Topology

Xiaolong Han

Department of Mathematics, California State University, Northridge, CA 91330, USA

Email address: xiaolong.han@csun.edu

Remark. You are entitled to a reward of 1 point toward a homework assignment if you are the first person to report a bona-fide mathematical mistake (i.e. not including language typos and grammatical errors.)

Contents

CHAPTER 1

Preliminaries

1.1. Sets and functions

We speak the language of sets and functions (or mappings) in mathematics.

Definition (Sets).

- Given a set A, $x \in A$ denotes that x is an element (or member, point) of A and $x \notin A$ denotes that x is not an element of A.
- We say that two sets A and B are equal, denoted by $A = B$, if they have the same elements.
- Given two sets A and B, we say that A is a subset of B, denoted by $A \subset B$ (or $A \subseteq B$), if each element of A is a member of B; we say that A is a proper subset of B, denoted by $A \subsetneq B$, if $A \subseteq B$ and $A \neq B$.

Remark. Given two sets A and B, $A = B$ if and only if (i.e. iff) $A \subset B$ and $B \subset A$.

Example.

- $\mathbb{N} = \{1, 2, 3, ...\}$ denotes the set of natural numbers. (Notice that in some other books, $\mathbb{N}$ refers to the set $\{0, 1, 2, 3, ...\}$.)
- $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$ denotes the set of integers.
- $\mathbb{Q} = \{p/q : p, q \in \mathbb{Z}, q \neq 0\}$ denotes the set of rational numbers.
- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{R} \setminus \mathbb{Q}$ is called the set of irrational numbers.

Definition (The empty set). The set that has no elements is called the empty set and is denoted by $\emptyset$. A set that is not equal to the empty set is called nonempty.

Definition (A singleton set). A set that contains a single element is called a singleton set.

Definition (Cartesian products). The Cartesian product $A_1 \times \cdots \times A_m$ of m sets $A_1, ...A_m$ is the set of all m-tuples $(x_1, ..., x_m)$ such that $x_i \in A_i$, $i = 1, ..., m$. That is,

$$A_1 \times \cdots \times A_m = \{(x_1, ..., x_m) : x_i \in A_i, i = 1, ..., m\}.$$

Definition (Relations). A relation on a set A is a subset C of the Cartesian product $A \times A$. Let C be a relation on a set A. We denote as xCy if $(x, y) \in C$ and say that x is in the relation C to y.

Example (Equivalence relations). An equivalence relation on a set A is a relation C that satisfies

(i). [Reflexivity] xCx for every $x \in A$.
(ii). [Symmetry] If xCy, then yCx.
(iii). [Transitiity] If xCy and yCz, then xCz.

In this case, we say x is equivalent to y if xCy and also denote by $x \sim y$.

Definition (Union, intersection, and difference). Let A and B be two sets.

- The union of A and B is $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
- The intersection of A and B is $A \cap B = \{x : x \in A \text{ and } x \in B\}$. We say that A and B are disjoint if $A \cap B = \emptyset$.
- The set difference of A and B is $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$, and is also denoted by $A - B$. It is sometimes called the complement of B relative to A. In particular, if all the set operations are within a universal set X, then for a set $A \subset X$, we simply call $X \setminus A$ the complement of A, and is also denoted by A^c.

Remark. Given a family of sets $\mathcal{F}$, we define

$$\bigcup_{E \in \mathcal{F}} E = \{x : x \in E \text{ for some } E \in \mathcal{F}\},$$

and

$$\bigcap_{E \in \mathcal{F}} E = \{x : x \in E \text{ for all } E \in \mathcal{F}\}.$$

In particular, if $\mathcal{F} = \{E_\lambda\}_{\lambda \in \Lambda}$, where λ is called the index and Λ is called the index set, then we define

$$\bigcup_{\lambda \in \Lambda} E_\lambda = \{x : x \in E_\lambda \text{ for some } \lambda \in \Lambda\},$$

and

$$\bigcap_{\lambda \in \Lambda} E_\lambda = \{x : x \in E_\lambda \text{ for all } \lambda \in \Lambda\}.$$

For example,

$$\bigcup_{i=1}^{n} E_i = \{x : x \in E_i \text{ for some } i = 1, ..., n\},$$

and

$$\bigcap_{i=1}^{n} E_i = \{x : x \in E_i \text{ for all } i = 1, ..., n\}.$$

THEOREM (De Morgan's Law). *Let $\mathcal{F}$ be a family of sets. Then*

$$\left(\bigcup_{E \in \mathcal{F}} E\right)^c = \bigcap_{E \in \mathcal{F}} E^c \quad \text{and} \quad \left(\bigcap_{E \in \mathcal{F}} E\right)^c = \bigcup_{E \in \mathcal{F}} E^c.$$

Definition (Functions). Let A and B be two sets. A function f from A to B, denoted by $f : A \to B$, is a correspondence that assigns to each element of A an element of B; for $x \in A$, we denote by $f(x)$ the assigned element in B.

Definition (Domain and range). Let f be a function from A to B. We call A the domain of f and B the codomain of f. Given a subset $E \subset A$, we define $f(E) = \{y \in B : y = f(x) \text{ for some } x \in E\}$ the image of E. We call $f(A)$ the range of f.

Definition (Preimage). Let f be a function from A to B. Given a subset $E \subset B$, we define $f^{-1}(E) = \{x \in A : f(x) \in E\}$ the preimage of E.

Notice that f^{-1} is not necessarily a function. In order to make f^{-1} a function, we need the following concepts.

Definition (Injective and surjective functions). Let f be a function from A to B.

- We say that f is injective (or one-to-one) if $x, y \in A$ and $x \neq y$ imply $f(x) \neq f(y)$.
- We say that f is surjective (or onto) if $f(A) = B$.

Definition (Bijective functions). Let f be a function from A to B. We say that f is bijective if f is injective and surjective. In this case, we say that A and B are equipotent.

Let f be a bijective function from A to B. Given each element $y \in B$, there is exactly one element $x \in A$ such that $f(x) = y$ and we denote by $f^{-1}(y) = x$. This defines a function $f^{-1} : B \to A$ and we call f^{-1} the inverse of f.

Problems.

1-1.[i] Let $f : A \to B$ be a function. Suppose that $B_1, B_2 \subset B$. Prove that
(1). If $B_1 \subset B_2$, then $f^{-1}(B_1) \subset f^{-1}(B_2)$.
(2). $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.
(3). $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.
(4). $f^{-1}(B_1 \setminus B_2) = f^{-1}(B_1) \setminus f^{-1}(B_2)$.

1-2.[ii] Let $f : A \to B$ be a function. Suppose that $A_1, A_2 \subset A$. Prove that
(1). If $A_1 \subset A_2$, then $f(A_1) \subset f(A_2)$.
(2). $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.
(3). $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$ and the equality holds if f is injective.
(4). $f(A_1 \setminus A_2) \supset f(A_1) \setminus f(A_2)$ and the equality holds if f is injective.

1-3. Let $f : A \to B$ be a function. Suppose that $A_1, A_2 \subset A$. Are $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$ and $f(A_1 \setminus A_2) = f(A_1) \setminus f(A_2)$? Prove your assertion.

1.2. The real number system

The real number system $\mathbb{R}$ is an example of a complete ordered field. We define its structure in three steps:

Step 1: $\mathbb{R}$ is a field.

Definition (Fields). A field F is a nonempty set together with two operators $+$ and $\cdot$, called addition and multiplication, which satisfy the following axioms.

(A0). The operations $+$ and $\cdot$ are binary operations, that is, if $a, b \in F$, then $a + b$ and $a \cdot b$ are uniquely determined elements of F.
(A1). Commutativity of addition: If $a, b \in F$, then $a + b = b + a$.
(A2). Associativity of addition: If $a, b, c \in F$, then $(a + b) + c = a + (b + c)$.
(A3). The additive identity: There is an element in F, denoted by 0, such that $0+a = a+0 = a$ for all $a \in F$.
(A4). The additive inverse: For each $a \in F$, there is an element $b \in F$, called the additive inverse of a and denoted by $-a$, such that $a + b = 0$.
(A5). Commutativity of multiplication: If $a, b \in F$, then $a \cdot b = b \cdot a$.
(A6). Associativity of multiplication: If $a, b, c \in F$, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
(A7). The multiplicative identity: There is an element in F, denoted by 1, such that $1 \cdot a = a \cdot 1 = a$ for all $a \in F$.
(A8). The multiplicative inverse: For each $a \in F$ and $a \neq 0$, there is an element $b \in F$, called the multiplicative inverse of a and denoted by a^{-1} or $1/a$, such that $a \cdot b = 1$.
(A9). The distributive property: If $a, b, c \in F$, then $a \cdot (b + c) = a \cdot b + a \cdot c$.

Remark. We will simply denote $a + (-b)$ by $a - b$, the multiplication $a \cdot b$ by ab, and $a \cdot b^{-1}$ by a/b.

[i] This problem shows that f^{-1} preserves inclusions, unions, intersections, and differences of sets.

[ii] This problem shows that f preserves inclusions and unions of sets; if in addition f is injective, then it also preserves intersections and differences of sets.

Question. Among $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{R} \setminus \mathbb{Q}$, which one(s) are fields, why not for $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R} \setminus \mathbb{Q}$, and why for $\mathbb{Q}$ and for $\mathbb{R}$?

After Step 1, we can do addition, subtraction (as added by the additive inverse), multiplication, and division (as multiplied by the multiplicative inverse) on $\mathbb{R}$, according to all the rules that we knew before.

Step 2: $\mathbb{R}$ is an ordered field.

Definition (Order relations). An order relation (or a simple order, or a linear order) on a set A is a relation C that satisfies

(i). [Comparability] For every $x, y \in A$ and $x \neq y$, either xCy or yCx.
(ii). [Nonreflexivity] There is no $x \in A$ such that xCx.
(iii). [Transitiity] If xCy and yCz, then xCz.

In this case, we also denote as $x < y$ and also $y > x$ if xCy.

Definition (Ordered field). Let F be a field. Then F is an ordered field if

(A10). it has an order relation $<$ that satisfies
  (i). If $x < y$, then $x + z < y + z$.
  (ii). If $x < y$ and $0 < z$, then $xz < yz$.

