Math 462; Practice for Exam 1

1. Prove or disprove: there exists a basis \((p_0, p_1, p_2, p_3)\) of \(P_3(\mathbb{C})\) such that none of the polynomials \(p_0, p_1, p_2, p_3\) has degree 2.

2. Prove that the set of all continuous real valued functions on the closed interval \([0, 1]\) is an infinite dimensional vector space.

3. Let \(S, T \in \mathcal{L}(V)\) and \((v_1, \ldots, v_n)\) be a basis in which \(S, T\) have a diagonal matrix. Prove that \(ST = TS\).

4. Give an example of a linear operator on \(\mathbb{R}^4\) which has no real eigenvalues.

5. Let \(V = P_2(\mathbb{C})\) and \(W = P_3(\mathbb{C})\). Moreover, let \(T: V \to W\) be the linear operator \(T: p(z) \mapsto (z = 1)p(z)\). Show that \(T\) is injective. Furthermore, find the matrix for \(T\) in the standard bases for \(V\) and \(W\).

6. Let \(T \in \mathcal{L}(V, W)\). Prove that
\[
\dim \ker T + \dim \text{range} T = \dim V
\]

7. Let \(T \in \mathcal{L}(V, W)\) be a surjective linear map. Show that
\[
\dim V \geq \dim V
\]