
Problem 1. Suppose that $U = \mathbb{R}^2 \setminus \{P_1, \ldots, P_n\}$ is the complement of $n$ points in the plane. Prove that the mapping that takes a closed 1-chain $\gamma$ to $(W(\gamma, P_1), \ldots, W(\gamma, P_n))$ determines an isomorphism of $H_1U$ with the free abelian group $\mathbb{Z}^n$.

Solution. Consider the map

$$
\Phi : Z_1U \longrightarrow \mathbb{Z}^n
$$

given by

$$
\Phi(\gamma) = (W(\gamma, P_1), \ldots, W(\gamma, P_n)).
$$

By the very definition of $W(\gamma, P)$ (winding number of a closed 1-chain $\gamma$ around a point $P$ not in $\text{supp} \gamma$), this map is a homomorphism of groups.

Let $r > 0$ be smaller than the distance between any pair of points $P_i$, and let $\gamma_i$ be a circle or radius $r$ and center $P_i$. If $m_1, m_2, \ldots, m_n$ are integers, then the closed 1-chain $\gamma = \sum_{i=1}^n m_i \gamma_i$ satisfies

$$
\Phi(\gamma) = (m_1, \ldots, m_n),
$$

that is to say, the homomorphism $\Phi$ is onto.

To compute the kernel of $\Phi$ we use Theorem 6.11. If $\gamma$ is a boundary, then $\Phi(\gamma) = 0$. Conversely, if $\Phi(\gamma) = 0$, then $\gamma$ has the same winding number around any point not in $U$ as the trivial 1-chain, i.e., $\gamma$ is homologous to 0, or $\gamma$ is a boundary in $U$. Thus the kernel of $\Phi$ is precisely $B_1(U)$.

By the first isomorphism theorem for groups, $Z_1U/B_1U \cong \mathbb{Z}^n$. \qed

Problem 2. (i) Prove that a continuous mapping $F : X \rightarrow Y$ determines a homomorphism from $Z_1X$ to $Z_1Y$ taking $B_1X \rightarrow B_1Y$, and thus it determines a homomorphism of abelian groups $F_* : H_1X \rightarrow H_1Y$.

(ii) Prove that if $F : X \rightarrow Y$ and $G : Y \rightarrow Z$ are continuous, then $(G \circ F)_* = G_* \circ F_*$ as homomorphism from $H_1X$ to $H_1Z$. In particular, prove that if $X$ and $Y$ are homeomorphic space, then $H_1X$ and $H_1Y$ are isomorphic abelian groups.

Solution. If $n_1P_1 + \cdots + n_kP_k$ is a 0-chain in $X$, where the $P_i$’s are points in $X$, define

$$
F_0(n_1 \gamma_1 + \cdots + n_k \gamma_k) = n_1F(P_1) + \cdots + n_kF(P_k).
$$

Since $C_0(X)$ is the free abelian group on the set $X$, this defines a homomorphism of abelian groups $F_0 : C_0(X) \rightarrow C_0(Y)$.

If $\gamma = n_1 \gamma_1 + \cdots + n_k \gamma_k$, where the $\gamma_i$ are continuous paths in $X$, then define

$$
F_1(\gamma) = n_1(F \circ \gamma_1) + \cdots + n_k(F \circ \gamma_k).
$$
Since the group of 1-chains on $X$ is the free abelian group on the (classes of) non-degenerate paths in $X$, it follows that $F_1$ induces a homomorphism of abelian groups $F_1 : C_1(X) \to C_1(Y)$.

Analogously, if $\Gamma = n_1 \Gamma_1 + \cdots + n_k \Gamma_k$, where the $\Gamma_i$’s are squares in $X$, set

$$F_2(\Gamma) = n_1(F \circ \Gamma_1) + \cdots + n_k(F \circ \Gamma_k)$$

and note that, by reasons similar to the above, this induces a homomorphism of abelian groups $F_2 : C_2(X) \to C_2(Y)$.

Now we show that if $\gamma$ is a 1-chain in $X$, then $F_0(\partial \gamma) = \partial F_1(\gamma)$. By linearity of the maps $F_j$, it suffices to prove this for $\gamma : [0, 1] \to X$ a continuous path in $X$. In this case, $\partial \gamma = \gamma(1) - \gamma(0)$, and

$$F_0(\partial \gamma) = F(\gamma(0)) - F(\gamma(1)) = \partial(F_1(\gamma)).$$

It follows that if $\gamma$ is a closed 1-chain in $X$, that is, if $\partial \gamma = 0$, then $\partial(F_1(\gamma)) = F_0(\partial \gamma) = F_0(0) = 0$, so that $F_1(\gamma)$ is a closed 1-chain in $Y$. Therefore, $F_1$ takes $Z_1 X$ into $Z_1 Y$.