Question. Among $\mathbb{Q}$ and $\mathbb{R}$, which one(s) are ordered fields?

After Step 2, we can compare the greatness of elements in $\mathbb{R}$. We can also define the intervals in $\mathbb{R}$. In fact, we can define the intervals in any set equipped with an order relation.

Definition (Intervals). Given two numbers $a, b \in \mathbb{R}$, we define an interval as one of the following sets.

$$\begin{aligned} (a,b) &= \{x : a < x < b\}, & (a,b] &= \{x : a < x \leq b\}, \\ [a,b) &= \{x : a \leq x < b\}, & [a,b] &= \{x : a \leq x \leq b\}, \\ (-\infty, a) &= \{x : x < a\}, & (-\infty, a] &= \{x : x \leq a\}, \\ (a, \infty) &= \{x : x > a\}, & [a, \infty) &= \{x : x \geq a\}. \end{aligned}$$

Definition (Absolute value). Let F be an ordered field. We define the absolute value $|a|$ of an element $a \in F$ by

$$|a| = \begin{cases} a & \text{if } x \geq 0, \\ -a & \text{if } x < 0. \end{cases}$$

Step 3: $\mathbb{R}$ is a complete ordered field.

There are several ways to define completeness equivalently. Here it is done through least upper bounds (or greatest lower bounds).

Definition (Upper and lower bounds). Let F be an ordered field.

- A set E in F is said to be bounded above provided there is an element $b \in F$ such that $x \leq b$ for all $x \in E$; the number b is called an upper bound of E.
- A set E in F is said to be bounded below provided there is an element $b \in F$ such that $x \geq b$ for all $x \in E$; the number b is called an lower bound of E.
- A set E in F is said to be bounded if E is bounded above and is bounded below.

Remark. If a set of real numbers E is not bounded above, then we denote $\sup E = \infty$; if a set of real numbers E is not bounded below, then we denote $\inf E = -\infty$.

Definition (Supremum and infimum).

- Let E be a set that is bounded above. We say b is the supremum (or least upper bound) of the set E, denoted by $\sup E$, if b is an upper bound of E and $b \leq c$ for any upper bound c of E.
- Let E be a set that is bounded below. We say b is the infimum (or greatest lower bound) of the set E, denoted by $\inf E$, if b is a lower bound of E and $b \geq c$ for any lower bound c of E.

THEOREM 1.1.

(i). A number $a = \sup E$ for a set of real numbers E iff a is an upper bound of E and for any $\varepsilon > 0$, there exists $x(\varepsilon) \in E$ such that $x(\varepsilon) > a - \varepsilon$.

(ii). A number $a = \inf E$ for a set of real numbers E iff a is a lower bound of E and for any $\varepsilon > 0$, there exists $x(\varepsilon) \in E$ such that $x(\varepsilon) < a + \varepsilon$.

Definition (Completeness by supremum). Let F be an ordered field. We say F is complete if it satisfies the following axiom.

(A11). For any subset E of F that is bounded above, there exists $\sup E \in F$.

Remark. Equivalently, we can define completeness by infimum: Let F be an ordered field. We say F is complete if for any subset E of F that is bounded below, there exists $\inf E \in F$. See Problem 1-5 to transform between supremum and infimum.

Question. Among $\mathbb{Q}$ and $\mathbb{R}$, which one(s) are complete ordered fields, why not for $\mathbb{Q}$ and why for $\mathbb{R}$?

Question. Let $E = \{x \in \mathbb{Q} : x^2 < 2\}$. Is $\sup E \in \mathbb{Q}$? In fact, we define $\sqrt{2}$ as a "new" real number (which is irrational.)

After Step 3, we can do limits on $\mathbb{R}$.

THEOREM 1.2. *Between any two distinct real numbers, there is a rational number and an irrational number.*

Question. Prove that each real number is the supremum of a set of rational number.

PROOF. If x is rational, then $x = \sup E$ for $E = \{x\}$. If x is irrational, then there is a rational number $x_1 \in (x-1, x)$ by the above theorem; there is a rational number $x_2 \in (x - 1/2, x)$ by the above theorem again. Inductively, there is a set of rational numbers $E = \{x_1, x_2, ...\}$ such that $x_n \in (x - 1/n, x)$. One can see that $\sup E = x$. □

Problems.

1-4. For a nonempty set of real numbers E, prove that $\inf E = \sup E$ iff E consists of a single point.

1-5. Use the completeness axiom to prove that every nonempty set of real numbers E that is bounded below has an infimum and that

$$\inf E = -\sup\{-x : x \in E\}.$$

1-6. Prove that each real number is the supremum of a set of irrational numbers.

1.3. Cardinality

Recall that two sets A and B are equipotent if there is a bijection between A and B, in this case, we say that A and B have the same cardinality, denoted by $\mathrm{Card}\,(A) = \mathrm{Card}\,(B)$. In fact,

Definition. Let A and B be two sets.

- If there is a function $f : A \to B$ that is surjective, then we say that the cardinality of A is greater than or equal to the cardinality of B, denoted by $\text{Card}\,(A) \geq \text{Card}\,(B)$;
- If there is a function $f : A \to B$ that is injective, then we say that the cardinality of A is less than or equal to the cardinality of B, denoted by $\text{Card}\,(A) \leq \text{Card}\,(B)$;
- $\text{Card}\,(A) > \text{Card}\,(B)$ if $\text{Card}\,(A) \geq \text{Card}\,(B)$ and $\text{Card}\,(A) \neq \text{Card}\,(B)$;
- $\text{Card}\,(A) < \text{Card}\,(B)$ if $\text{Card}\,(A) \leq \text{Card}\,(B)$ and $\text{Card}\,(A) \neq \text{Card}\,(B)$.

Cardinality generalizes the simple concept of "number of elements of a set".

Definition (Finite, countable, uncountable sets).

- A set A is said to be finite if either it is empty or there is a natural number n for which E is equipotent to the set $\{1, ..., n\}$.
- A set A is said to be countably infinite if it is equipotent to $\mathbb{N}$.
- A set A is said to be countable if it is either finite or countable infinite, that is, it is equipotent with a subset of $\mathbb{N}$.
- A set A is said to be uncountable if it is not countable.

A set E is finite iff its element can be enumerated as $x_1 = f(1)$, ..., $x_n = f(n)$ by the bijection $f : \{1, ..., n\} \to E$; E is countably infinite iff its elements can be enumerated by $x_1 = f(1)$, ..., $x_n = f(n)$, ..., by the bijection $f : \mathbb{N} \to E$.

Question. Examples of countable and uncountable sets? In particular, are $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}\backslash\mathbb{Q}$ countable?

THEOREM 1.3. *Any subset of a countable set is countable. In particular, any set of natural numbers is countable.*

Definition (The power set). Given a set A, the set of all the subsets of A is called the power set of A and is denoted by $\mathcal{P}(A)$.

Example. Let $A = \{1, 2, 3\}$. Then

$$\mathcal{P}(A) = \big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\big\}.$$

Remark. If A has N elements, then $\mathcal{P}(A)$ has 2^N elements, which is strictly greater than N.

THEOREM 1.4.

(i). The Cartesian product of countable sets is countable.
(ii). The countable union of countable sets is countable.
(iii). For any set A, $\text{Card}\,(\mathcal{P}(A)) > \text{Card}\,(A)$. Here, $\mathcal{P}(A)$ is the power set of A.

Example. $\text{Card}\,(\mathcal{P}(\mathbb{N})) = \text{Card}\,(\mathbb{R})$.

THEOREM 1.5. *A nonempty interval of real numbers is uncountable. In particular, $\mathbb{R}$ is uncountable.*

Problems.

1-7. Let $a, b, c, d \in \mathbb{R}$, $a < b$, and $c < d$. Prove that the two intervals (a, b) and (c, d) are equipotent.

CHAPTER 2

Topological spaces and continuous functions

2.1. Topological spaces

Definition (Topological spaces). A topology $\mathcal{T}$ on a set X is a collection of subsets of X that satisfies

(i). $\emptyset, X \in \mathcal{T}$.

(ii). The union of the elements in any subcollection of $\mathcal{T}$ is in $\mathcal{T}$. That is, for any $\mathcal{F} \subset \mathcal{T}$,

$$\bigcup_{E \in \mathcal{F}} E \in \mathcal{T}.$$

(iii). The intersection of the elements of any finite subcollection of $\mathcal{T}$ is in $\mathcal{T}$. That is, for any $E_1, ..., E_n \in \mathcal{T}$,

$$\bigcap_{i=1}^{n} E_i \in \mathcal{T}.$$

A set X for which a topology $\mathcal{T}$ has been specified is called a topological space. The elements in $\mathcal{T}$ are called the open sets.

Example (Discrete topology and trivial topology). Let X be a set. The discrete topology is its power set $\mathcal{P}(X)$; the trivial (or indiscrete) topology is $\{\emptyset, X\}$.

Question. Provide a topology on $X = \{a, b, c\}$. Observe the many topologies on such a set.

Example (Finite complement topology). Let X be a set. Let $\mathcal{T}_f$ be the collection of all subsets $U \subset X$ such that U^c is finite or is all of X. Then $\mathcal{T}_f$ is a topology on X, called the finite complement topology.

Definition. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on a set X. We say that $\mathcal{T}_2$ is finer (resp. strictly finer) than $\mathcal{T}_1$ if $\mathcal{T}_2 \supset \mathcal{T}_1$ (resp. $\mathcal{T}_2 \supsetneq \mathcal{T}_1$). In this case, we also say that $\mathcal{T}_1$ is coarser (resp. strictly coarser) than $\mathcal{T}_2$. We say that $\mathcal{T}_1$ and $\mathcal{T}_2$ are comparable if $\mathcal{T}_1 \subset \mathcal{T}_2$ or $\mathcal{T}_2 \subset \mathcal{T}_1$.

Definition (Basis). Let X be a set. A basis for a topology on X is a collection $\mathcal{B}$ of subsets of X (called the basis elements) such that

(i). For each $x \in X$, there is at least one basis element B containing x.

