Math 512A. Homework 9. Due 11/14/07

(Revised 11/10)

**Problem 1.**  (i) Suppose that \( g(x) = f(x + c) \) for all \( x \). Prove, starting from the definition of the derivative, that \( g'(x) = f'(x + c) \) for all \( x \).

(ii) Prove that if \( g(x) = f(cx) \), then \( g'(x) = c \cdot f'(cx) \).

(iii) Suppose that \( f \) is differentiable and periodic, with period \( a \), i.e., \( f(x + a) = f(x) \) for all \( x \). Prove that \( f' \) is also periodic with period \( a \).

(iv) (Not required) Prove that if \( f \) is even, i.e., \( f(x) = f(-x) \), then \( f'(x) = -f'(-x) \).

(v) (Not required) Prove that if \( f \) is odd, i.e., \( f(-x) = -f(x) \), then \( f'(x) = f'(-x) \).

**Problem 2.**  (i) Let \( f(x) = x^2 \) if \( x \) is rational, and \( f(x) = 0 \) if \( x \) is irrational. Prove that \( f \) is differentiable at 0.

(ii) Let \( f \) be a function such that \( |f(x)| \leq x^2 \) for all \( x \). Prove that \( f \) is differentiable at 0.

(iii) (Not required) Let \( \alpha > 1 \). Prove that if \( f \) satisfies \( |f(x)| \leq |x|^\alpha \), then \( f \) is differentiable at 0.

**Problem 3.** Suppose that \( a \) and \( b \) are two consecutive roots of the polynomial function \( f \), but that \( a \) and \( b \) are not double roots, so that we can write \( f(x) = (x-a)(x-b)g(x) \) where \( g(a) \neq 0 \) and \( g(b) \neq 0 \).

(i) Prove that \( g(a) \) and \( g(b) \) have the same sign.

(ii) Prove that there is some number \( x \) with \( a < x < b \) and \( f'(x) = 0 \).

(iii) (Not required) Prove that (ii) holds true even if \( a \) and \( b \) are multiple roots. Hint: If \( f(x) = (x-a)^n(x-b)^m g(x) \) where \( g(a) \neq 0 \) and \( g(b) \neq 0 \), consider the polynomial function \( h(x) = f'(x)/(x-a)^{n-1}(x-b)^{m-1} \).

**Problem 4.**  (i) If \( a_1 < a_2 < \cdots < a_n \), find the minimum value of \( f(x) = \sum_{i=1}^{n} (x-a_i)^2 \).

(ii) Find the minimum value of \( f(x) = \sum_{i=1}^{n} |x - a_i| \).

(iii) (Not required) Let \( a > 0 \). Prove that the maximum value of

\[
    f(x) = \frac{1}{1 + |x|} + \frac{1}{1 + |x-a|}
\]

is \((2 + a)/(1 + a)\).

**Problem 5.**  (i) Suppose that \( |f(x) - f(y)| \leq |x - y|^\alpha \) for some \( \alpha > 1 \). Prove that \( f \) is constant.

(ii) Find a function \( f \) other than a constant function such that \( |f(x) - f(y)| \leq |x - y| \).