Problem Set 8
Quantum Field Theory and Many Body Physics (SoSe2015)

Due: Monday, June 8, 2015 at the beginning of the lecture

In this problem set, we study the relation between spin 1/2 and fermions. We also introduce important models of magnetism and discuss them in one spatial dimension.

Problem 1: Jordan-Wigner transformation (10+15 points)

A single fermionic state can be empty or occupied. Similarly, a spin-1/2 can point either up or down. This suggests that one might be able to map a spin-1/2 degree of freedom to a fermion mode. However, a little thought reveals that this is not so simple. The problem is that operators corresponding to different spins commute while different fermion operators anticommute. In this problem, we will establish that one can in fact find an exact and useful general mapping between spins and fermions in *one dimension* by attaching an additional string operator to a fermion. This transformation is known as Jordan-Wigner transformation.

(i) Let's start with a single spin-1/2 degree of freedom and write the spin operator in terms of a fermion operator. The spin operator is defined by

$$S^j = \frac{1}{2}\sigma^j, \tag{1}$$

where $j = x, y, z$ and σ^j denotes a Pauli matrix. The spin operators satisfy the angular-momentum algebra

$$[S^j, S^k] = i\epsilon^{jkl} S^l \tag{2}$$

and the anticommutation relations

$$\{S^j, S^k\} = \frac{1}{2}\delta^{jk}. \tag{3}$$

We will also use the usual raising and lowering operators $S^\pm = S^x \pm iS^y$.

Now we identify the spin-down state $|\downarrow\rangle$ (satisfying $S^z|\downarrow\rangle = -(1/2)|\downarrow\rangle$) with the vacuum state $|0\rangle$ for a fermion operator f (i.e., $f|0\rangle = 0$) and the spin-up state $|\uparrow\rangle$ with the occupied fermion state $|1\rangle = f^\dagger|0\rangle$. Show that we can make the identifications

$$\begin{aligned}
S^x &= \frac{1}{2}\left(f + f^\dagger\right) && (4)\\
S^y &= \frac{i}{2}\left(f - f^\dagger\right) && (5)\\
S^z &= f^\dagger f - \frac{1}{2} && (6)\\
S^+ &= f^\dagger && (7)\\
S^- &= f. && (8)
\end{aligned}$$

Confirm that these operators indeed satisfy the commutation and anticommutation relations of the spin operators.

(ii) Now consider a one-dimensional lattice with sites labeled by $j = \ldots -2, -1, 0, 1, 2 \ldots$. Define a spin operator $\mathbf{S}_j$ and a fermion operator f_j on every site. We can no longer directly use the previous mapping between spin and fermion because the spin operators belonging to different sites commute while the corresponding fermion operators anticommute. To fix this, consider the string operator

$$e^{i\phi_j} = e^{i\pi \sum_{k<j} n_k}, \tag{9}$$

where $n_k = f_k^\dagger f_k$. Explain why this is a hermitian operator. Now show that a spin can be thought of as a fermion with an attached string operator by verifying that the Jordan-Wigner transformation

$$S_j^z = f_j^\dagger f_j - \frac{1}{2} \tag{10}$$

$$S_j^+ = f_j^\dagger e^{i\phi_j} \tag{11}$$

$$S_j^- = f_j e^{-i\phi_j} \tag{12}$$

preserves the (anti)commutation relation on each site and correctly yields commuting spin operators on different sites. You may find it helpful to first show that the string operator anticommutes with each fermion operator to the left of its open end and commutes with fermion operators at or to the right of its open end,

$$[f_k, e^{i\phi_j}] = 0 \quad ; \quad k < j \tag{13}$$

$$\{f_k, e^{i\phi_j}\} = 0 \quad ; \quad k \geq j \tag{14}$$

Problem 2: Quantum XXZ model (5+10+10 points)

In this problem, we want to use the Jordan-Wigner transformation to discuss the XXZ Hamiltonian in one dimension,

$$H = -\sum_j \{J[S_j^x S_{j+1}^x + S_j^y S_{j+1}^y] + J_z S_j^z S_{j+1}^z\}. \tag{15}$$

For the isotropic case with $J_z = J$, this is known as the quantum Heisenberg model. For $J_z = 0$, the model becomes the quantum xy model. For $J, J_z > 0$, the model describes a ferromagnet in which it is energetically favorable for neighboring spins to be parallel. Antiferromagnetic coupling corresponds to $J, J_z < 0$. We can also write this Hamiltonian as

$$H = -\sum_j \left\{ \frac{J}{2}[S_j^+ S_{j+1}^- + S_{j+1}^+ S_j^-] + J_z S_j^z S_{j+1}^z \right\}. \tag{16}$$

(i) Show that the Jordan-Wigner transformation maps the XXZ Hamiltonian H to

$$H = -\frac{J}{2}\sum_j [f_{j+1}^\dagger f_j + f_j^\dagger f_{j+1}] - J_z \sum_j (n_j - \frac{1}{2})(n_{j+1} - \frac{1}{2}). \tag{17}$$

This Hamiltonian describes spinless fermions in one dimension with nearest-neighbor hopping and interactions.