(ii). If x belongs to the intersection of two basis elements B_1 and B_2, then there is a basis elements B_3 containing x such that $B_3 \subset B_1 \cap B_2$.

If $\mathcal{B}$ satisfies these two conditions, then we define the topology $\mathcal{T}$ generated by $\mathcal{B}$ as follows: A subset $U \subset X$ is said to be open in X (that is, to be an element of $\mathcal{T}$) if for each $x \in U$, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Note that each basis element is itself an element of $\mathcal{T}$.

Example (Topology generated by basis). The discrete topology is generated by the basis B as the collection of all singleton subsets of X.

Example. Let $\mathcal{B}_1$ be the collection of all circular regions (interiors of circles) in the plane. Then $\mathcal{B}_1$ is a basis and generates a topology $\mathcal{T}_1$ on the plane. That is, a subset U of the plane is open if every $x \in U$ lies in some circular region contained in U.

Let $\mathcal{B}_2$ be the collection of all rectangular regions (interiors of rectangles) in the plane. Then $\mathcal{B}_2$ is a basis and generates a topology $\mathcal{T}_2$ on the plane. That is, a subset U of the plane is open if every $x \in U$ lies in some rectangular region contained in U.

In fact, $\mathcal{T}_1 = \mathcal{T}_2$ (by Proposition 2.3). This shows that the basis of a topology is not unique.

Proof of $\mathcal{T}$ generated by a basis X is a topology.

(i). It is vacuously true that $\emptyset$; it is also true that $X \in \mathcal{T}$ by Condition (i) in the definition of basis.

(ii). Let $\mathcal{F} \subset \mathcal{T}$ and $x \in \cup_{U\in\mathcal{F}} U$. Then there exists $U \in \mathcal{F}$ such that $x \in U$. Since U is open, there is a basis element $B \in \mathcal{B}$ such that $x \in B$ and $B \subset U$. Hence, $\cup_{U\in\mathcal{F}} U \supset B$ and therefore is open.

(iii). We prove by induction.

First we show the intersection of two open sets is open. Let $x \in U_1 \cap U_2$ for some $U_1, U_2 \in \mathcal{T}$. Then there exist basis elements $B_1 \subset U_1$ and $B_2 \subset U_2$ such that $x \in B_1$ and $x \in \mathcal{B}_2$. Hence, $x \in B_1 \cap B_2$. By Condition (ii) in the definition of basis, there is an basis element $B_3 \subset B_1 \cap B_2$ such that $x \in B_3$. Therefore, $U_1 \cap U_2 \supset B_3$ and is then open.

Suppose that the intersection of $n-1$ open sets is open. Since

$$U_1 \cap \cdots \cap U_n = (U_1 \cap \cdots \cap U_{n-1}) \cap U_n,$$

in which $U_1 \cap \cdots \cap U_{n-1}$ is open by induction, $U_1 \cap \cdots \cap U_n$ is open by the first step.

□

Remark (Principle of mathematical induction). For each natural number n, let $S(n)$ be some mathematical assertion. Suppose that $S(1)$ is true and $S(k)$ is true implies that $S(k+1)$ is also true for all natural numbers (or the statement that $S(1)$,...,$S(k-1)$ are all true implies that $S(k+1)$ is true.) Then $S(n)$ is true for all natural numbers.

Proposition 2.1. *Let $\mathcal{B}$ be a basis on a set X. Then the topology $\mathcal{T}$ generated by $\mathcal{B}$ is the collection of all unions of elements of $\mathcal{B}$.*

Proof. Since $\mathcal{B} \subset \mathcal{T}$, it is then clear that any union of basis elements is in $\mathcal{T}$. On the other hand, let U be open and $x \in U$, there is a basis element B_x such that $B_x \ni x$ and $B_x \subset U$. Observe that $\cup_{x\in U} B_x = U$ (Both inclusions are immediate.) and the proof is finished. □

Proposition 2.2. *Let $\mathcal{T}$ be a topology on a set X. Suppose that $\mathcal{C}$ is the collection of open sets such that for each open set U of X and each $x \in U$, there is an element C of $\mathcal{C}$ such that $x \in C \subset U$. Then $\mathcal{C}$ is a basis of $\mathcal{T}$.*

Proof. See Problem 2-1. □

Proposition 2.3. *Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two bases for topologies $\mathcal{T}_1$ and $\mathcal{T}_2$ on a set X, receptively. Then the following statements are equivalent.*

(i). $\mathcal{T}_2$ is finer than $\mathcal{T}_1$.

(ii). For each $x \in X$ and each basis element $B_1 \in \mathcal{B}_1$ containing x, there is a basis element $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subset B_1$.

Proof.

(ii)⇒(i). Let $U \in \mathcal{T}_1$. For any $x \in U$, there exists $B_1 \in \mathcal{B}_1$ such that $B_1 \subset U$ since $\mathcal{B}_1$ generates $\mathcal{T}_1$. By Condition (ii), there exists $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subset B_1$. Hence, $U \in \mathcal{T}_2$ since $\mathcal{B}_2$ generates $\mathcal{T}_2$.

(i)⇒(ii). For each $x \in X$ and each basis element $B_1 \in \mathcal{B}_1$ containing x, $B_1 \in \mathcal{T}_1$ and hence $B_1 \in \mathcal{T}_2$ since $\mathcal{T}_2$ is finer than $\mathcal{T}_1$. Because $\mathcal{B}_2$ generates $\mathcal{T}_2$, there is $B_2 \in \mathcal{B}_2$ such that $x \in B_2 \subset B_1$.

□

Example (Topologies on $\mathbb{R}$). We define several topologies on $\mathbb{R}$:

(1). Let $\mathcal{B}_1 = \{(a, b) : -\infty < a < b < \infty\}$. The topology generated by $\mathcal{B}_1$ is called the standard topology on $\mathbb{R}$ and is denoted as $\mathbb{R}$ (if no confusion is caused). Check that only the intervals of the forms (a, b), (a, ∞), and $(-\infty, a)$ are open in this topology.

(2). Let $\mathcal{B}_2 = \{[a, b) : -\infty < a < b < \infty\}$. The topology generated by $\mathcal{B}_2$ is called the lower limit topology on $\mathbb{R}$ and is denoted as $\mathbb{R}_l$. (It is sometimes called the Sorgenfrey line.)

(3). Let $K = \{1/n : n \in \mathbb{N}\}$ and $\mathcal{B}_3 = \{(a, b) : -\infty < a < b < \infty\} \cup \{(a, b) \setminus K : -\infty < a < b < \infty\}$. The topology generated by $\mathcal{B}_3$ is called the K-topology on $\mathbb{R}$ and is denoted as $\mathbb{R}_K$.

(4). Let $\mathcal{B}_4 = \{(a, b] : -\infty < a < b < \infty\}$. The topology generated by $\mathcal{B}_4$ is called the upper limit topology on $\mathbb{R}$ and is denoted as $\mathbb{R}_u$.

We have that $\mathbb{R} \subsetneq \mathbb{R}_l$. First, $\mathbb{R} \subset \mathbb{R}_l$: For each $x \in \mathbb{R}$ and each basis element $(a, b) \in \mathcal{B}_1$ containing x, there is a basis element $[x, b) \in \mathcal{B}_2$ such that $x \in [x, b) \subset (a, b)$. Second, $\mathbb{R}_l \not\subset \mathbb{R}$: For each $x \in \mathbb{R}$ and the basis element $[x, b) \in \mathcal{B}_2$ containing x, there is no basis element $(a, b) \in \mathcal{B}_1$ such that $x \in (a, b) \subset [x, b)$.

We have that $\mathbb{R} \subsetneq \mathbb{R}_K$. First, $\mathbb{R} \subset \mathbb{R}_K$: For each $x \in \mathbb{R}$ and each basis element $(a, b) \in \mathcal{B}_1$ containing x, there is a basis element $(a, b) \in \mathcal{B}_3$ such that $x \in (a, b) \subset (a, b)$. Second, $\mathbb{R}_K \not\subset \mathbb{R}$: For the basis element $(-1, 1) \setminus K \in \mathcal{B}_3$ containing 0, there is no basis element $(a, b) \in \mathcal{B}_1$ such that $0 \in (a, b) \subset (-1, 1) \setminus K$.

We have that $\mathbb{R}_l$ and $\mathbb{R}_u$ are not comparable. First, $\mathbb{R}_u \not\subset \mathbb{R}_l$: For each $x \in \mathbb{R}$ and each basis element $(c, x] \in \mathcal{B}_4$ containing x, there is no basis element $[a, b) \in \mathcal{B}_2$ such that $x \in [a, b) \subset (c, x]$. Second, $\mathbb{R}_l \not\subset \mathbb{R}_u$: For each $x \in \mathbb{R}$ and the basis element $[x, c) \in \mathcal{B}_2$ containing x, there is no basis element $(a, b] \in \mathcal{B}_4$ such that $x \in (a, b] \subset [x, c)$.

We have that $\mathbb{R}_l$ and $\mathbb{R}_K$ are not comparable. First, $\mathbb{R}_K \not\subset \mathbb{R}_l$: For the basis element $(-1, 1) \setminus K \in \mathcal{B}_3$ containing 0, there is no basis element $[a, b) \in \mathcal{B}_2$ such that $0 \in [a, b) \subset (-1, 1) \setminus K$. Second, $\mathbb{R}_l \not\subset \mathbb{R}_K$: For the basis element $[2, b) \in \mathcal{B}_2$ containing 2, there is no basis element $(c, d) \in \mathcal{B}_3$ such that $2 \in (c, d) \subset [2, b)$ and there is no basis element $(c, d) \setminus K \in \mathcal{B}_3$ such that $2 \in ((c, d) \setminus K) \subset [2, b)$.

Definition (Subspace topology). Let X be a topological space with topology $\mathcal{T}$. If Y is a subset of X, then the collection $\mathcal{T}_Y = \{U \cap Y : U \in \mathcal{T}\}$ is a topology on Y, called the subspace topology. With this topology, Y is called the subspace of X; the open sets consist of all intersections of open sets of X with Y. We say U is open in Y (or open relative to Y) if $U \in \mathcal{T}_Y$.