Next we show that if $\Gamma$ is a 2-chain in $X$, then $\partial(F_2(\Gamma)) = F_1(\partial \Gamma)$. As above, it suffices to show that this is the case for $\Gamma : [0, 1] \times [0, 1] \to X$ is a square in $X$. Recall that the boundary of such square $\Gamma$ is $\partial \Gamma = \gamma_1 + \gamma_2 - \gamma_3 - \gamma_4$, where $\gamma_i$ are the paths in $X$ determined by the restrictions of $\Gamma$ to the sides of $[0, 1] \times [0, 1]$. Then $F_1(\partial \Gamma) = F \circ \gamma_1 + F \circ \gamma_2 - F \circ \gamma_3 - F \circ \gamma_4$. The boundary of the rectangle $F_2(\Gamma) = F \circ \Gamma$ is $\sigma_1 + \sigma_2 - \sigma_3 - \sigma_4$, where the $\sigma_i$’s are the restrictions of the square $F \circ \Gamma$ to the sides of $[0, 1] \times [0, 1]$. But then it is easy to see that $\sigma_i = F \circ \gamma_i$, and the identity $\partial(F_2(\Gamma)) = F_1(\partial \Gamma)$ follows. Hence, if $\gamma$ is a 1-boundary in $X$, that is $\gamma = \partial \Gamma$, then $F_1(\gamma) = F_1(\partial \Gamma) = \partial F_2(\Gamma)$, so that $F_1(\gamma)$ is a 1-boundary in $Y$. Thus $F_1$ takes $B_1 X$ into $B_1 Y$ and so it induces a homomorphism $F_* : H_1 X \to H_1 Y$.

The fact that $(G \circ F)_* = G_* \circ F_*$ follows immediately from the associativity of composition of mappings.

Finally, if $F$ is a homomorphism $X \to Y$, then there is a homomorphism $G : Y \to X$ such that $G \circ F = \text{id}_X$ and $F \circ G = \text{id}_Y$. Then $G_* \circ F_* = \text{id}_{H_1 X}$ and $F_* \circ G_* = \text{id}_{H_1 Y}$, which implies that $F_*$ is one-one and onto.

Problem 3. Find examples of continuous mappings $F : X \to Y$ such that:

(i) $F$ is one-one, but $F_*$ is not one-one.

(ii) $F$ is surjective, but $F_*$ is not surjective.

Solution. (i) Take $X = \mathbb{R}^2 \setminus \{0\}$, $Y = \mathbb{R}^2$, and $F$ the inclusion mapping. Then $F$ is one-one, but $F_* = 0$. (ii) Take $X = Y = \mathbb{R}^2 \setminus \{0\}$ and $F(z) = z^2$. Then $F$ is surjective, but $F_* : \mathbb{Z} \to \mathbb{Z}$ is the homomorphism $F_*(n) = 2n$, which is not surjective.

Problem 4. Let $K$ be a compact subset of the plane and let $U = \mathbb{R}^2 \setminus K$. Prove that if $K$ is not connected, and $P$ and $Q$ are in different components of $K$, then the classes $[\omega_P]$ and $[\omega_Q]$ are linearly independent in $H^1 U$. In particular, prove that if $K$ has $n$ connected components, then $H^1 U$ is a vector space of dimension $n$.

Solution. The connected components of $K$ are closed subsets of $K$, and therefore are compact subsets of the plane. If $A$ and $B$ are different components of $K$, then, by Lemma 9.1, there are closed 1-chains $\gamma_A$ and $\gamma_B$ in the complement of $K$ such that $W(\gamma_A, P) = 1$ for all $P$ in $A$ and $W(\gamma_B, P) = 0$ for all $P$ in $B$, and similarly for $\gamma_B$. 

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Therefore, if $P$ is in $A$, then $\int_{\gamma_A} \omega_P = 1$ and $\int_{\gamma_B} \omega_P = 0$, and if $Q$ is in $B$, then $\int_{\gamma_B} \omega_Q = 1$ and $\int_{\gamma_Q} \omega_Q = 0$. This implies that $[\omega_P]$ and $[\omega_Q]$ are linearly independent in $H^1U$.

If $K$ has $n$ components $K_1, \ldots, K_n$, apply Lemma 9.1 with $A = K_i$ and $B = K \setminus K_i$ (which is closed, as finite union of closed sets) to obtain closed 1-chains $\gamma_1, \ldots, \gamma_n$ in $\mathbb{R}^2 \setminus K$ such that $W(\gamma_i, P) = 1$ for all $P$ in $K_i$ and $W(\gamma, P) = 0$ for all $P$ in $K \setminus K_i$.

Consider the map

$$\Psi : \text{Closed 1-forms on } U \longrightarrow \mathbb{R}^n$$

given by

$$\Psi(\omega) = \left( \int_{\gamma_1} \omega, \ldots, \int_{\gamma_n} \omega \right).$$

This map is linear. By an argument similar to that in Problem 1, it is easy to prove that $\Psi$ is surjective. Moreover, if $\omega$ is a closed 1-form on $U$ such that $\Psi(\omega) = 0$, then it follows that $\int_{\gamma} \omega = 0$ for every closed path $\gamma$ in $U$, because any closed path $\gamma$ in $U$ is homologous to a linear combination $m_1 \gamma_1 + \cdots + m_n \gamma_n$, and the integral of a closed 1-form along a closed 1-chain only depends on the homology class of the 1-chain. It follows from this that the kernel of $\Psi$ is the space of exact 1-forms on $U$, and thus that $H^1U \cong \mathbb{R}^n$. \qed

**Problem 5.** Compute the integral $\int_{\gamma} \omega$, where $\omega$ is the 1-form

$$\omega = \sum_{n=1}^{17} \frac{1}{(x-n)^2 + y^2} \left( -ydx + (x-n)dy \right),$$

and $\gamma(t) = (t \cos(t), t \sin(t)), 0 \leq t \leq 6\pi$.

**Solution.** The integral is $53\pi$. \qed

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