Math 512A. Homework 3. Solutions

**Problem 1.** Find all the accumulation points of the following sets:

(i) The interval $[0, 1)$.

(ii) The set of all the irrational numbers.

(iii) The set of the natural numbers.

**Solution.** (i) The interval $[0, 1)$. (ii) The set of all real numbers. (iii) The empty set.

**Problem 2.** A sequence $(a_n)$ is said to be Cauchy if, for every $\varepsilon > 0$, there is a natural number $N$ such that whenever $n, m > N$, $|a_n - a_m| < \varepsilon$.

(i) Prove that a convergent sequence of real numbers is Cauchy.

(ii) Prove that a Cauchy sequence is bounded.

**Proof.** Solution (i) Suppose that $a_n \to l$. Given $\varepsilon > 0$ there is a natural number $N$ such that if $n > N$, then $|a_n - l| < \varepsilon/2$. Therefore, if $p, q > N$,

$$|a_p - a_q| = |a_p - l + l - a_q| \leq |a_p - l| + |a_q - l| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

thus showing that $(a_n)$ is Cauchy.

(ii) If there is a natural number $N$ such that $a_N > l$, then $a_n \geq a_N > l$ for all natural numbers $n \geq N$, and $a_n$ cannot converge to $l$.

**Problem 3.** Prove or give a counterexample:

(i) If $(a_n)$ is an increasing sequence (that is, $a_1 < a_2 < a_3 < \cdots$) such that $\lim_{n \to \infty} (a_{n+1} - a_n) = 0$, then $(a_n)$ is convergent.

(ii) If $(a_n)$ is increasing and bounded above, and $\lim_{n \to \infty} a_n = l$, then $a_n \leq l$.

**Proof.** Solution (i) Let $a_n = \sqrt{n}$. Then $a_n < a_{n+1}$ and (rationalizing)

$$0 < a_{n+1} - a_n = \sqrt{n+1} - \sqrt{n} = \frac{1}{\sqrt{(n+1)n}} \leq \frac{1}{n+1}$$

which implies that $\lim_{n \to \infty} (a_{n+1} - a_n) = 0$.

(ii) If there is a natural number $N$ such that $a_N > l$, then $a_n \geq a_N > l$ for all natural numbers $n \geq N$, and $a_n$ cannot converge to $l$.

**Problem 4.** (i) Give an example of a sequence of real numbers with subsequences converging to every integer.
(ii) Give an example of a sequence of real numbers with subsequences converging to every real number.

Proof. Solution (i) 0, -1, 0, 1, -2, -1, 0, 1, -3, -2, -1, 0, 1, 2, 3 ..., and so on. Each integer appear infinitely many times in this sequence and thus you can extract a subsequence which converges to any integer (in fact, a constant sequence).

Problem 5. Prove that if the subsequences \((a_{2n})\) and \((a_{2n+1})\) of a sequence \((a_n)\) of real numbers both converge to the same limit \(l\), then \((a_n)\) converges to \(l\).

Proof. Solution Given \(\varepsilon > 0\) there are natural numbers \(N_e\) and \(N_o\) such that if \(n\) is an even natural number and \(n > N_e\), then \(|a_n - l| < \varepsilon\), and if \(n\) is an odd natural number and \(n > N_o\), then \(|a_n - l| < \varepsilon\). Let \(N = \max\{N_e, N_o\}\). If \(n\) is a natural number and \(n > N\), then \(n\) is either even and \(> N_e\), or \(n\) is odd and \(> N_o\). In either case, \(|a_n - l| < \varepsilon\).