Problem Set 1
Quantum Field Theory and Many Body Physics (SoSe2015)

Due: Monday, April 20, 2015 before the beginning of the class

One of the most important techniques of quantum field theory are Gaussian integrals. In this problem set, we want to discuss Gaussian integrals over real variables. We will use the results over and over again throughout the class. The importance of doing this problem set carefully cannot be overstated.

Many Gaussian integrals of quantum field theory are functional integrals over continuous fields. However, in many cases we can reduce these integrals to invole only a finite and discrete number of integration variables. The simplest way is to put the field theory on a lattice, say with periodic boundary conditions. Then, there are only a finite number of lattice points at which the field is defined. This is the case that we want to discuss at length in this problem set. It actually turns out that ignoring mathematical subtleties, appropriate versions of our results remain valid for integrals over continuous fields.

Problem 1: Gaussian integrals (5+5+10 points)

It is actually very instructive to start with the very simplest case of Gaussian integrals over a *single* variable. This will be done in this problem.

(a) Let's begin with showing that

$$\int_{-\infty}^{\infty} \mathrm{d}x \exp\left(-\frac{1}{2}ax^2\right) = \sqrt{\frac{2\pi}{a}} \tag{1}$$

for any $a > 0$. You may remember that the trick to do this integral is to consider its square and to introduce polar coordinates.

(b) Now add a linear term to the exponent. This integral can be reduced to the previous one by completing the square in the exponent and shifting the integration variable. Do this to find

$$\int_{-\infty}^{\infty} \mathrm{d}x \exp\left(-\frac{1}{2}ax^2 + bx\right) = \sqrt{\frac{2\pi}{a}} \exp\left(\frac{b^2}{2a}\right). \tag{2}$$

(c) The last integral can be used to compute many more integrals. Starting from this integral, show that

$$\int_{-\infty}^{\infty} \mathrm{d}x x^n \exp\left(-\frac{1}{2}ax^2\right) = \sqrt{\frac{2\pi}{a}} \times \left\{ \begin{array}{cc} 0 & n \text{ odd} \\ \frac{(n-1)!!}{a^{n/2}} & n \text{ even} \end{array} \right. . \tag{3}$$

Here, $(n-1)!! = 1 \cdot 3 \cdot 5 \cdot \ldots \cdot (n-1)$ denotes the product over all odd numbers up to $(n-1)$. The basic idea is to use the variable b in Eq. (2) as a "source field," i.e., to take n derivatives with respect to b on both sides of Eq. (2) and to subsequently set $b = 0$. Since the integral is highly convergent, one can freely interchange integration and differentiation on the left-hand side. [Perhaps the simplest way to perform the derivatives of the right-hand side is to expand $\exp(b^2/2a)$ into a Taylor series and to take the derivatives term by term. (Only a single term contributes for $b = 0$.)]

We can look at Eq. (3) in another way by treating the Gaussian factor as a probability distribution. Then, we can define the averages

$$\langle x^n \rangle = \frac{\int_{-\infty}^{\infty} \mathrm{d}x x^n \exp\left(-\frac{1}{2}ax^2\right)}{\int_{-\infty}^{\infty} \mathrm{d}x \exp\left(-\frac{1}{2}ax^2\right)} \tag{4}$$

and Eq. (3) gives the result

$$\langle x^n \rangle = \left\{ \begin{array}{cc} 0 & n \text{ odd} \\ \frac{(n-1)!!}{a^{n/2}} & n \text{ even} \end{array} \right. . \tag{5}$$

We will see below that the combinatorial factor $(n-1)!!$ is just the number of ways in which one can pair the n factors of x. The first x can be paired with $n-1$ other factors. One of the remaining $n-2$ factors can then be paired with $n-3$ factors and so on. Then, we can write this as

$$\langle x^n \rangle = (\text{number of ways to pair } n \text{ factors of } x) \times \langle x^2 \rangle^{n/2}. \tag{6}$$

for even n. This is an example of a very general result for Gaussian probability distributions which is known as Wick's theorem and which underlies the derivation of Feynman diagrams.

