Problem Set 10
Quantum Field Theory and Many Body Physics (SoSe2016)

Due: Monday, June 30, 2016 at the beginning of the lecture

In this problem set, we first discuss the effective theory of two interacting species of fermions. We find that the presence of the one fermion species induces an effective interaction between the particles of the other fermion species. This interaction is given by the polarization operator of the fermion species. In the second problem, we continue our study of the polarization operator, this time considering it for strictly one-dimensional systems. Finally, we discuss how connecting a superconductor and a normal metal induces superconducting correlations in the normal metal. This is known as proximity effect.

Problem 1: Effective action for interacting species of fermions (10+5+10 points)

Consider the partition function of two species of fermions, labeled a and b, that interact via a local, spatially-uniform, and repulsive density-density interaction. We assume that particles of type a interact with particles of type b and vice versa, but particles do not interact with other particles of their own kind. This is described by the action

$$S = \int_0^\beta d\tau \int \mathrm{d}\mathbf{r} \left[\sum_{j=a,b} \psi_j^*(\mathbf{r}) G_0^{-1} \psi_j(\mathbf{r}) + v n_a(\mathbf{r}) n_b(\mathbf{r}) \right] \tag{1}$$

where $G_0^{-1} = \partial_\tau + \mathbf{p}^2/2m - \mu$, $v > 0$ is the strength of the repulsive local interaction, $n_j(\mathbf{r}) = \psi_j^*(\mathbf{r})\psi_j(\mathbf{r}) - n_0$, and n_0 is a constant equal to the density of either species of particles.

(a) Express the partition function (normalized to the partition function of the non-interacting system) in terms of a functional integral and integrate out the ψ_b field. Express your result explicitly in terms of determinants.

(b) Derive an effective action for the fermions of type a by expansion of the result in (a) in powers of v. Show that the first order term in the expansion of the effective action is cancelled by the background term. *Hint:* Use the fact that $G_0(\mathbf{r}, \tau; \mathbf{r}, \tau) = -n_0$ (prove that!), due to charge neutrality.

(c) Show that to quadratic order in v, you obtain an effective interaction between particles of type a. Show that this interaction is governed by the polarization operator Π of the particles of type b, i.e., that the effective action of the a fermions to quadratic order in v is given by

$$S = \int \mathrm{d}\tau \mathrm{d}\mathbf{r}\, \psi_a^*(\mathbf{r}) G_0^{-1} \psi_a(\mathbf{r}) + \int \mathrm{d}\tau \mathrm{d}\tau' \int \mathrm{d}\mathbf{r} \mathrm{d}\mathbf{r}' n_a(\mathbf{r}, \tau) v^2 \Pi(\mathbf{r}, \tau; \mathbf{r}'\tau') n_a(\mathbf{r}', \tau'). \tag{2}$$

Thus, even though there is no bare interaction between particles of the same kind, the presence of the other kind of particles effectively mediates an interaction.

Problem 2: Polarization operator of (non-interacting) 1d electron systems

Consider a one-dimensional system of electrons with Hamiltonian

$$H = \int dx \psi^\dagger(x) \left(-\frac{\nabla^2}{2m} \right) \psi(x). \tag{3}$$

We want to study the density response of the system to external perturbations.

(a) Use linear-response theory and exploit the relations between the various correlation functions to compute its polarization operator defined through

$$\delta n(x, t) = -\int \mathrm{d}x' \mathrm{d}t' \Pi(x, t; x', t') e\phi(x', t') \tag{4}$$

in momentum and frequency representation. Here, $\delta n(x,t)$ is the change in density relative to the ground state and $\phi(x,t)$ is an applied scalar potential. You should find

$$\Pi(q,\omega) = \frac{m}{2\pi q}\left\{\ln\frac{\omega + i\eta + q^2/2m + qk_F/m}{\omega + i\eta - q^2/2m + qk_F/m} - \ln\frac{\omega + i\eta + q^2/2m - qk_F/m}{\omega + i\eta - q^2/2m - qk_F/m}\right\}. \tag{5}$$

(b) Specify to the static limit and consider what happens to it in the limits of $q \to 0$ and $q \to 2k_F$. Show that the polarization operator becomes equal to the density of states in the former case and diverges logarithmically at $2k_F$.

