Problem 1. Let $\gamma : [a, b] \to \mathbb{R}^2 \setminus \{0\}$ be a continuous path. Prove that there are continuous functions $r : [a, b] \to \mathbb{R}_+$ (the positive real numbers) and $\theta : [a, b] \to \mathbb{R}$, so that

$$\gamma(t) = (r(t) \cos \theta(t), r(t) \sin \theta(t)), \quad a \leq t \leq b.$$ 

Prove also that $r$ is uniquely determined, and $\theta$ is uniquely determined up to an integer multiple of $2\pi$.

**Hint.** Show in fact that $r(t) = |\gamma(t)|$, and that if $\gamma'$ denotes the restriction of $\gamma$ to the interval $[a, t]$, for $a \leq t \leq b$, and $\theta_a$ is an angle for $\gamma(a)$, then one may take

$$\theta(t) = \theta_a + 2\pi W(\gamma', 0).$$

**Definition 1.** Let $\gamma : [a, b] \to \mathbb{R}^2 \setminus \{0\}$ be a continuous path. By Problem 1, for any choice of angle $\theta_a$ for the initial point $P_a = \gamma_a$, there is a unique continuous path $\tilde{\gamma} : [a, b] \to \{(r, \theta) \mid r > 0\}$ such that $p \circ \tilde{\gamma} = \gamma$ and $\tilde{\gamma}(a) = (r(a), \theta_a)$, where $p$ is the polar coordinate mapping at the origin, $p(r, \theta) = (r \cos \theta, r \sin \theta)$. Such path $\tilde{\gamma}$ is called a lifting of $\gamma$ with starting point $(r(a), \theta_a)$.

Problem 2. Let $\gamma, \delta : [a, b] \to \mathbb{R}^2 \setminus \{0\}$ be two continuous paths with the same endpoints. Prove that the following are equivalent:

(i) $\gamma$ and $\delta$ are homotopic in $\mathbb{R}^2 \setminus \{0\}$ relative to endpoints;

(ii) $W(\gamma, 0) = W(\delta, 0)$; and

(iii) if $\tilde{\gamma}$ and $\tilde{\delta}$ are liftings of $\gamma$ and $\delta$ with the same initial point (as in Definition 1 above), then $\tilde{\gamma}$ and $\tilde{\delta}$ have the same final point.

Problem 3. Identify $\mathbb{R}^2$ with the complex numbers $\mathbb{C}$, so that the vectors $(x, y)$ corresponds to the complex number $z = x + iy$. Let $C$ be the unit circle $\{|z| = 1\}$ in $\mathbb{C}$. Determine the winding number $W(F, 0)$ for the following mappings $F : C \to \mathbb{C}$.

(i) $F(z) = z^n$, $n$ an integer.

(ii) $F(z) = -z$.

(iii) $F(z) = \overline{z}$.

Problem 4. Let $C$ be the circle centered at the origin, and let $F : C \to \mathbb{R}^2$ be a continuous mapping such that the vector $F(P)$ is never tangent to the curve $C$ at $P$, i.e., the dot product $P \cdot F(P) \neq 0$ for all $P$ in $C$. Show that $W(F, 0) = 1$. 

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