Math 512A. Homework 2. Solutions

Problem 1. Recall that the absolute value $|a|$ of a real number $a$ is given by

$$|a| = \begin{cases} 
    a, & a \geq 0 \\
    -a, & a \leq 0.
\end{cases}$$

Prove the following:

(i) $|a + b| \leq |a| + |b|$.
(ii) $|a - b| \leq |a| + |b|$.
(iii) $|a| - |b| \leq |a - b|$.
(iv) $||a| - |b|| \leq |a - b|$.

Solution. (i) We consider 4 cases:

1. $a \geq 0$, $b \geq 0$
2. $a \geq 0$, $b \leq 0$
3. $a \leq 0$, $b \geq 0$
4. $a \geq 0$, $b \leq 0$

In case (1) we have $|a| = a$ and $|b| = b$, and we also have $a + b \geq 0$, so $|a + b| = a + b$, making the result obvious: $|a + b| = a + b = |a| + |b|$, so that in fact we have equality. Case (4) is similar to case (1). Indeed we have $|a| = -a$, $|b| = -b$ and also $|a + b| = -(a + b) = -a - b$ because $a + b \leq 0$.

In case (2) we have $a \geq 0$ and $b \leq 0$, hence $|a| = a$ and $|b| = -b$, so we must prove that

$$|a + b| \leq a - b.$$  

We divide the proof into two subcases

1. $a + b \leq 0$
2. $a + b \geq 0$

If case (2a) holds, then $|a + b| = -(a + b) = -a - b$, and we must show that $-a - b \leq a - b$, or that $-a \leq a$. This is certainly true because $a \geq 0$ implies that $-a \leq 0 \leq a$.

If case (2b) holds, then $|a + b| = a + b$, and we must show that $a + b \leq a - b$, or that $b \leq -b$. But the hypothesis for case (2) is that $b \leq 0$, so that $-b \geq 0$ and hence $-b \leq 0 \leq b$.

Case (3) requires no additional work; it follows by applying case (2) with $a$ and $b$ interchanged.

(ii) By (i),

$$|a - b| = |a + (-b)| \leq |a| + | -b| = |a| + |b|$$

(iii) By (i),

$$|a| = |a - b + b| \leq |a - b| + |b|$$

and subtracting $|b|$

$$|a| - |b| \leq |a - b|$$

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(iv) By (iii) we have that $|a| - |b| \leq |a - b|$, and by interchanging $a$ and $b$ we also have that $|b| - |a| \leq |b - a| = |a - b|$. Therefore,

$$||a| - |b|| = \begin{cases} |a| - |b| \leq |a - b| & \text{if } |a| - |b| \geq 0, \\ |b| - |a| \leq |a - b| & \text{if } |a| - |b| \leq 0. \end{cases}$$

**Problem 2.** Suppose that $\lim_{n \to \infty} a_n = a$ and $\lim_{n \to \infty} b_n = b$. Prove the following:

(i) $a_n + b_n \to a + b$.

(ii) $a_n \cdot b_n \to a \cdot b$.

**Solution.** (ii) By adding and subtracting $a_n b$ and then rearranging terms, we obtain

$$|a_n b_n - ab| = |a_n (b_n - b) + (a_n - a)b| \leq |a_n||b_n - b| + |b||a_n - a|$$

Because the sequences $a_n$ and $b_n$ both converge, they are both bounded: there is $M > 0$ such that $|a_n| \leq M$ and $|b_n| \leq M$ for all $n$. Therefore:

$$|a_n b_n - ab| \leq M|a_n - a| + M|b_n - b|,$$

for all natural numbers $n$.

Let $\varepsilon > 0$. There is a natural number $N_1$ such that if $n > N_1$, then $|a_n - a| < \varepsilon/2M$, and there is a natural number $N_2$ such that if $n > N_2$, then $|b_n - b| < \varepsilon/2M$. Let $N = \max\{N_1, N_2\}$. If $n > N$, then

$$|a_n b_n - ab| \leq M \frac{\varepsilon}{2M} + M \frac{\varepsilon}{2M} = \varepsilon.$$

**Problem 3.** (i) Prove that if $a_n \leq b_n$, if $\lim_{n \to \infty} a_n = a$ and $\lim_{n \to \infty} b_n = b$, then $a \leq b$.

(ii) Prove that if $a_n \leq c_n \leq b_n$ and $\lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = l$, then $\lim_{n \to \infty} c_n = l$.

