Math 512A. Homework 5. Due 10/10/07

(Work out any 5 problems)

(Revised 10/4)

Problem 1. (i) Define “countable set.”

(ii) Determine (either prove or give a counterexample) whether the following statements are true: (a) The union of two uncountable sets is uncountable. (b) The intersection of two uncountable sets is uncountable.

Problem 2. (i) Define “\( \lim_{n \to \infty} a_n = l \).”

(ii) Prove, using the definition in (i), that \( a_n = 2^n - \frac{1}{n} + 3 \) converges to \( l = 2 \).

Problem 3. Let \( a_n \) be the Fibonacci sequence, \( a_1 = a_2 = 1, a_{n+2} = a_n + a_{n+1} \).

(i) If \( r_n = \frac{a_{n+1}}{a_n} \), then prove that \( r_{n+1} = 1 + \frac{1}{r_n} \).

(ii) Prove that \( r = \lim_{n \to \infty} r_n \) exists, and \( r = 1 + \frac{1}{2} \). Conclude that \( r = \frac{1 + \sqrt{5}}{2} \).

Problem 4. (i) Find all the accumulation points of the set \( \{ \frac{1}{n} + \frac{1}{m} | n \text{ and } m \in \mathbb{N} \} \).

(ii) Prove that \( p \) is an accumulation point of a set \( S \subset \mathbb{R}^n \) if and only if every ball about \( p \) contains infinitely many points of \( S \).

Problem 5. (i) Let \( a_n \) be a bounded sequence of real numbers. Prove that if \( p \) is the only accumulation point of the set \( A = \{a_n | n \in \mathbb{N}\} \), then the sequence \( a_n \) converges and \( \lim_{n \to \infty} a_n = p \).

(ii) Show by a counterexample that this property is not true for unbounded sequences.

Problem 6. (i) Define the concept “bounded sequence.”

(ii) Prove that a set \( S \subset \mathbb{R} \) is bounded if and only if every sequence of points in \( S \) has a convergent subsequence.

Problem 7. (i) Define the concept “\( f \) is a continuous function at the point \( p \).”

(ii) Let \( f : \mathbb{R} \to \mathbb{R} \) be the function given by \( f(x) = x \) if \( x \) is rational, and \( f(x) = -x \) if \( x \) is irrational. Prove that \( f \) is continuous only at \( p = 0 \).

Problem 8. (i) If \( \lim_{x \to a} f(x) \) and \( \lim_{x \to a} g(x) \) do not exist, can \( \lim_{x \to a} \left[ f(x) + g(x) \right] \) or \( \lim_{x \to a} (f \cdot g)(x) \) exist?

(ii) If \( \lim_{x \to a} f(x) \) exists and \( \lim_{x \to a} \left[ f(x) + g(x) \right] \) exists, must \( \lim_{x \to a} g(x) \) exist?

(iii) If \( \lim_{x \to a} f(x) \) exists and \( \lim_{x \to a} g(x) \) does not exist, can \( \lim_{x \to a} \left[ f(x) + g(x) \right] \) exist?

(iv) If \( \lim_{x \to a} f(x) \) exists and \( \lim_{x \to a} f(x)g(x) \) exists, does it follow that \( \lim_{x \to a} g(x) \) exists?

Problem 9. (i) Define the concepts “closed set” and “closure of a set.”

(ii) Prove that the closure of a set \( S \subset \mathbb{R}^n \) is the smallest closed subset of \( \mathbb{R}^n \) which contains \( S \).