Remark. It is straightforward to see that if Y is open in X, then the open sets in Y are also open in X. Otherwise, it is not necessarily true. For example, let $\mathbb{R}$ be equipped with the standard topology and $Y = [0, \infty) \subset \mathbb{R}$. Then $[0, b)$ is open in Y (since $[0, b) = (-\infty, b) \cap Y$ in which $(-\infty, b)$ is open in $\mathbb{R}$) but is not open in $\mathbb{R}$.

PROPOSITION 2.4. *If $\mathcal{B}$ is a basis for the topology $\mathcal{T}$ of X, then the collection $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$ is a basis for the subspace topology $\mathcal{T}_Y$ on Y.*

PROOF. Let $U \cap Y$ be an open set in $\mathcal{T}_Y$, in which U is open in X. For each $y \in U \cap Y$, $y \in U$ and there exists a basis element $B \in \mathcal{B}$ such that $x \in B \subset U$, since $\mathcal{B}$ is a basis for $\mathcal{T}$. Hence, $B \cap Y \in \mathcal{B}_Y$ and $y \in B \cap Y \subset U \cap Y$. By Proposition 2.2, $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$ is a basis for $\mathcal{T}_Y$. □

Problems.

2-1. Prove Proposition 2.2.

2-2. Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on a set X. Prove that $\cap_\alpha \mathcal{T}_\alpha$ is a topology on X.

2-3. Let $\{\mathcal{T}_\alpha\}$ be a family of topologies on a set X. Is $\cup_\alpha \mathcal{T}_\alpha$ is a topology on X? Prove your assertion.

2-4. On $\mathbb{R}$, let $\mathcal{T}_1$ be the standard topology, $\mathcal{T}_2 = \mathbb{R}_l$, $\mathcal{T}_3 = \mathbb{R}_K$, $\mathcal{T}_4 = \mathbb{R}_u$, and $\mathcal{T}_5$ be the finite complement topology. Determine for each of these topologies, which of others it contains.

2-5. Let $\mathcal{T}$ be a topology on a set X and $A \subset Y \subset X$. Prove that $(\mathcal{T}_Y)_A = \mathcal{T}_A$, that is, the topology A inherits as a subspace of Y is the same as the topology it inherits as a subspace of X.

2-6. Let $\mathcal{T}_2$ be a strictly finer topology than $\mathcal{T}_1$ on a set X. Suppose that $Y \subset X$ and $Y \neq \emptyset$. Is $(\mathcal{T}_2)_Y$ strictly finer than $(\mathcal{T}_1)_Y$? Prove your assertion.

2.2. Closed sets and limit points

Definition (Closed sets). Let X be a topological space. A subset $V \subset X$ is said to be closed if $X \setminus V$ is open.

Example.

(1). In any topology on a set X, the sets $\emptyset$ and X are both open and closed.[i]

(2). In the discrete topology $\mathcal{T} = \mathcal{P}(X)$, every set is both open and closed. In fact, every set in a topology $\mathcal{T}$ is both open and closed iff $\mathcal{T} = \mathcal{P}(X)$.

(3). In the finite complement topology, the closed sets consists of X and all the finite subsets of X.

(4). In the standard topology on $\mathbb{R}$, the intervals $[a, b]$, $(-\infty, a]$, and $[a, \infty)$ are closed; the intervals $(a, b]$ and $[a, b)$ are neither open nor closed.

(5). In the lower limit topology $\mathbb{R}_l$, classify all the intervals as open or/and closed sets. See Problem 2-7.

Using De Morgan's law, we can establish

THEOREM 2.5. *Let X be a topological space. Then*

(i). $\emptyset$ and X are closed.

(ii). Arbitary intersections of closed sets are closed

(iii). Finite unions of closes sets are closed.

Remark. Instead of using open sets, one could just as well specify a topology on a space by giving a collections of sets (to be called "closed sets") satisfying the three properties of the above theorem. One could then define open sets as the complements of closed sets and proceed just as before. The two procedures using open sets or closed sets are equivalent, however, most mathematicians prefer to use open sets to define topologies.

THEOREM 2.6. *Let X be a topological space and $Y \subset X$ be equipped with the subspace topology. Then $A \subset Y$ is closed iff $A = V \cap Y$ for some closed set $V \subset X$. In this case, we say that A is closed in Y.*

[i]A set that is both open and closed are sometimes called a clopen set. This terminology is however not widely adapted.

PROOF. Sufficiency: Let $A = V \cap Y$ in which $V \subset X$ is closed. Then $X \setminus V$ is open and $Y \setminus A = Y \setminus (V \cap Y) = Y \cap (X \setminus V)$ is open in the subspace topology.

Necessity: Let A be closed in Y. Then $Y \setminus A$ is open in the subspace topology. There exists an open set $U \subset X$ such that $Y \setminus A = U \cap Y$. Hence, $A = Y \setminus U = Y \cap (X \setminus U)$, in which $X \setminus U$ is closed. □

Remark. It is straightforward to see that if Y is closed in X, then the closed sets in Y are also closed in X. Otherwise, it is not necessarily true. For example, let $\mathbb{R}$ be equipped with the standard topology and $Y = (0, \infty) \subset \mathbb{R}$. Then $(0, b]$ is closed in Y (since $(0, b] = (-\infty, b] \cap Y$ in which $(-\infty, b]$ is closed in $\mathbb{R}$) but is not closed in $\mathbb{R}$.

Definition (Interior and closure). Let X be a topological space and $A \subset X$. Define

- The interior of A
$$\operatorname{Int} A = \bigcup_{U \subset A,\ U \text{ open}} U.$$
That is, the interior of A is the union of all open sets contained in A.
- the closure of A
$$\overline{A} = \bigcap_{V \supset A,\ V \text{ closed}} V.$$
That is, the closure of A is the intersection of all closed sets containing A.

Remark. It is clear that

- $x \in \operatorname{Int} A$ iff there is an open set $U \ni x$ such that $U \subset A$.
- $\operatorname{Int} A$ is open and $\overline{A}$ is closed.
- $\operatorname{Int} A \subset A \subset \overline{A}$.
- A is open iff $\operatorname{Int} A = A$; A is closed iff $\overline{A} = A$.

Example. Notice that the interior and closure of a set in a topological space X depend on the given topology. For example,

(1). Let $\mathbb{R}$ be the standard topology on $\mathbb{R}$. Then $\operatorname{Int}[a, b) = (a, b)$ and $\overline{[a, b)} = [a, b]$; $\operatorname{Int}(a, b] = (a, b)$ and $\overline{(a, b]} = [a, b]$.

(2). Let $\mathbb{R}_l$ be the lower limit topology on $\mathbb{R}$. Then $\operatorname{Int}[a, b) = [a, b)$ and $\overline{[a, b)} = [a, b)$; $\operatorname{Int}(a, b] = (a, b)$ and $\overline{(a, b]} = [a, b]$.

THEOREM 2.7. *Let X be a topological space and $Y \subset X$ be equipped with the subspace topology. Then for any $A \subset Y$, the closure of A in Y equals $\overline{A} \cap Y$, in which $\overline{A}$ is the closure of A in X.*

PROOF. Denote B the closure of A in Y. We need to show that $B = \overline{A} \cap Y$.

$B \subset \overline{A} \cap Y$: Since $\overline{A}$ is closed in X, $\overline{A} \cap Y$ is closed in Y by Theorem 2.6. Notice that $A \subset \overline{A} \cap Y$. So $B \subset \overline{A} \cap Y$ since the closure of A in Y is the intersection of all closed sets containing A.

$\overline{A} \cap Y \subset B$: Since B is closed in Y, $B = V \cap Y$ for some closed set $V \subset X$ by Theorem 2.6. Notice that $A \subset V$. So $\overline{A} \subset V$ since the closure of A in X is the intersection of all closed sets containing A. Therefore, $\overline{A} \cap Y \subset V \cap Y = B$. □

The definition of the interior if A implies that $x \in \operatorname{Int} A$ iff there is an open set U such that $x \in U \subset A$. While the definition of the closure of a set A does not provide a convenient way to find the closure, since all closed sets containing A are too big to work with. Hence, the following theorem gives two equivalent statements for the closure.

THEOREM 2.8. *Let X be a topological space and $A \subset X$. Then*

(i). $x \in \overline{A}$ iff every open set $U \ni x$ intersects A, i.e. $U \cap A \neq \emptyset$.

(ii). Suppose that the topology of X is given by a basis $\mathcal{B}$. Then $x \in \overline{A}$ iff every $B \in \mathcal{B}$ and $B \ni x$ intersects A.

Remark. We say an open set $U \ni x$ is a neighborhood of x. Then (i) can be rephrased as "$x \in \overline{A}$ iff every neighborhood of x intersects A."

PROOF.

(i). Let (P) be $x \in \overline{A}$ and (Q) be every open set $U \ni x$ intersects A. Then it suffices to prove that $\neg(P) \Leftrightarrow \neg(Q)$.

If $\neg(P)$, i.e. $x \notin \overline{A}$, then $x \in X \setminus \overline{A}$, which is an open set since $\overline{A}$ is closed, i.e. $\neg(Q)$.

If $\neg(Q)$, i.e. there is an open set $U \ni x$ such that $U \cap \overline{A} = \emptyset$, then $\overline{A} \subset (X \setminus U)$, in which $X \setminus U$ is closed. But $\overline{A}$ is the intersection of all closed sets containing A. Hence, $(X \setminus U) \supset \overline{A}$ so $x \notin \overline{A}$, i.e. $\neg(P)$.

(ii). If every open set containing x intersects A, then every basis element containing x intersects A; if every basis element containing x intersects A, then every open set $U \ni x$ there is a basis element $B \subset U$ and $B \ni x$, so $U \cap A \supset B \cap A \neq \emptyset$.

□

Example. Let $\mathbb{R}$ be equipped with the standard topology. Then $\overline{(a,b)} = [a,b]$ and $\overline{\{1/n : n \in \mathbb{N}\}} = \{0\} \cup \{1/n : n \in \mathbb{N}\}$.

