Problem Set 9
Quantum Field Theory and Many Body Physics (SoSe2015)

Due: Monday, June 15, 2015 at the beginning of the lecture

In our discussion of charged superfluids, we briefly touch upon the action for electromagnetic fields. In this problem set, we discuss the Lagrangian and Hamiltonian formalism for (classical) electromagnetism is some more detail.

Problem 1: Maxwell action (5+10+10 points)

Hamilton's principle is not restricted to mechanical systems, but can be extended to field theories. The field theory can be defined by a Lagrangian (or, equivalently, the corresponding action), which, through Hamilton's principle, leads to the correct equations of motion of the field when varying the field configurations.

In this problem, we discuss the Lagrange formalism for the electromagnetic field whose equations of motion are Maxwell's equations (setting $c = 1$),

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0\,, \tag{1}$$

$$\nabla \cdot \mathbf{B} = 0\,, \tag{2}$$

$$\nabla \cdot \mathbf{E} = 4\pi\rho\,, \tag{3}$$

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = 4\pi\mathbf{j}\,. \tag{4}$$

The first two equations are referred to as homogeneous Maxwell equations.

(a) The homogeneous Maxwell equations are solved by the introduction of the scalar and vector potentials, $\phi(x,t)$ and $\mathbf{A}(x,t)$. Write explicitly the solution of Eqs. (1) and (2) in terms of ϕ, $\mathbf{A}$.

(b) While the six components of the $\mathbf{E}$ and $\mathbf{B}$ fields are not independent and thus not suitable as generalized coordinates, the potentials are. For this reason, write the Lagrangian of the magnetic field

$$L = \int \mathrm{d}\mathbf{r}\mathcal{L} = \int \mathrm{d}\mathbf{r} \left[\frac{\mathbf{E}^2 - \mathbf{B}^2}{8\pi} - \rho\phi + \mathbf{j}\mathbf{A}\right] \tag{5}$$

in terms of ϕ and $\mathbf{A}$, using the expressions for $\mathbf{E}$ and $\mathbf{B}$ in terms of ϕ and $\mathbf{A}$. ($\mathcal{L}$ is referred to as Lagrangian density.) Then, use Hamilton's principle, applied to the action $S = \int dt\, L$, to derive the inhomogeneous Maxwell equations, i.e., Eqs. (3,4).

(c) Show that the Lagrangian for the electromagnetic field can equivalently be written as

$$\mathcal{L} = -\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu \tag{6}$$

in terms of the electromagnetic field tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, the four-potential $A^\mu = (\phi, \mathbf{A})$, and the four-current $j^\mu = (\rho, \mathbf{j})$.

This expression for the Lagrangian can be deduced (up to the numerical coefficients) by demanding that (i) the action is a Lorentz scalar, (ii) the free-field term is quadratic in the velocity, i.e., in $\partial^\beta A^\alpha$, and (iii) the coupling to the source j^μ is linear in agreement with the Lagrangian of a particle in the electromagnetic field.

Problem 2: Particle in an electromagnetic field (10+5+5+5 points)

Now consider the Lagrangian of a charged particle in an electromagnetic field, described by a scalar potential $\phi(\mathbf{r},t)$ and a vector potential $\mathbf{A}(\mathbf{r},t)$,

$$L = \frac{1}{2}m\mathbf{v}^2 - e\phi(\mathbf{r},t) + e\dot{\mathbf{r}}\cdot\mathbf{A}(\mathbf{r},t) \tag{7}$$

(a) Show that the corresponding Euler-Lagrange equation takes the form

$$m\ddot{\mathbf{r}} = e\mathbf{E} + e\mathbf{v}\times\mathbf{B} \tag{8}$$

in terms of the electric field $\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}$ and the magnetic field $\mathbf{B} = \nabla\times\mathbf{A}$.

(b) Use Hamilton's principle to show that the Euler-Lagrange equations (say, for a single particle) remain invariant when adding a total time derivative to the Lagrangian,

$$L \to L + \frac{\mathrm{d}}{\mathrm{d}t}\Lambda(\mathbf{r},t). \tag{9}$$

(c) Show directly from the Lagrangian that the result formulated in (b) implies that the equations of motion are invariant under the gauge transformation

$$\phi(\mathbf{r},t) \quad\to\quad \phi(\mathbf{r},t) - \frac{\partial\Lambda(\mathbf{r},t)}{\partial t} \tag{10}$$

$$\mathbf{A}(\mathbf{r},t) \quad\to\quad \mathbf{A}(\mathbf{r},t) + \nabla\Lambda(\mathbf{r},t) \tag{11}$$

of the potentials. In addition check explicitly that this leaves the fields $\mathbf{E}$ and $\mathbf{B}$ unchanged.