Problem 2: Multidimensional Gaussian integrals (10+10+5+5 points)

Now, we want generalize the results of the first problem to Gaussian integrals over N variables $x_1, \ldots, x_N$. For notational simplicity, it is sometimes convenient to collect these N variables into a vector $\mathbf{x} = (x_1, x_2, \ldots, x_N)$.

(a) First, consider the integral

$$\int \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_N \exp\left(-\frac{1}{2}\sum_{i,j=1}^{N} M_{ij} x_i x_j\right), \tag{7}$$

where the integrals over all x_j extend from $-\infty$ to ∞. We can consider the real M_{ij} as the entries of a symmetric (why?) matrix M. Then, we can write the quadratic form in the exponent in vector notation as $\sum_{i,j=1}^{N} M_{ij} x_i x_j = \mathbf{x}^T \mathrm{M} \mathbf{x}$. Now, show that

$$\int [\mathrm{d}\mathbf{x}] \exp\left(-\frac{1}{2}\mathbf{x}^T \mathrm{M}\mathbf{x}\right) = \sqrt{\frac{(2\pi)^N}{\det \mathrm{M}}}, \tag{8}$$

where we introduced the shorthand $[\mathrm{d}\mathbf{x}] = \mathrm{d}x_1 \mathrm{d}x_2 \ldots \mathrm{d}x_N$.

The basic idea to do this integral is to diagonalize $\mathrm{M} = \mathrm{O}^T \Lambda \mathrm{O}$, where the matrix Λ is a diagonal and O an orthogonal matrix. Now, introduce new integration variables such that the integral decouples into N independent one-dimensional Gaussian variables. Be sure to consider the Jacobian of this change of integration variables. Also discuss the conditions that the matrix M has to satisfy for the integral to be well defined.

(b) Next, we introduce a linear term $\mathbf{J}^T \mathbf{x} = \sum_{j=1}^{N} J_j x_j$ into the exponent with a "source field" $\mathbf{J} = (J_1, J_2, \ldots, J_N)$. Show that the resulting integral becomes

$$\int [\mathrm{d}\mathbf{x}] \exp\left(-\frac{1}{2}\mathbf{x}^T \mathrm{M}\mathbf{x} + \mathbf{J}^T \mathbf{x}\right) = \sqrt{\frac{(2\pi)^N}{\det \mathrm{M}}} \exp\left(\frac{1}{2}\mathbf{J}^T \mathrm{M}^{-1} \mathbf{J}\right). \tag{9}$$

To show this, you can follow closely the corresponding calculation for the single-variable Gaussian integral in Eq. (2). The only difference is that you have to be careful with the ordering of factors in the present case because matrix multiplications do not commute.

(c) The result in Eq. (9) can again be used to compute many other integrals. Let us introduce averages by

$$\langle \ldots \rangle = \frac{\int [\mathrm{d}\mathbf{x}] \ldots \exp\left(-\frac{1}{2}\mathbf{x}^T \mathrm{M}\mathbf{x}\right)}{\int [\mathrm{d}\mathbf{x}] \exp\left(-\frac{1}{2}\mathbf{x}^T \mathrm{M}\mathbf{x}\right)} \tag{10}$$

as in problem 1. Here, $\ldots$ stands for the function of the x_j which we want to average. Now, use Eq. (9) to show that

$$\langle x_i x_j \rangle = [M^{-1}]_{ij}. \tag{11}$$

To derive this result, you should again take derivatives of Eq. (9) with respect to appropriate elements of $\mathbf{J}$. This is another very important result that we will use over and over again.

(d) Finally, you are in a position to derive Wick's theorem

$$\langle x_{j_1} x_{j_2} \dots x_{j_{2n}} \rangle = \sum_P \langle x_{P_1} x_{P_2} \rangle \dots \langle x_{P_{2n-1}} x_{P_{2n}} \rangle. \tag{12}$$

Here, P refers to the set of all distinct pairings $(P_1, P_2), (P_3, P_4), \dots (P_{2n-1}, P_{2n})$ of the indices $j_1, j_2, \dots, j_{2n}$. One of these pairings is for instance $(j_1, j_2), (j_3, j_4), \dots (j_{2n-1}, j_{2n})$, another $(j_1, j_{2n}), (j_2, j_{2n-1}), \dots (j_n, j_{n+1})$. As explained above, there are altogether $(2n-1)!!$ distinct pairings.