Definition (Limit points). Let X be a topological space and $A \subset X$. We say that x is a limit point (or cluster point, point of accumulation) of A if every neighborhood U of x intersects A in some point other than x, i.e. $(U \cap A) \setminus \{x\} \neq \emptyset$. The set of limit points of A is denoted by A'.

Example. Let $\mathbb{R}$ be equipped with the standard topology. Then the set of limit points of (a,b) is $[a,b]$; the only limit point of $\{1/n : n \in \mathbb{N}\}$ is 0.

THEOREM 2.9. *Let X be a topological space and $A \subset X$. Then $\overline{A} = A \cup A'$.*

PROOF. $A \cup A' \subset \overline{A}$: $A \subset \overline{A}$ follows from the definition of closure, while $A' \subset \overline{A}$ follows Theorem 2.8, that is, if $x \in A'$, then every neighborhood of x intersects A so $x \in \overline{A}$.

$\overline{A} \subset A \cup A'$: Let $x \in \overline{A}$. If $x \in A$, then $x \in A \cup A'$; if $x \notin A$, since every neighborhood U of x intersects A, $U \cap A \neq \emptyset$. But $x \notin A$, $(U \cap A) \setminus \{x\} \neq \emptyset$ so $x \in A'$. □

Since a set is closed iff it contains its closure, we have the following corollary.

COROLLARY 2.10. *Let X be a topological space and $A \subset X$. Then $\overline{A} = A$ iff A contains all its limit points, i.e. $A' \subset A$.*

Definition (Convergence of a sequence). Let X be a topological space and $\{x_n\}_{n=1}^{\infty} \subset X$ be a sequence. We say that the sequence $\{x_n\}$ converges to $x \in X$, denoted by $x_n \to x$, if for every neighborhood U of x there is $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. In this case, we say that x is a limit of the sequence.

It is not always true that convergent sequence has a unique limit point. For example,[i] let $X = \{a,b,c\}$ be equipped with the topology $\{\emptyset, \{a,b\}, \{b,c\}, \{b\}, X\}$. Then the sequence $\{x_n\}$

[i] In the trivial topology, every sequence is convergent and all the points are its limit point. In the discrete topology, a sequence $x_n \to x$ iff $x_n = x$ with finite exceptions of $n \in \mathbb{N}$.

for $x_n = b$ converges to b (since all the points in the sequence are in the neighborhood $\{b\}$ of b), to a (since all the points in the sequence are in the neighborhood $\{a, b\}$ of b), and to c (since all the points in the sequence are in the neighborhood $\{b, c\}$ of c).[i] To avoid such situation, we need the following concept.

2.2.1. Hausdorff spaces.

Definition (Hausdorff space). We say that a topological space X is a Hausdorff space if for every $x, y \in X$ and $x \neq y$, there are open sets $U \ni x$ and $W \ni y$ such that $U \cap W = \emptyset$.

Remark. Notice that $X = \{a, b, c\}$ equipped with the topology $\{\emptyset, \{a, b\}, \{b, c\}, \{b\}, X\}$ is not a Hausdorff space since there are no disjoint open sets containing b and c, respectively.

Example. Among $\mathbb{R}$, $\mathbb{R}_l$, $\mathbb{R}_u$, $\mathbb{R}_K$, $\mathbb{R}_f$, which ones are Hausdorff spaces?

LEMMA 2.11. *Every finite set in a Hausdorff space is closed.*

PROOF. It suffices to show that the singleton set $A = \{x_0\}$ is closed in a Hausdorff space X, since finite union of closed sets is closed.

Let $x \in X$ and $x \neq x_0$, i.e. $x \notin A$. Then there are open sets $U \ni x_0$ (i.e. $U \supset A$) and $W \ni x$ such that $U \cap W \neq \emptyset$. Hence, $W \cap A \subset W \cap U = \emptyset$. By Theorem 2.8, $x \notin \overline{A}$. This is true for all $x \in X$ and $x \neq x_0$ so $\overline{A} = A = \{x_0\}$, which means that $\{x_0\}$ is closed. □

THEOREM 2.12. *Let X be a Hausdorff space and $A \subset X$. Then x is a limit point of A iff every neighborhood of x contains infinitely many points of A.*

PROOF. Sufficiency: If every neighborhood of x contains infinitely many points of A, then x is a limit point of A by the definition.

Necessity: Let x be a limit point of A. Suppose that there is a neighborhood U of x intersecting A at finitely many points. Let $U \cap (A \setminus \{x\}) = \{x_1, ..., x_n\}$. Then $U \setminus \{x_1, ..., x_n\} \ni x$ and is open by the previous lemma. But $(U \setminus \{x_1, ..., x_n\}) \cap A = \emptyset$, contradicting with the fact that x is a limit point of A. □

THEOREM 2.13. *Let X be a Hausdorff space. Then a sequence of points of X converges to at most one point in X.*

PROOF. See Problem 2-11. □

Problems.

2-7. In the lower limit topology $\mathbb{R}_l$, classify all the intervals as open or/and closed sets.

2-8. Let X be a topological space and $Y \subset X$ be equipped with the subspace topology. Suppose that Y is closed in X and $A \subset Y$. Prove A is closed in Y iff A is closed in X.

2-9. Let X be a topological space and $Y \subset X$ be equipped with the subspace topology. Suppose that $A \subset Y$. Then is that A is closed in Y iff A is closed in X? Prove your assertion.

2-10. Let X be a topological space and A, B, A_α be the subsets of X. Prove that

(1). If $A \subset B$, then $\overline{A} \subset \overline{B}$.

(2). $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(3). $\overline{\cup_\alpha A_\alpha} \supset \cup_\alpha \overline{A}_\alpha$; provide an example that the equality fails.

(4). $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$; provide an example that the equality fails.

---

[i]It is interesting to ask if it is possible for a constant sequence not to converge to the constant. The answer is no: Let $\{x_n\}$ be a constant sequence in a topological space X such that $x_n = c$ for all $n \in N$. Suppose that $x_n \not\to c$. Then there is a neighborhood U of c such that for every $N \in \mathbb{N}$, there exists some $m \geq N$ such that $x_m \notin U$. But this means that $c = x_m \notin U$, contradicting with the fact that U is a neighborhood of c.

(5). $\overline{A \setminus B} \supset \overline{A} \setminus \overline{B}$; provide an example that the equality fails.

2-11. Let X be a Hausdorff space. Prove that a sequence of points of X converges to at most one point in X.

2-12. Let X be a topological space and $Y \subset X$ be equipped with the subspace topology. Suppose that X is a Hausdorff space. Prove that Y is also a Hausdorff space.

2-13. Let X be a topological space and $A \subset X$. Define the boundary of A by

$$\partial A = \overline{A} \cap \overline{X \setminus A}.$$

(1). Prove that $\operatorname{Int} A \cap \partial A = \emptyset$ and $\overline{A} = \operatorname{Int} A \cup \partial A$.

(2). Prove that $\partial A = \emptyset$ iff A is both open and closed.

(3). Prove that U is open iff $\partial U = \overline{U} \setminus U$.

(4). If U is open, is $U = \operatorname{Int}(\overline{U})$? Prove your assertion.

2.3. Continuous functions

Definition (Continuous functions). Let X and Y be two topological spaces. We say a function $f : X \to Y$ is continuous if $f^{-1}(U)$ is open in X for every open set U in Y.

Remark. Continuity of a function $f : X \to Y$ of course depends on the function itself, but also on the topologies on the domain X and on the codomain Y. For example, let $\mathbb{R}$ and $\mathbb{R}_l$ be the standard and lower limit topologies on $\mathbb{R}$. Define the function $f : \mathbb{R} \to \mathbb{R}$ be $f(x) = x$. Then

- f is continuous relative to the topologies $\mathbb{R}$ and $\mathbb{R}$.
- f is not continuous relative to the topologies $\mathbb{R}$ and $\mathbb{R}_l$, since the preimage of the open set $[a, b)$ in $\mathbb{R}_l$, $f^{-1}([a, b)) = [a, b)$, is not open in $\mathbb{R}$.
- f is continuous relative to the topologies $\mathbb{R}_l$ and $\mathbb{R}$, since the preimage of any open set in $\mathbb{R}$ is itself and is open in $\mathbb{R}_l$ (because $\mathbb{R} \subset \mathbb{R}_l$).

Therefore, any discussion about the topological properties (open, closed, continuous, etc) is under the premise of topologies. If one wishes to emphasize the dependence of continuity on topologies, one can say $f : X \to Y$ is continuous relative to the topology $\mathcal{T}_X$ on X and the topology $\mathcal{T}_Y$ on Y, and denote the function by $f : (X, \mathcal{T}_X) \to (Y, \mathcal{T}_Y)$.

THEOREM 2.14. *Let X and Y be two topological spaces and $f : X \to Y$ be a function. Then the following statements are equivalent.*

(i). f is continuous.

(ii). For every closed set $B \subset Y$, $f^{-1}(B)$ is closed in X.

(iii). For every $A \subset X$, $f(\overline{A}) \subset \overline{f(A)}$.

(iv). For every $x \in X$ and every neighborhood W of $f(x)$, there is a neighborhood U of x such that $f(U) \subset W$.

(v). Suppose that $\mathcal{B}_Y$ generates the topology on Y. For every $B \in \mathcal{B}_Y$, $f^{-1}(B)$ is open in X.

PROOF. We follow the program that (i)⇔(ii), (i)⇒(iii), (iii)⇒(ii), (i)⇔(iv), and (i)⇔(v).

(i)⇒(ii). Recall that f^{-1} preserves differences of sets. Since $f^{-1}(Y) = X$, $X \setminus f^{-1}(B) = f^{-1}(Y) \setminus f^{-1}(B) = f^{-1}(Y \setminus B)$. So if B is closed, then $Y \setminus B$ is open and $f^{-1}(Y \setminus B)$ is open since f is continuous. It thus follows that $f^{-1}(B)$ is closed.

(ii)⇒(i). Similarly as before, if B is open, then $Y \setminus B$ is closed and $f^{-1}(Y \setminus B)$ is closed by assumption. It follows that $f^{-1}(Y \setminus B) = f^{-1}(Y) \setminus f^{-1}(B) = X \setminus f^{-1}(B)$ is closed. Hence, $f^{-1}(B)$ is open so f is continuous.

