Math 462; Assignment 6

1. Let $V$ be a finite dimensional vector space over $F$ and $S, T \in \mathcal{L}(V)$. Moreover, let $(v_1, \ldots, v_n)$ be a basis such that both $S$ and $T$ have upper triangular matrices in this basis. Show that $ST$ also has an upper triangular matrix in this basis.

**Solution:** Since $S$ and $T$ have upper triangular matrices in the given bases we have that for any $k \in \{1, \ldots, n\}$ $U_k = \text{span}(v_1, \ldots, v_k)$ is an invariant subspace for both $S$ and $T$. Since $TU_k \in u_K$ and $SU_K \in U_k$ it follows that $STU_k \in U_k$, so $u_k$ is an invariant subspace of $ST$ for all $k$, and $ST$ has an upper triangular matrix in the given basis.

2. Repeat the previous problem with diagonal instead of upper triangular matrices for $S$ and $T$.

**Solution:** Since both $S$ and $T$ have diagonal matrices in the given basis there are numbers $\tau_1, \ldots, \tau_n$ and $\sigma_1, \ldots, \sigma_n$ such that
\[
Sv_j = \sigma_j v_j, \quad Tv_j = \tau_j v_j, \quad \forall j \in \{1, \ldots, n\}
\]
But then
\[
STv_j = S\tau_j v_j = \tau_j S v_j = \tau_j \sigma_j v_j, \quad \forall j \in \{1, \ldots, n\}
\]
Therefore $ST$ has a diagonal matrix.

3. Give an example of an operator whose matrix with respect to some basis contains only zeros on the main diagonal, but the operator is invertible.

**Solution:** Consider the operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ which maps $T : (x, y) \mapsto (y, x)$. This operator is clearly invertible, in fact $T^{-1} = T$ and its matrix in the standard basis is
\[
\begin{pmatrix}
0 & 1 \\
1 & 0 
\end{pmatrix}
\]

4. Give an example of an operator whose matrix with respect to some basis contains only non-zero numbers on the main diagonal, but the operator is not invertible.

**Solution:** Consider the operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ which maps $T : (x, y) \mapsto (x + y, x + y)$. This operator is not invertible, since every vector of the form $(a, -a)$ is in the kernel of $T$, and $\ker T \neq \{0\}$. Its matrix in the standard basis is
\[
\begin{pmatrix}
1 & 1 \\
1 & 1 
\end{pmatrix}
\]

5. Suppose $S, T \in \mathcal{L}(V)$ and $S$ is invertible. Prove that if $p$ is a polynomial, then
\[
p(STS^{-1}) = Sp(T)S^{-1}.
\]

**Solution:** Observe that for any $k \in \{0, \ldots n\}$
\[
ST^k S^{-1} = ST(SS^{-1})T \cdots (SS^{-1})T S^{-1} = (STS^{-1})^k.
\]
The coefficients of $p$ are numbers which commute with any linear operator. The result follows immediately.