Math 462; Exam 2 Solutions

Do five of the following eight problems, at least two from each group. Do all problems for homework due Wednesday. If you do more than five problems clearly indicate, which problems you would like to have graded. All spaces are inner product spaces, unless otherwise stated.

Group I:

1. Let $T \in \mathcal{L}(V)$ be a self-adjoint positive isometry. Show that $T$ must be the identity.

   Solution: Since $T$ is isometry all eigenvalues of $T$ have absolute value 1 and $T$ is normal. Hence by the complex Spectral Theorem there is an orthonormal basis in which $T$ is represented by a diagonal matrix. But $T$ is positive therefore all eigenvalues are positive. So 1 is the only eigenvalue, and $T$ is represented by a unit matrix, hence $T$ is the identity.

2. Let $V$ be a complex inner-product space and $T \in \mathcal{L}(V)$ a normal operator such that $T^8 = T^9$. Show that $T$ is self-adjoint, and that $T^2 = T$.

   Solution: Let $v \in V$. Then $T^8v - T^9v = (I - T)T^8v = 0$. So either $v \in \ker T^8$ or $u = T^8v$ is an eigenvector to the eigenvalue 1. But if $v \in \ker T^8 = \ker T$ (by a homework problem) Then $(T - T^2)v = 0$. If not $u = \text{range } T^8 = \text{range } T$ and $(T - T^2)v = 0$, i.e. $T = T^2$. Since $T$ is normal, there is an orthonormal basis of $T$ in which $T$ has a diagonal matrix. Since 0 and 1 are the only eigenvalues, and are real $T$ must be self-adjoint.

3. Let $S \in \mathcal{L}(V)$ be such that there is an orthonormal basis $(e_1, \ldots, e_n)$ of $V$ with $\|Se_j\| = 1$. Is $S$ an isometry?

   Solution: No. Consider, for example $T : \mathbb{R}^3 \to \mathbb{R}^3$ which maps $T : e_j \mapsto e_1$ for $j = 1, 2, 3$. This operator satisfies the hypothesis, but it is no isometry. $\|T(e_1 + e_2)\| = \|2e_1\| = 2$, but $\|e_1 + e_2\| = \sqrt{2}$.

4. State and prove the Real Spectral Theorem.

   Solution: Refer to the notes and the book.

Group II:

5. Suppose that $U$ is a subspace of $V$. Prove that

   $$\dim U + \dim U^\perp = \dim V.$$

   Solution: Since $U$ is a subspace of $V$ we have $V = U \oplus U^\perp$. And the result follows immediately.

6. Let $u, v \in V$. Prove that

   $$\|u + v\|^2 + \|u - v\|^2 = 2 (\|u\|^2 + \|v\|^2).$$
Solution:

\[
\|\mathbf{u} + \mathbf{v}\|^2 + \|\mathbf{u} - \mathbf{v}\|^2 = [\mathbf{u} + \mathbf{v}, \mathbf{u} + \mathbf{v}] + [\mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v}]
\]
\[
= [\mathbf{u}, \mathbf{u}] + [\mathbf{v}, \mathbf{u}] + [\mathbf{u}, \mathbf{v}] + [\mathbf{v}, \mathbf{v}]
\]
\[
= [\mathbf{u}, \mathbf{u}][\mathbf{v}, \mathbf{u}] - [\mathbf{u}, \mathbf{v}] - [\mathbf{v}, \mathbf{v}]
\]
\[
= 2 (\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2)
\]

7. Suppose \( T \in \mathcal{L}(V) \) and \( \lambda \in \mathbb{F} \). Prove that \( \lambda \) is an eigenvalue of \( T \) if and only if \( \overline{\lambda} \) is an eigenvalue of \( T^* \).

**Solution:** Let \( \lambda \) be eigenvalue of \( T \) and \( \mathbf{v} \) be and eigenvector for \( \lambda \). We have

\[
[\mathbf{v}, \overline{\lambda} \mathbf{v}] = [\lambda \mathbf{v}, \mathbf{v}]
\]
\[
= [T \mathbf{v}, \mathbf{v}]
\]
\[
= [\mathbf{v}, T^* \mathbf{v}]
\]

and so \( \overline{\lambda} \) is an eigenvalue for \( T^* \). The opposite direction follows from the same equation.

8. Let \( T \in \mathcal{L}(\mathbb{R}^2) \) be given by:

\[
T(x_1, x_2) = (2x_1 + x_2, x_1 + 2x_2).
\]

Show that \( T \) is a positive operator.

**Solution:** Recall that \( a^2 + b^2 \geq 2|a||b| \) for all real \( a, b \). This follows from \((a \pm b)^2 \geq 0\).

Now we have:

\[
T(x_1, x_2) \cdot (x_1, x_2) = 2x_1^2 + x_1x_2 + x_1x_2 + 2x_2^2
\]
\[
= 2(x_1^2 + x_2^2 + x_1x_2) > 0
\]