(i)⇒(iii). Let $y \in f(\overline{A})$. We need to show that $y \in \overline{f(A)}$. Since $y \in f(\overline{A})$, there is $x \in \overline{A}$ such that $y = f(x)$. Let W be a neighborhood of y in Y. $f^{-1}(W) := U$ is open and contains

x. Hence, U is a neighborhood of x in X and so $U \cap A \neq \emptyset$ since $x \in \overline{A}$. It then follows that $W \cap f(A) \supset f(U) \cap f(A) \supset f(U \cap A) \neq \emptyset$ so $y \in \overline{f(A)}$.

(iii)⇒(ii). Let $B \subset Y$ be closed and denote $A = f^{-1}(B)$. We need to show A is closed and it suffices to prove that $\overline{A} = A$. Notice that $f(A) = f(f^{-1}(B)) \subset B$.

Let $x \in \overline{A}$. Then $f(x) \in f(\overline{A}) \subset \overline{f(A)}$ by assumption. But $\overline{f(A)} \subset \overline{B} = B$ since B is closed. Then $f(x) \in B$ so $x \in f^{-1}(B) = A$, completing the proof.

(i)⇒(iv). Let $x \in X$ and W be a neighborhood of $f(x)$ in Y. Then $U := f^{-1}(W)$ is open by (i) and contains x. Hence, $f(U) \subset W$ and U is a neighborhood of x in X.

(iv)⇒(i). Let $W \subset Y$ be open and $U = f^{-1}(W)$. For each $x \in U$, $f(x) \in f(U)$. By (iv), there is a neighborhood U_x of x such that $f(U_x) \subset W$. But this means that $U_x \subset U$ and $U = \cup_x U_x$ is therefore open.

(i)⇒(iv). It is straightforward since any basis element in $\mathcal{B}_Y$ is open.

(iv)⇒(i). It is also straightforward since any open set in Y is a union of basis element in $\mathcal{B}_Y$ and f^{-1} preserves unions. □

Definition (Homeomorphisms). Let $f : X \to Y$ be a bijection between two topological spaces. We say f is a homeomorphism if f and f^{-1} are both continuous. Two topological spaces are said to be homeomorphic if there is a homeomorphism between them.

Remark. One can say that $f : X \to Y$ is a homeomorphism between $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ if one wishes to emphasize the dependence of a homeomorphism on the topologies.

Example.

(1). It is easy to see the identity map $f : (X, \mathcal{T}) \to (X, \mathcal{T})$ defined by $f(x) = x$ is a homeomorphism.

(2). Let $\mathbb{R}$ be equipped with the standard topology. Then any linear function $f : \mathbb{R} \to \mathbb{R}$ with nonzero slope is a homeomorphism.

(3). Let (a, b) and (c, d) be equipped with the subspace topology from the standard topology $\mathbb{R}$. Then the linear function from (a, b) to (c, d) is a homeomorphism.

(4). Let $\mathbb{R}$ be equipped with the standard topology and $(-\pi/2, \pi/2) \subset \mathbb{R}$ be equipped with the subspace topology. Then $f : (-\pi/2, \pi/2) \to \mathbb{R}$ by $f(x) = \tan x$ is a homeomorphism.

(5). The identity map $f : \mathbb{R} \to \mathbb{R}_l$ is not a homeomorphism. In fact, $\mathbb{R}$ and $\mathbb{R}_l$ are not homeomorphic, i.e. there are no homeomorphism between them.

Theorem 2.15. *Let X, Y, Z be topological spaces.*

(i). The constant function $f : X \to Y$ defined by $f(x) = y_0$ for all $x \in X$ and some $y_0 \in Y$ is continuous.

(ii). Let $A \subset X$ be equipped with the subspace topology. Then the inclusion function $i : A \to X$ defined by $i(x) = x$ for all $x \in A$ is continuous.

(iii). Let $f : X \to Y$ and $g : Y \to Z$ be continuous. Then the composition $g \circ f : X \to Z$ defined by $g \circ f(x) = g(f(x))$ for $x \in X$ is continuous.

(iv). Let $f : X \to Y$ be continuous and $A \subset X$ be equipped with the subspace topology. Then the restriction function $f|_A : A \to Y$ defined by $f|_A(x) = f(x)$ for all $x \in A$ is continuous.

Proof.

(i). Let $f(x) = y_0$ for all $x \in X$ and some $y_0 \in Y$. Let $U \subset Y$ be open. If $y_0 \in U$, then $f^{-1}(U) = X$ is open in X; if $y_0 \notin U$, then $f^{-1}(U) = \emptyset$ is open in X. This means that f is continuous.

(ii). Let $i(x) = x$ be the inclusion map from A to X. Notice that $i^{-1}(U) = U \cap A$ is open in A from the definition of subspace topology. This means that i is continuous.

(iii). Notice that $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$. Let $U \subset Z$ be open. Then $(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U))$, in which $g^{-1}(U)$ is open in Y since g is continuous. So $f^{-1}(g^{-1}(U))$ is open in X since f is continuous. This means that $g \circ f$ is continuous.

(iv). Let $U \in Y$ be open. Then $(f|_A)^{-1}(U) = f^{-1}(U) \cap A$, in which $f^{-1}(U)$ is open since f is continuous. It follows that $f^{-1}(U) \cap A$ is open in A from the definition of subspace topology. This means that $f|_A$ is continuous.

Alternatively, $f|_A = f \circ i$, in which $i : A \to X$ is the inclusion function. Then $f|_A$ is continuous because both i and f are continuous.

□

THEOREM 2.16 (The Pasting Lemma). *Let X be a topological space. Suppose that $X = A \cup B$ with two closed sets $A, B \subset X$. Let $f : A \to Y$ and $g : B \to Y$ be continous. If $f(x) = g(x)$ for all $x \in A \cap B$, then f and g combine to define a continuous function $h : X \to Y$, defined by*

$$h(x) = \begin{cases} f(x) & \text{for } x \in A, \\ g(x) & \text{for } x \in B. \end{cases}$$

PROOF. Let $W \subset Y$ be closed. It suffices to show that $h^{-1}(W)$ is closed in X. Notice that

$$h^{-1}(W) = f^{-1}(W) \cup g^{-1}(W),$$

in which $f^{-1}(W)$ is closed in A and $g^{-1}(W)$ is closed in B since f and g are continuous. It then follows that $f^{-1}(W)$ and $g^{-1}(W)$ are closed in X since A and B are closed. Hence, $h^{-1}(W)$ is closed. □

Remark. The Pasting Lemma is also valid if A and B are open. It is a special case of Problem 2-17. Note that $f(x) = g(x)$ on the $A \cap B$ is only required so that h is well defined.

Example.

(1). The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x & \text{for } x \leq 0, \\ x/2 & \text{for } x \geq 0, \end{cases}$$

is continuous, by The Pasting Lemma since both pieces are continuous.

(2). The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x - 2 & \text{for } x < 0, \\ x + 2 & \text{for } x \geq 0, \end{cases}$$

is not continuous even though both pieces are continuous.

Problems.

2-14. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous iff for every $x \in \mathbb{R}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(y) - f(x)| < \varepsilon$ if $y \in (x - \delta, x + \delta)$.

2-15. Let X and Y be two topological spaces and $f : X \to Y$ be continuous. Suppose that $A \subset X$ and $x \in A'$. (Recall that A' is the set of limit points of A.) Then is $f(x) \in f(A)'$? Prove your assertion.

2-16. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on a set X. Let $f : (X, \mathcal{T}_1) \to (X, \mathcal{T}_2)$ be the identity map. Prove that

(1). f is continuous iff $\mathcal{T}_1$ is finer than $\mathcal{T}_2$.

(2). f is a homeomorphism iff $\mathcal{T}_1 = \mathcal{T}_2$.

2-17. Let X and Y be two topological spaces and $f : X \to Y$ be a function. Suppose that $X = \cup_\alpha U_\alpha$ in which $U_\alpha \subset X$ are open sets. If $f|_{U_\alpha}$ is continuous for all α, then prove that f is continuous.

2-18. Let X be a topological space and $f, g : X \to \mathbb{R}$ be two continuous functions. Define $h(x) = \min\{f(x), g(x)\}$. Prove that h is also continuous.

2.4. Metric topology

One of the most important and frequently used ways of imposing a topology on a set is to define the topology via a metric on the set, called the metric topology. Metric topology lie at the heart of modern analysis, the examples of which include the Euclidean (or standard) topology on $\mathbb{R}^n$.

Definition (Metric). A metric on a set X is a function

$$d : X \times X \to [0, \infty)$$

that satisfies

(i). $d(x, y) = 0$ iff $x = y$,

(ii). $d(x, y) = d(y, x)$ for all $x, y \in X$,

(iii). $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$ (triangle inequality).

Given a metric d on X, $d(x, y)$ is often called the distance between x and y.

Example. In $\mathbb{R}^n$, denote a point $x = (x_1, ..., x_n) \in \mathbb{R}^n$. Then

$$d(x, y) = \Big(|x_1 - y_1|^2 + \cdots + |x_n - y_n|^2\Big)^{\frac{1}{2}}$$

is a metric on $\mathbb{R}^n$. In fact, for any $1 \leq p \leq \infty$,

$$d_p(x, y) = \Big(|x_1 - y_1|^p + \cdots + |x_n - y_n|^p\Big)^{\frac{1}{p}}$$

is a metric on $\mathbb{R}^n$. In particular,

$$d_1(x, y) = |x_1 - y_1| + \cdots + |x_n - y_n|;$$

and

$$\rho(x, y) := d_\infty(x, y) = \sup_{i=1,\dots,n} \{|x_i - y_i|\}$$

is called the square metric.

Definition (Metric topology). Let X be a set equipped with a metric d. For $x \in X$ and $\varepsilon > 0$, denote

$$B_d(x, \varepsilon) = \{y \in X : d(x, y) < \varepsilon\}$$

the ε-ball centered at x. Then the metric topology induced by the metric d on X is generated by the basis

$$\mathcal{B}_d = \{B_d(x, \varepsilon) : x \in X, \varepsilon > 0\}.$$

We say X is a metric space if it is a topological space with the topology induced by some metric d on X.

