Problem 1. Let $U, V$ be open subsets of $\mathbb{R}^2$. Prove that the coboundary map $\delta : H^0(U \cap V) \to H^1(U \cup V)$ is a homomorphism of vector spaces.

Problem 2. Let $F : U \to V$ be a smooth map from the open set $U \subset \mathbb{R}^2$ into the open set $V \subset \mathbb{R}^2$. In a previous homework set we showed that $F$ induces a “pull-back” operator $F^*$ that sends $n$-forms on $V$ into $n$-forms on $U$. Prove that $F^*$ induces a linear map of vector spaces $F^* : H^n V \to H^n U$, for $n = 0, 1$.

Moreover, prove that if $F$ is a diffeomorphism, then $F^* : H^n V \to H^n U$ is an isomorphism of vector spaces.

Problem 3. Prove that if $U$ and $V$ are connected, and $H^1(U \cup V) = 0$, then $U \cap V$ is connected.

Problem 4. Prove that if the open set $U \subset \mathbb{R}^2$ can be written as an union $U = U_1 \cup \cdots \cup U_n$, where each $U_j$ is a convex open set, then $H^1 U$ is a finite dimensional vector space.

Problem 5. Let $U \subset \mathbb{R}^2$ be the complement of $n$ points. Prove that $H^1 U$ is a vectors space of dimension $n$, and find a basis for it.