Problem Set 4
Theoretical Solid State Physics (SoSe2017)

Due: Due Thursday, May 18, 2017; **at the beginning of class**

Problem 1: Berry phase – an example

Consider a spin 1/2 in an adiabatically varying Zeeman field $\mathbf{B}(t)$, as described by the Hamiltonian

$$H = -\mathbf{B}(t) \cdot \sigma, \tag{1}$$

where σ denotes the vector of Pauli matrices. Parametrize the magnetic field as

$$\mathbf{B} = B(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta). \tag{2}$$

(a) Find the eigenvectors of the Hamiltonian for an arbitrary, but fixed magnetic field. Make sure that your result is single valued as function of the magnetic field.

(b) Use your result in (a) to compute the corresponding Berry connection. Relate your result explicitly to the vector potential for a magnetic monopole which we encountered earlier.

(c) Compute the corresponding Berry curvature and show that the Berry phase accumulated when the magnetic field varies in a closed loop is equal to half the solid angle subtended by the magnetic field loop.

Problem 2: Shortcuts to adiabaticity

Shortcuts to adiabaticity are currently a popular topic in research and address the following question: Imagine that you find the adiabatic dynamics of some Hamiltonian $H_0(t)$ attractive and want to implement it (e.g., because there are no transitions between instaneous eigenstates $|\psi_n(t)\rangle$ with instantaneous eigenenergies $E_n(t)$), but you do not have the patience to wait long enough so that the dynamics is really adiabatic. Then you would like to find a way to implement the adiabatic dynamics in a finite (and possibly short) time. One possibility is to search for a modified Hamiltonian $H(t) = H_0(t) + H_1(t)$ which implements the desired adiabatic dynamics as its *exact* quantum dynamics. It turns out that this problem can be solved quite generally.

(a) Show that the time evolution operator describing the adiabatic dynamics of $H_0(t)$ can be written as

$$\mathcal{U}(t) = \sum_n \mathrm{e}^{-i\int_0^t \mathrm{d}t' E_n(t') + i\gamma_n(t)} |\psi_n(t)\rangle\langle\psi_n(0)|, \tag{3}$$

where $\gamma_n(t) = i\int_0^t \mathrm{d}t' \langle\psi_n(t')|\partial_{t'}\psi_n(t')\rangle$ denotes the Berry phase.

(b) Show that given a time evolution operator $\mathcal{U}(t)$, one can find the corresponding Hamiltonian for which $\mathcal{U}(t)$ describes the exact quantum dynamics through

$$H(t) = i[\partial_t\mathcal{U}]\mathcal{U}^\dagger. \tag{4}$$

(c) Now insert the desired adiabatic time evolution operator from (a) into the general expression in (b) to obtain

$$H_1(t) = i\sum_n \left(|\partial_t\psi_n\rangle\langle\psi_n| - |\psi_n\rangle\langle\psi_n|\partial_t\psi_n\rangle\langle\psi_n|\right). \tag{5}$$

for the so-called counter-diabatic term $H_1(t)$.

(d) Explicitly evaluate the counter-diabatic term for the system which you treated in problem 1.

This version of shortcuts to adiabaticity is due to Michael Berry, J. Phys. A: Math. Theor., **42** 365303 (2009) (which you can turn to if you need help in solving this problem). Such protocols can actually be implemented experimentally, see, e.g., M.G. Bason *et al.*, Nat. Phys. **8**, 147 (2012).

Problem 3: Dimerized site energies

Solve for the band structure of the 1D tight-binding Hamiltonian

$$H = \sum_j \left\{ (-1)^j \epsilon |j\rangle\langle j| - t[|j\rangle\langle j+1| + |j+1\rangle\langle j|] \right\} \tag{6}$$

with dimerized site energies $\pm\epsilon$. At half filling, is the model a metal or an insulator?