The above definition implies that on a set X equipped with a metric topology induced by a metric d, a set $U \subset X$ is open iff for every $x \in U$, there is $\delta > 0$ such that $B_d(x, \delta) \subset U$.

Remark. We show that $\mathcal{B}_d$ is a basis.

(i). For each $x \in X$, there is $B_d(x, \varepsilon) \in \mathcal{B}_d$ that contains x.

(ii). Let $x \in B_d(x_1, \varepsilon_1) \cap B_d(x_2, \varepsilon_2)$. First, $x \in B_d(x_1, \varepsilon_1)$ implies that $d(x, x_1) < \varepsilon_1$. Choose $\delta_1 = \varepsilon_1 - d(x, x_1)$. For every $y \in B_d(x, \delta_1)$, $d(y, x) < \delta_1$ so triangle inequality implies that $d(y, x_1) \leq d(y, x) + d(x, x_1) < \delta_1 + d(x, x_1) = \varepsilon_1$. Hence, $y \in B_d(x_1, \varepsilon_1)$ so $B_d(x, \delta_1) \subset B_d(x_1, \varepsilon_1)$. Similarly, we can choose $\delta_2 = \varepsilon_2 - d(x, x_2)$ so $B_d(x, \delta_2) \subset B_d(x_2, \varepsilon_2)$.

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $B_d(x, \delta) \subset B_d(x, \delta_1) \subset B_d(x_1, \varepsilon_1)$ and $B_d(x, \delta) \subset B_d(x, \delta_2) \subset B_d(x_2, \varepsilon_2)$. This means that $x \in B_d(x, \delta) \subset B_d(x_1, \varepsilon_1) \cap B_d(x_2, \varepsilon_2)$.

Example.

(1). The Euclidean topology on $\mathbb{R}^n$ is induced by the metric d. In fact, the metric topologies induced by the metrics d_p are identical for all $1 \leq p \leq \infty$.

(2). Let X be a nonempty set. Define the metric

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } x \neq y. \end{cases}$$

Then this metric induces the discrete topology on X. It is because the metric balls $B_d(x, \varepsilon) = \{x\}$ if $\varepsilon \leq 1$ and $B_d(x, \varepsilon) = X$ if $\varepsilon > 1$.

(3). Denote $\mathbb{R}^\omega = \Pi_{i=1}^{\infty} X_n$ with $X_n = \mathbb{R}$ for all $n \in \mathbb{N}$. Define $\overline{d}(x, y) = \min\{|x - y|, 1\}$ for $x, y \in \mathbb{R}$ as the standard bounded metric on $\mathbb{R}$. Then for $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$,

$$\overline{\rho}(x, y) := \sup_{i \in \mathbb{N}} \{\overline{d}(x_i, y_i)\}$$

defines a metric on $\mathbb{R}^\omega$. We call $\overline{\rho}$ the uniform metric on $\mathbb{R}^\omega$ and the topology induced by $\overline{\rho}$ the uniform topology.

(4).

$$D(x, y) := \sup_{i \in \mathbb{N}} \left\{ \frac{\overline{d}(x_i, y_i)}{i} \right\}$$

defines a metric on $\mathbb{R}^\omega$.

The following discussion is on the interpretation of topological statements (continuity, convergence, etc) in metric spaces.

2.4.1. Continuous functions involving metric spaces.

THEOREM 2.17. *Let X and Y be equipped with metric topologies $\mathcal{T}_X$ and $\mathcal{T}_Y$, induced by metrics d_X and d_Y, respectively. Then $f : X \to Y$ is continuous iff for any $x \in X$ and $\varepsilon > 0$, there exists $\delta > 0$ such that $d_Y(f(x), f(y)) < \varepsilon$ if $d_X(x, y) < \delta$.*

PROOF. See Problem 2-20. □

LEMMA 2.18. *Let X be a topological space and $A \subset X$.*

(i). If there is a sequence $x_n \to x$ for $\{x_n\} \subset A$ and some $x \in X$, then $x \in \overline{A}$.

(ii). If in addition X is a metric space, then $x \in \overline{A}$ implies that there is a sequence $x_n \to x$ for $\{x_n\} \subset A$.

PROOF.

(i). Suppose that there is a sequence $x_n \to x$ for $\{x_n\} \subset A$ and some $x \in X$. Then for every neighborhood U of x, there is $N \in \mathbb{N}$ such that $x_n \in U$ for all $n \geq N$. This means that $\{x_n : n \geq N\} \subset U \cap A$ and $x \in \overline{A}$.

(ii). Suppose that X is a metric space. Let $x \in \overline{A}$. Then for each $n \in \mathbb{N}$, $B_d(x, 1/n) \cap A \neq \emptyset$. So there is $x_n \in A$ such that $d(x_n, x) < 1/n$. Notice that $x_n \to x$.

□

THEOREM 2.19. *Let X and Y be two topological spaces. Let $f : X \to Y$.*

(i). If f is continuous, then for every convergence sequence $x_n \to x$, we have that $f(x_n) \to f(x)$.

(ii). Suppose that in addition X is a metric space. If for every convergent sequence $x_n \to x$, we have that $f(x_n) \to f(x)$, then f is continuous.

PROOF.

(i). Suppose that f is continuous. Let $x_n \to x$. For every $W \ni f(x)$ open in Y, $f^{-1}(W)$ is open in X since f is continuous. Now $f^{-1}(W)$ is a neighborhood of x in X because $f(x) \in W$. Since $x_n \to x$, there is $N \in \mathbb{N}$ such that $x_n \in f^{-1}(W)$ for all $n \geq N$. This means that $f(x_n) \in W$ for all $n \geq N$ so $f(x_n) \to f(x)$.

(ii). By (iii) in Theorem 2.14, it suffices to show that $f(\overline{A}) \subset \overline{f(A)}$ for each $A \subset X$. Let $x \in \overline{A}$, by (ii) in the previous lemma, there is $\{x_n\} \subset A$ such that $x_n \to x$. By the assumption, $f(x_n) \to f(x)$. By (i) in the previous lemma, that $f(x) \in \overline{f(A)}$ since $f(x_n) \in f(A)$. Hence, $f(\overline{A}) \subset \overline{f(A)}$. Notice that here we do not need to require that Y is a metric space.

□

Definition (Uniform convergence). Let X and Y be two topological spaces. Let $f_n : X \to Y$ be a sequence of functions and $f : X \to Y$ be a function. We say that $f_n \to f$ pointwisely if $f_n(x) \to f(x)$ for all $x \in X$.

Let Y be equipped with a metric topology induced by metric d. We say that $f_n \to f$ uniformly if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that

$$d(f_n(x), f(x)) < \varepsilon$$

for all $x \in X$ and $n \geq N$.

THEOREM 2.20. *Let $f_n : X \to Y$ be a sequence of continuous functions from a topological space X to a metric space Y. If $f_n \to f$ uniformly for some function $f : X \to Y$. Then f is continuous.*

PROOF. Let W be open in Y. We need to show that $f^{-1}(W)$ is open in X. Choose $x_0 \in f^{-1}(W)$. It suffices to prove that there is a neighborhood U of x_0 such that $U \subset f^{-1}(W)$ (i.e. $f(U) \subset W$) so $f^{-1}(W)$ is open in X.

Denote $y_0 = f(x_0)$. Then $y_0 \in W$ since $x_0 \in f^{-1}(W)$. First choose $\varepsilon > 0$ such that $B(y_0, \varepsilon) \subset W$.

By uniform convergence of $f_n \to f$, there exists $N \in \mathbb{N}$ such that $d(f_n(x), f(x)) < \varepsilon/3$ for all $x \in X$ and $n \geq N$.

Since f_N is continuous, $f_N^{-1}(B(y_0, \varepsilon/3)) := U$ is open in X. Hence, $d(f_N(x), f_N(x_0)) < \varepsilon/3$ if $x \in U$. Now we claim that $f(U) \subset W$. Compute for $x \in U$ that

$$d(f(x), f(x_0)) \leq d(f(x), f_N(x)) + d(f_N(x), f_N(x_0)) + d(f_N(x_0), f(x_0)) < \varepsilon.$$

So $f(x) \in B(y_0, \varepsilon) \subset W$ if $x \in U$. □

Problems.

2-19. Prove that every metric space is a Hausdorff space.

2-20. Prove Theorem 2.17.

2-21. Let X be a topological space and $A \subset X$. Let $x \in \overline{A}$. Is there a sequence $x_n \to x$ for $\{x_n\} \subset A$? Prove your assertion.

2-22. Let $f_n : X \to Y$ be a sequence of continuous functions from a topological space X to a metric space Y. If $f_n \to f$ pointwisely for some function $f : X \to Y$, then is f continuous? Prove your assertion.

2.5. Product topology

Definition (Product topology). Let X_i be topological spaces generated by the basis $\mathcal{B}_i$ for $i = 1, ..., n$. Then the product topology is defined to be a topology on $X_1 \times \cdots \times X_n$ that is generated by the basis

$$\mathcal{B} := \{B_1 \times \cdots \times B_n : B_i \in \mathcal{B}_i, i = 1, ..., n\}.$$

Notice that $X_1 \times \cdots \times X_n = \emptyset$ if $X_i = \emptyset$ for some $i = 1, ..., n$. To avoid the trivial cases, we assume all the sets X_i are nonempty.

THEOREM 2.21. *$\mathcal{B}$ in the above definition is a basis.*

PROOF. Denote $X = X_1 \times \cdots \times X_n$.

Let $x = (x_1, ..., x_n) \in X$. Then $x_i \in X_i$ and there is a basis element $B_i \in \mathcal{B}_i$ such that $x_i \in B_i$. Hence, $x \in B_1 \times \cdots \times B_n \in \mathcal{B}$.