**Solution.** (i) Suppose that $a > b$ and apply the definition of limit to $\varepsilon = \frac{a - b}{2}$. There is a natural number $N_1$ such that if $n > N_1$, then $|a_n - a| < \frac{a - b}{2}$ and a natural number $N_2$ such that if $n > N_2$, then $|b_n - b| < \frac{a - b}{2}$. If $n > \max\{N_1, N_2\}$, then

$$b_n < \frac{a - b}{2} + b = a - \frac{a - b}{2} < a,$$

which contradicts the hypothesis.

**Problem 4.** Verify the following limits

(i) $\lim_{n \to \infty} \frac{3n^3 + 7n^2 + 1}{4n^3 - 8n + 63} = \frac{3}{4}$.

(ii) $\lim_{n \to \infty} \frac{2^n + (-1)^n}{2n+1 + (-1)^n+1} = \frac{1}{2}$.

(iii) $\lim_{n \to \infty} \sqrt[n]{a} = 1$. (Hint: put $\sqrt[n]{a} = 1 + a_n$, prove that $a_n > 0$ for $n > 1$, deduce that $n - 1 \geq \frac{1}{2} n(n - 1)a_n^2$ for $n > 1$, hence that $0 \leq a_n^2 \leq 2/n$.)
Solution. (i) Divide numerator and denominator by $n^3$ and obtain:

\[
\frac{3n^3 + 7n^2 + 1}{4n^3 - 8n + 63} = \frac{3 + \frac{7}{n} + \frac{1}{n^3}}{4 - \frac{8}{n^2} + \frac{63}{n^3}}.
\]

The limit of the numerator is (by using the algebraic properties of limits):

\[
\lim_{n \to \infty} 3 + \frac{7}{n} + \frac{1}{n^3} = \lim_{n \to \infty} 3 + \frac{7}{n} + \lim_{n \to \infty} \frac{1}{n^3} = 3
\]

and by similar arguments, the limit of the denominator is

\[
\lim_{n \to \infty} 4 - \frac{8}{n^2} + \frac{63}{n^3} = 4
\]

Since this limit is $\neq 0$, we have

\[
\lim_{n \to \infty} \frac{3n^3 + 7n^2 + 1}{4n^3 - 8n + 63} = \frac{3}{4}
\]

(ii) Divide numerator and denominator by $2^{n+1}$.

(iii) Let $\sqrt[n]{n} = 1 + a_n$. It is clear that $a_n \geq 0$, for if $a_n < 0$, then $n = (1 + a_n)^n < 1$. By the binomial theorem

\[
n = (1 + a_n)^n = 1 + na_n + \frac{n(n-1)}{2} a_n^2 + \cdots
\]

\[
\geq 1 + \frac{n(n-1)}{2} a_n^2
\]

(we removed positive terms form the sum above),

which implies that

\[
0 \leq a_n \leq \sqrt[2]{\frac{2}{\sqrt{n}}}
\]

and thus that It follows from Problem 3(ii) that $\lim_{n \to \infty} a_n = 0$, and hence that $\lim_{n \to \infty} \sqrt[n]{n} = 1$.

Problem 5. Does the sequence converge or diverge? If it converges, what is the limit?

(i) $a_n = \frac{n}{n+1} - \frac{n+1}{n}$.

(ii) $a_n = \frac{2^n}{n!}$.

(iii) $a_n = \text{the } n\text{th decimal digit of } \pi \text{ (thus } a_1 = 1, a_2 = 4, a_3 = 1, \text{ and so on).}$

Solution. (ii) We have the inequalities

\[
0 \leq \frac{2^n}{n!} = \frac{2 \cdot 2 \cdot \ldots \cdot 2}{n(n-1) \cdot \ldots \cdot 2 \cdot 1} \leq \frac{4}{n}
\]

so Problem 3(ii) implies that $\lim_{n \to \infty} \frac{2^n}{n!} = 0$.

(iii) This assumes that you know that $\pi$ is not a rational number, thus that its decimal digit expansion does not eventually repeat.

Let $a_n$ be the $n$th decimal digit of $\pi$, and suppose that $(a_n)$ converges, say $\lim a_n = a$. Because all $a_n$ are decimal digits, $a$ must also be a decimal digit. Indeed, if not, $|a_n - a| \geq \min \{|a - d| \mid d = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9\} > 0$, contradicting $a_n \to a$. In fact, we must have $a_n = a$ eventually. Indeed, for $\varepsilon = 1$ there is a natural number $N$ such that if $n > N$, then $|a_n - a| < 1$, hence that $a_n = a$ because two decimal digits either are equal or their difference is at least 1 in absolute value. The fact that $a_n = a$ eventually implies that $\pi$ is a rational number, a contradiction.