(d) Derive the corresponding Hamiltonian $H(\mathbf{p},\mathbf{r})$. Is the canonical momentum $\mathbf{p}$ gauge invariant? How about the kinematic momentum $\boldsymbol{\pi} = m\dot{\mathbf{r}}$?

Problem 3: Hamiltonian formulation for electrodynamics (5+10+10 points)

Next we discuss the Hamiltonian formulation of electrodynamics, starting with the Langrangian

$$L = \int \mathrm{d}\mathbf{r}\mathcal{L} = \int \mathrm{d}\mathbf{r}\left[\frac{\mathbf{E}^2 - \mathbf{B}^2}{8\pi} - \rho\phi + \mathbf{j}\mathbf{A}\right]. \tag{12}$$

This is a necessary step in the canonical quantization of electromagnetism. In this case, quantization is not really straight-forward because of the gauge freedom of the theory.

(a) Compute the canonically conjugate momentum fields to the potentials ϕ and $\mathbf{A}$. Show that $\pi_{\mathbf{A}} = -\mathbf{E}/4\pi$ is canonically conjugate to $\mathbf{A}$ and that the canonically conjugate π_ϕ of ϕ vanishes. Use your result to derive the Hamiltonian

$$H = \int \mathrm{d}\mathbf{r}\mathcal{H} = \int \mathrm{d}\mathbf{r}\left[\frac{\mathbf{E}^2 + \mathbf{B}^2}{8\pi} + \frac{1}{4\pi}\mathbf{E}\cdot\nabla\phi + \rho\phi - \mathbf{j}\mathbf{A}\right]. \tag{13}$$

Note that this is essentially written in terms of the fields and the momenta as it does not contain any of the velocities (time derivatives of the potentials).

(b) First consider the scalar potential ϕ. As the canonically conjugate momentum vanishes, we can obtain the corresponding Hamilton equation by minimizing the time integral of the Hamiltonian over ϕ. (Remember that Hamilton's equations follow from Hamilton's principle written in phase space variables.) Show that this reproduces the Maxwell equation

$$\nabla\cdot\mathbf{E} = 4\pi\rho. \tag{14}$$

This is not a dynamical equation of the theory but rather a constraint that must be satisfied for all times.

(c) Next consider Hamilton's equations for $\mathbf{A}$ and $\pi_{\mathbf{A}}$ and relate your results to Maxwell's equations.

(d) Now let's choose the Coulomb gauge $\nabla\cdot\mathbf{A} = 0$. By choosing this gauge, we effectively fix one component of $\mathbf{A}$ so that $\mathbf{A}$ has only two independent components. Writing $\mathbf{A}$ in terms of its longitudinal and its transverse components, $\mathbf{A} = \mathbf{A}_\parallel + \mathbf{A}_\perp$ (defined[1] by $\nabla\times\mathbf{A}_\parallel = 0$ and $\nabla\cdot\mathbf{A}_\perp = 0$), we see that the

[1] A simple way to think about this is in Fourier space, where $\mathbf{A}_\parallel$ is parallel to and $\mathbf{A}_\perp$ perpendicular to $\mathbf{q}$.

gauge choice sets the parallel component to zero while the transverse component remains finite.

Since $\mathbf{A}$ has only two independent components, so should the cononically conjugate momentum. Thus, we also decompose $\mathbf{E}$ into its longitudinal and transverse parts, $\mathbf{E} = \mathbf{E}_{\parallel} + \mathbf{E}_{\perp}$ so that $\mathbf{E}_{\parallel} = -\nabla\phi$ and $\mathbf{E}_{\perp} = -\partial\mathbf{A}/\partial t$. Thus, it is the transverse part which is canonically conjugate to $\mathbf{A}$.

To discuss the longitudinal part, show that with our gauge choice, the constraint becomes

$$-\nabla^2\phi = 4\pi\rho \tag{15}$$

and can be solved as

$$\phi(\mathbf{r},t) = \int \mathrm{d}\mathbf{r}' \frac{\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|} \tag{16}$$

Now, collect the terms in the Hamiltonian which depend on $\mathbf{E}_{\parallel}$ and express them using the constraint. Show that this yields the Hamiltonian

$$H = \int \mathrm{d}\mathbf{r} \left[\frac{\mathbf{E}_{\perp}^2 + \mathbf{B}^2}{8\pi} - \mathbf{jA}\right] + \frac{1}{2}\int \mathrm{d}\mathbf{r} \int \mathrm{d}\mathbf{r}' \frac{\rho(\mathbf{r},t)\rho(\mathbf{r}',t)}{|\mathbf{r}-\mathbf{r}'|}. \tag{17}$$

This form of the Hamiltonian makes the Coulomb interaction explicit and can be used as a starting point for quantizing the electromagnetic field.