(ii) First consider the xy model with $J_z = 0$. We see that this model maps to non-interacting fermions and can thus be readily solved exactly. Specifically, the xy model maps to the fermion Hamiltonian

$$H = -\frac{J}{2}\sum_j [f_{j+1}^\dagger f_j + f_j^\dagger f_{j+1}]. \tag{18}$$

Show that this simple tight-binding Hamiltonian can be diagonalized by transforming to the momentum representation (translation invariance),

$$H = \sum_k \epsilon_k \, c_k^\dagger c_k \tag{19}$$

with $\epsilon_k = -J\cos k$ and $k \in [-\pi, \pi]$. Give explicit expressions for the operators c_k. Discuss the ground state of the fermion model and its excitation spectrum. (Be sure to notice that some of the single-particle energies are negative!) Use this to compute the ground state energy of the original xy model and to explain that its excitation spectrum is characterized by a linear magnon dispersion. Does the ground state have a spontaneous magnetization?

(iii) Next consider the isotropic Heisenberg model. In this case, we cannot easily find an exact solution because of the nearest-neighbor interaction between the fermions. Nevertheless, we can discuss the properties of this model approximately. Let us first neglect the interaction term $\sum_j n_j n_{j+1}$ and discuss the resulting non–interacting Hamiltonian. First find the single-particle spectrum of this non-interacting problem. You should find that the ground state corresponds to the fermion vacuum which is equivalent to all spins pointing in the spin-down direction. Thus, we actually find a ferromagnetic ground state in this model. Show that the magnon dispersion above this ground state has a quadratic dispersion (unlike the xy model which had a linear dispersion). Now return to the interacting Hamiltonian and explain why one may be tempted to conclude that the interaction term is weak and that it may be a good approximation to neglect it.

Problem 3: Transverse field Ising model (5+10+10 points)

The simplest spin model exhibiting a quantum phase transition (i.e., a transition between different zero-temperature phases as a function of some parameter in the Hamiltonian) is the transverse field Ising model

$$H = -J \sum_j S_j^x S_{j+1}^x - h \sum_j S_j^z \tag{20}$$

in one dimension.

(i) Describe the ground states of the model qualitatively in the limits $h = 0$ (ferromagnetic phase) and $J = 0$ (paramagnetic phase). It turns out that there is a phase transition between these phases when $J = h$.

(ii) Use the Jordan-Wigner transformation to map the transverse field Ising model to a fermion model. Show that this fermion model has the structure of the Kitaev chain

$$H = -\sum_j \left[\Delta (f_j^\dagger - f_j)(f_{j+1}^\dagger + f_{j+1}) + \mu f_j^\dagger f_j \right] \tag{21}$$

(Convince yourself that the first term is actually hermitian.) This fermion model is the drosophila of the theory of topological superconductors describing a one-dimensional p-wave superconductor. Because it has so many remarkable properties, there is currently enormous interest in realizing this model experimentally (and it is quite possible that these experiments were already successful).

(iii) Let us finally discuss the quantum phases in the language of the fermion model. In the limit of large μ, we can neglect the coupling Δ between neighboring sites. In this limit, the model describes a trivial (band) insulator with all particles stuck on their sites. Now consider the limit in which Δ dominates over μ so that we can set $\mu = 0$. In this limit, we can solve the problem by writing fermion operators in terms of Majorana fermion operators,

$$f_j = \frac{1}{2}(\gamma_{2j} + i\gamma_{2j+1}). \tag{22}$$

These Majorana operators are hermitian satisfying $\gamma = \gamma^\dagger$ (unlike the fermionic operators f and $f^\dagger$). Explain why one can always find such a decomposition. Show that the Majorana operators satisfy the algebra

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij}. \tag{23}$$

Show also that for any two Majorana operators γ_1 and γ_2 and the associated fermion operator $f = (1/2)(\gamma_1 + i\gamma_2)$, we have the relation $f^\dagger f = (1/2)(1 + i\gamma_1\gamma_2)$ or $i\gamma_1\gamma_2 = 2f^\dagger f - 1$. These are basic relations which are often needed for diagonalizing Majorana Hamiltonians. Finally show that you can diagonalize the Kitaev chain for $\mu = 0$ by recombining Majorana operators from neighboring sites into new fermion operators d_j. Thus, in the fermionic language, the phase transition of the transverse field Ising model becomes a dimerization transition of these Majorana fermion operators.