Math 102. Fall 2006. Practice 2nd Midterm

1 Solve \( \frac{x}{x-1} \leq \frac{1}{x} \). Write your answer using interval notation.

**Solution.** \((0, 1]\)

2 Let \( P(x) = 2x^3 - 5x^2 - 4x + 3 \).
   (i) List all the possible rational zeros of \( P \).
   (ii) Verify that 3 is a zero of \( P \).
   (iii) Find all other zeros of \( P \).
   (iv) Find the complete factorization of \( P \).

**Solution.** (i) \( \pm 1, \pm 1/2, \pm 3, \pm 3/2 \)
   (iv) \( 2(x-3)(x+1)(x-1/2) \)

3 Find a fourth degree polynomial, \( P(x) \), with real coefficients which has zeros 1 + 3i and -1, with -1 a zero of multiplicity 2, and such that its constant coefficient is 20.

**Solution.** \( 2x^4 + 14x^3 + 36x + 20 \)

4 Find a polynomial \( P \) with real coefficients that has degree 4, zeros \( i \) and 2i, and whose graph passes through the point (1, 5).

**Solution.** \( \frac{1}{2}x^4 + \frac{5}{2}x^3 + 2 \)

5 Given that the complex number \( 2 + i \) is a zero of the polynomial \( P(x) = 2x^4 - 8x^3 + 28x^2 - 72x + 90 \), find
   (i) all the zeros of \( P \);
   (ii) the complete factorization of \( P \).

**Solution.** (ii) \( 2(x-3)(x+3i)(x-2-i)(x-2+i) \)

6 For the functions
   \( f(x) = \frac{1}{x+3} \) and \( g(x) = \frac{1}{x-2} \)
   (i) Find \( (f \circ g)(x) \), \( (g \circ f)(x) \), \( (f \circ f)(x) \) and \( (g \circ g)(x) \)
   (ii) Find the domains of \( f \circ g \), \( g \circ f \), \( f \circ f \), and \( g \circ g \).

**Solution.**
\[
\begin{align*}
(f \circ g)(x) &= \frac{x - 2}{3x - 5}, & \text{Domain: } x \neq 2, \frac{5}{3} \\
g \circ f(x) &= \frac{x + 3}{-2x - 5}, & \text{Domain: } x \neq -3, \frac{5}{2} \\
f \circ f(x) &= \frac{x + 3}{3x + 10}, & \text{Domain: } x \neq -3, \frac{10}{3} \\
g \circ g(x) &= \frac{x - 2}{5 - 2x}, & \text{Domain: } x \neq 2, \frac{5}{2}
\end{align*}
\]

7 Let \( f(x) = 1 - (x + 2)^3 \).
   (i) Use transformations and basic shapes to sketch the graph of \( f \).
   (ii) Prove that \( f(x) \) is one-to-one.
   (iii) Find an equation for the inverse function \( f^{-1} \) of \( f \).
   (iv) Sketch the graph of \( f^{-1} \) and the graph of \( f \) (you did this in part (ii)) in the same coordinate axes.

**Solution.** \( f^{-1}(x) = -2 + \sqrt{1 - x} \)

8 (i) If \( f(x) = 2 - \sqrt{3 - x} \), find the inverse function \( f^{-1} \).
   (ii) Sketch the graphs of \( f \) and \( f^{-1} \) on the same coordinate axes.

9 For the function \( f(x) = 4 - e^{-x} \).
   (i) Graph \( f \) using transformations. State domain, range, intercepts, and horizontal asymptote.
   (ii) Determine the inverse \( f^{-1} \). State domain, range, intercepts, and vertical asymptote.
   (iii) Graph \( g \) and \( g^{-1} \) on the same pair of coordinate axes.

10 Sketch the graph of \( f(x) = 3 - \log(1 - x) \). Label intercepts and asymptotes.

11 Simplify the following expression (write as a single logarithm): \( 20 \log_2 \sqrt{x} + \log_2 (4x^2) - \log_2 4 \).

12 Solve for \( x \): \( \log_3 x + \log_3 (x + 3) = 4 \).

**Solution.** \(-2 + \sqrt{85}\)

13 Solve for \( x \): \( e^{2x} + 3e^x - 4 = 0 \).

**Solution.** \(0\)

14 It has been estimated that the rat population of a major city follows an exponential growth. The population triplicated in size during an 18 month period. If the rat population is currently 10 million, what will it be 2 years from now?

**Solution.** \(80/3\)

15 At the beginning of an experiment, a scientist has 364 grams of radioactive goo. After two hours, her sample has decayed to 24 grams.
   (i) What is the half life (in minutes) of the goo?
   (ii) Find a formula for \( G(t) \), the amount of goo remaining \( t \) minutes after the experiment started.
   (iii) How many grams of goo will remain after 14 minutes?

**Solution.** (i) \(80/3\)