - v_0, \ y = u_\infty - v_0 \) and \( z = u_\infty - v_\infty \), let \( g_A \) be the locally constant function on \( \mathbb{R}^2 \setminus A \) given by the pair of numbers \( g_A = (x, y) \) (again, this means that \( g_A(P) = x = u_0 - v_0 \) if \( P \)
is in $U_0$ and $g_A(P) = y = u_\infty - v_0$ if $P$ is in $U_\infty$) and let $g_B$ be given by $g_B = (0, z)$ (and this means that $g_B(P) = 0$ if $P$ is in $V_0$ and $g_B(P) = z = u_\infty - v_\infty$). If $P$ is in $\mathbb{R}^2 \setminus X$, then we compute the values $g_A(P) - g_B(P)$:

$$g_A(P) - g_B(P) = \begin{cases} 
  x - 0 = u_0 - v_0, & \text{if } P \text{ is in } U_0 \cap V_0; \\
  x - z = u_0 - u_\infty, & \text{if } P \text{ is in } U_0 \cap V_\infty; \\
  y - 0 = u_\infty - v_0, & \text{if } P \text{ is in } U_\infty \cap V_0; \\
  y - z = (u_\infty - v_0) - (v_\infty - v_0) = u_\infty - v_\infty, & \text{if } P \text{ is in } U_\infty \cap V_\infty.
\end{cases}$$

By comparing this expression with the expression for $f$ previously displayed we see that $f = g_A - g_B$.

You cannot do with less than 3 numbers because $\mathbb{R}^2 \setminus X$ has at least three components. This is because $\mathbb{R}^2 \setminus X$ is the disjoint union of the four sets $U_0 \cap V_0$, $U_\infty \cap V_0$, $U_0 \cap V_\infty$ and $U_\infty \cap V_\infty$, and at most one of these sets can be empty. Indeed, $U_\infty \cap V_\infty$ is never empty because, as $X$ is compact, there is a disk $D$ that contains $X$ and therefore $U_\infty \cap V_\infty$ contains the complement of $D$. If one of the other three intersections is empty, then the other two are not. For example, if $U_0 \cap V_0 = \emptyset$, then $U_0$ must be contained in $V_\infty$ and $V_0$ must be contained in $U_\infty$. If $U_0 \cap V_\infty = \emptyset$, then $U_0$ must be contained in $V_0$, and properly so because $A \cap B = \{P\}$; thus $V_0 \cap U_\infty \neq \emptyset$.

**Method 2.** The kernel of $\delta$ consists of the locally constants functions $f$ on $\mathbb{R}^2 \setminus X$ such that $f = f_A - f_B$, where $f_A$ is locally constant on $\mathbb{R}^2 \setminus A$ and $f_B$ is locally constant on $\mathbb{R}^2 \setminus B$. Consider the map

$$\varphi : H^0(\mathbb{R}^2 \setminus A) \times H^0(\mathbb{R}^2 \setminus B) \to H^0((\mathbb{R}^2 \setminus A) \cap (\mathbb{R}^2 \setminus B)) = H^0(\mathbb{R}^2 \setminus X)$$

defined by $\varphi(f, g) = f - g$, for $f$ locally constant on $\mathbb{R}^2 \setminus A$ and $g$ locally constant on $\mathbb{R}^2 \setminus B$. It is apparent that $\varphi$ is linear at that its image is precisely the kernel of $\delta$ (which is $H^0(\mathbb{R}^2 \setminus X)$).

I claim that the kernel of $\varphi$ consists of all the pairs $(f, g)$ such that $f$ is constant on $\mathbb{R}^2 \setminus A$, $g$ is constant on $\mathbb{R}^2 \setminus B$, and $f - g = 0$ on $\mathbb{R}^2 \setminus X$. This claim implies that the kernel of $\varphi$ has dimension 1. The Jordan curve theorem implies that the vector spaces $H^0(\mathbb{R}^2 \setminus A)$ and $H^0(\mathbb{R}^2 \setminus B)$ have dimension 2 and thus the product $H^0(\mathbb{R}^2 \setminus A) \times H^0(\mathbb{R}^2 \setminus B)$ has dimension 4. Therefore, the rank-nullity theorem applied to $\varphi$ implies that $1 + \dim H^0(\mathbb{R}^2 \setminus X) = 4$, or that $H^0(\mathbb{R}^2 \setminus X)$ has dimension 3.

To prove the claim, suppose first a pair $(f, g)$ is in the kernel of $\varphi$, that is, suppose that $f$ is locally constant on $\mathbb{R}^2 \setminus A$, $g$ is locally constant on $\mathbb{R}^2 \setminus B$, and that $f - g = 0$ on $(\mathbb{R}^2 \setminus A) \cap (\mathbb{R}^2 \setminus B)$. This implies that the functions $f$ and $g$ can be glued together (because $f = g$ on $(\mathbb{R}^2 \setminus A) \cap (\mathbb{R}^2 \setminus B)$) to obtain a locally constant function on $(\mathbb{R}^2 \setminus A) \cup (\mathbb{R}^2 \setminus B)$ (cf. Problem 1, Homework 2). But $(\mathbb{R}^2 \setminus A) \cup (\mathbb{R}^2 \setminus B) \supset \mathbb{R}^2 \setminus \{P\}$ is connected, and so this locally constant function on it must be constant. This implies that if a pair $(f, g)$ is such that $\varphi(f, g) = f - g = 0$, then $f$ and $g$ are both constant and $f = g$. Conversely, if $f$ and $g$ are constant and $f = g$, then $\varphi(f, g) = 0$. \qed