The latter divergence suggests that a one-dimensional interacting system might want to spontaneously form density modulations with wave vector $2k_F$. Such a state with spontaneous density modulations is known as a charge density wave.

Problem 3: Proximity effect (25 points)

Consider a 3d superconductor which is coupled to a one-dimensional normal electron system in the sense that electrons can pass between the two systems. For energies below the gap of the superconductor, electrons can enter the superconductor only virtually. These virtual excursions into the superconductor effectively induce superconducting correlations in the normal electron system. In this problem, we want to discuss the physics of this proximity effect in more detail. It turns out that it provides a very nice example of the concept of quasiparticle weight.

Consider the Hamiltonian

$$\mathcal{H} = \mathcal{H}_{\rm sc} + \mathcal{H}_{1d} + \mathcal{H}_t \tag{6}$$

describing a mean-field superconductor

$$\mathcal{H}_{\rm sc} = \int d\mathbf{r}\psi^\dagger_\sigma(\mathbf{r})\xi(-i\nabla)\psi_\sigma(\mathbf{r}) + \Delta\psi^\dagger_\uparrow(\mathbf{r})\psi^\dagger_\downarrow(\mathbf{r}) + \Delta^*\psi_\uparrow(\mathbf{r})\psi_\downarrow(\mathbf{r}), \tag{7}$$

the one-dimensional normal metal (say at $y = z = 0$)

$$\mathcal{H}_{1d} = \int dx\phi^\dagger_\sigma(\mathbf{r})\xi(-i\nabla)\phi_\sigma(\mathbf{r}), \tag{8}$$

and the tunnel coupling between the systems,

$$\mathcal{H}_{1d} = t\int dx[\phi^\dagger_\sigma(x)\psi_\sigma(x,0,0) + \psi^\dagger_\sigma(x,0,0)\phi_\sigma(x)]. \tag{9}$$

Now, introduce two-component Nambu spinors Ψ for the syperconductor and Φ for the 1d system and show that the system is described by the action

$$S = \Phi^\dagger\mathcal{G}^{-1}_{1d}\Phi + \Psi^\dagger\mathcal{G}^{-1}_{\rm sc}\Psi + \Psi^\dagger T\Phi + \Phi^\dagger T\Psi. \tag{10}$$

in matrix notation and an appropriate definition of the operator T. Integrate out the superconductor and obtain the effective action

$$S = \Phi^\dagger(\mathcal{G}^{-1}_{1d} + \Sigma)\Phi \tag{11}$$

in terms of the self energy

$$\Sigma(\mathbf{k}, i\omega_n) = \frac{t^2}{A_\perp}\sum_{k_y,k_z}\mathcal{G}_{\rm sc}(\mathbf{k}, i\omega_n) \tag{12}$$

when written explicitly in momentum space (with $A_\perp$ the cross-sectional area of the superconductor in the yz-plane). The momentum sum can be converted into an integral and performed explicitly. Do this integral to obtain

$$\Sigma(\mathbf{k}, i\omega_n) \simeq -\frac{\pi\nu_{2d}t^2}{\sqrt{\Delta^2+\omega^2}}\begin{bmatrix} i\omega & \Delta \\ \Delta & i\omega \end{bmatrix}. \tag{13}$$

Here, we assumed that the self energy depends only weakly on k_x which is valid for sufficiently large μ.

Note that the self energy has off-diagonal contributions which correspond to superconducting correlations induced in the normal metal. How large is the gap induced in the normal metal? What is the correlation length of the proximity-induced superconductivity? Ask your tutor about the quasiparticle weight and its physical interpretation.