Let $C_1 = B_{11} \times \cdots \times B_{1n}$, $C_2 = B_{21} \times \cdots \times B_{2n}$, and $x = (x_1, ..., x_n) \in C_1 \cap C_2$. Then $x_i \in B_{1i} \cap B_{2i}$. Since $\mathcal{B}_i$ is a basis, there is $B_{3i} \in \mathcal{B}_i$ such that $x_i \in B_{3i} \subset B_{1i} \cap B_{2i}$. Now $C_3 := B_{31} \times \cdots \times B_{3n} \in \mathcal{B}$ and $x \in C_3 \subset C_1 \cap C_2$. □

Remark. Let X_i be topological spaces and $X = X_1 \times \cdots \times X_n$ be equipped with the product topology. Then the boxes $U = U_1 \times \cdots \times U_n \subset X$ for which $U_i \subset X_i$ open are open in X. But there are other open sets different with the boxes, e.g. the union of two boxes.

Definition (Projection maps). Let $X = X_1 \times \cdots \times X_n$. The projection map $\pi_i : X \to X_i$ is defined as

$$\pi_i(x) = x_i \quad \text{for } x = (x_1, ..., x_n) \in X.$$

THEOREM 2.22. *Let X_i be topological spaces and $X = X_1 \times \cdots \times X_n$ be equipped with the product topology. Then the projections maps $\pi_i : X \to X_i$ are continuous for all $i = 1, ..., n$.*

PROOF. One only needs to observe that $\pi_i^{-1}(U) = X_1 \times \cdots \times U \times \cdots \times X_n$ if open in X if U is open in X_i. □

The product topology on $\mathbb{R}^n$ are generated by the basis

$$\{(a_1, b_1) \times \cdots \times (a_n, b_n) : -\infty < a_i < b_i < \infty, i = 1, ..., n\}.$$

THEOREM 2.23. *The product topology on $\mathbb{R}^n$ equals its Euclidean topology.*

PROOF. Notice that in the ρ-metric on $\mathbb{R}_n$,

$$B_\rho(x, \varepsilon) = (x_1 - \varepsilon, x_1 + \varepsilon) \times \cdots \times (x_n - \varepsilon, x_n + \varepsilon)$$

for $x = (x_1, ..., x_n) \in \mathbb{R}^n$ and $\varepsilon > 0$. It immediately follows that the product topology is finer than the ρ-metric topology since $B_\rho(x, \varepsilon)$ is a basis element for the product topology.

For every $x = (x_1, ..., x_n) \in (a_1, b_1) \times \cdots \times (a_n, b_n)$, $x_i \in (a_i, b_i)$. So there is $\varepsilon_i > 0$ such that $(x_i - \varepsilon_i, x_i + \varepsilon_i) \in (a_i, b_i)$. Choose $\varepsilon = \min\{\varepsilon_1, ..., \varepsilon_n\}$. Then $(x_i - \varepsilon, x_i + \varepsilon) \subset (a_i, b_i)$ for all $i = 1, ..., n$. This means that $x \in B_\rho(x, \varepsilon) \subset (a_1, b_1) \times \cdots \times (a_n, b_n)$. Hence, the ρ-metric topology is finer than the product topology. □

Problems.

2-23. Let X_i be Hausdorff spaces and $X = X_1 \times \cdots \times X_n$ be equipped with the product topology. Prove that X is also Hausdorff.

2-24. Let X and Y be topological spaces. Let $X \times Y$ and $Y \times X$ be equipped with the product topology. Prove that $X \times Y$ is homeomorphic to $Y \times X$.

CHAPTER 3

Connectedness and compactness

In the study of calculus on $\mathbb{R}$, there are three basic theorems about continuous functions, and on these theorems the rest of calculus depends. They are

- *Intermediate value theorem.* If $f : [a, b] \to \mathbb{R}$ is continuous and $r \in \mathbb{R}$ is between $f(a)$ and $f(b)$, then there exists $c \in [a, b]$ such that $f(c) = r$.
- *Maximum value theorem.* If $f : [a, b] \to \mathbb{R}$ is continuous, then there exists an element $c \in [a, b]$ such that $f(x) \leq f(c)$ for all $x \in [a, b]$, i.e. $f(c) = \max_{x \in [a,b]} f(x)$.
- *Uniform continuity theorem.* If $f : [a, b] \to \mathbb{R}$ is continuous, then f is uniformly continuous, i.e. for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ if $|x - y| < \delta$ and $x, y \in [a, b]$.

These theorems are used in a number of places. For example, the intermediate value theorem is used in constructing inverse functions, such as $\sqrt[3]{x}$ and $\arcsin x$; the maximum value theorem is used for proving the mean value theorem for derivatives, upon which the fundamental theorem of calculus depend; the uniform continuity theorem is sued for proving that every continuous function is Riemann integrable.

These theorems are about continuous functions, but also depend on the closed interval $[a, b]$ in $\mathbb{R}$. In particular, the intermediate value theorem depends on the connectedness of the topological space $[a, b]$; while the other two theorems depend on the compactness of the topological space $[a, b]$. In this chapter, we shall define these properties for arbitrary topological spaces and prove the appropriate generalized versions of these theorems.

Connectedness and compactness are fundamental in areas beyond calculus, e.g. analysis, geometry, topology, etc.

3.1. Connected spaces

Definition (Connected spaces). Let X be a topological space. A separation of X is a pair A and B of disjoint nonempty open subsets of X such that $A \cup B = X$. The space X is said to be connected if there does not exists a separation of X.

Notice that if there is a separation A and B of X. Then $A = B^c$ and $B = A^c$ are both closed as well. That is, A and B are both open and closed. Therefore, a topological space X is connected iff there is no nontrivial (i.e. not equal to $\emptyset$ or X) open and closed sets in X.

Notice also that if there is a separation A and B of X. Then $\overline{A} = \operatorname{Int} A = A$ and $\overline{B} = \operatorname{Int} B = B$. Hence, $\overline{A}$ and $\overline{B}$ form a separation of X; $\operatorname{Int} A$ and $\operatorname{Int} B$ form a separation of X.

Example.

(1). Let X be a set that contains more than one element and be equipped with the discrete topology. Then X is not connected.

(2). Let $\mathbb{R}$ be equipped with the standard topology and $X = [-1, 0) \cup (0, 1]$ be equipped with the subspace topology. Then X is not connected since $[-1, 0)$ and $(0, 1]$ form a separation of X. In fact, any union of disjoint intervals in $\mathbb{R}$ is not connected. It demonstrates the intuitive picture of "connectedness".

(3). Let $\mathbb{R}$ be equipped with the standard topology, $\mathbb{Q}$ is not connected.
(4). The real numbers $\mathbb{R}$ equipped with the lower limit topology $\mathbb{R}_l$ is not connected since $\mathbb{R} = (-\infty, a) \cup [a, \infty)$, in which both $(-\infty, a)$ and $[a, \infty)$ are open in $\mathbb{R}_l$.
(5). The real numbers equipped with the finite complement topology is connected. This is because any two open sets in the finite complement topology intersect. In fact, let X be a set that contains infinitely many elements. Then X connected with respect to the finite complement topology.

THEOREM 3.1. *The real numbers $\mathbb{R}$ equipped with the standard topology is connected.*

PROOF. Suppose $\mathbb{R}$ is not connected. Then we can write $\mathbb{R} = A \cup B$, where A and B are both open, non-empty, and $A \cap B = \emptyset$.

Fix $a \in A$ and $b \in B$. Without loss of generality, we may assume that $a < b$. Define a set

$$C = \{x \in \mathbb{R} : [a, x] \subset A\}.$$

This is nonempty since $a \in C$. Also C is bounded above by b (otherwise A and B intersect). Because $\mathbb{R}$ is complete, $s := \sup C \in \mathbb{R}$. Next we show that $s \notin A$ and $s \notin B$, therefore contradicting with the fact that $\mathbb{R} = A \cup B$.

(1). $s \notin A$. Assume that $s \in A$. Then since A is open, there is $\delta > 0$ such that $(s-\delta, s+\delta) \subset A$. Notice that $s = \sup C$ so there is $y \in (s - \delta, s]$ such that $y \in C$, meaning that $[a, y] \subset A$. But this implies that $[a, y] \cup (s - \delta, s + \delta/2] = [a, s + \delta/2] \subset A$. So $s + \delta/2 \in C$, which is impossible since $s = \sup C$.

(2). $s \notin B$. Assume that $s \in B$. Then since B is open, there is $\delta > 0$ such that $(s-\delta, s+\delta) \subset B$. Notice that $s = \sup C$ so there is $y \in (s-\delta, s] \subset B$ such that $y \in C$, meaning that $[a, y] \subset A$. But this implies that $y \in A \cap B$, which is impossible since $A \cap B = \emptyset$. □

Question. How does the above argument fail for $\mathbb{R}_l$ and $\mathbb{R}_u$?

LEMMA 3.2. *Let A and B form a separation of a topological space X. If $Y \subset X$ and is a connected subspace, then $Y \subset A$ or $Y \subset B$.*

PROOF. Notice that $A \cap Y$ and $B \cap Y$ are both open in Y. If both of them are nonempty, then they form a separation of Y, which is impossible since Y is connected. This means that either $A \cap Y = \emptyset$ so $Y \subset A^c = B$, or $B \cap Y = \emptyset$ so $Y \subset B^c = A$. □

THEOREM 3.3. *Let $A \subset X$ be a connected subspace of a topological space X. Suppose that $A \subset B \subset \overline{A}$. Then B is also connected. In particular, the closure of a connect subspace is also connected.*

PROOF. Suppose that B is not connected. Then there is a separation C and D of B. Since $A \subset B$, by the previous lemma, either $A \subset C$ or $A \subset D$. If $A \subset C$, then $\overline{A} \subset \overline{C}$. But since $B \subset \overline{A}$, $B \subset \overline{C} = C$. This means that $D = B \setminus C = \emptyset$, which is impossible. Similarly, $A \not\subset D$, posing a contradiction with the previous lemma. □

An immediate consequence of this theorem is that all the intervals in $\mathbb{R}$ are connected.

THEOREM 3.4. *Let X be a topological space. Then the union of any collection of connected subspaces of X that have a point in common is connected.*

PROOF. Let $Y = \cup_\alpha A_\alpha$, in which A_α is connected for each α. Let $p \in \cap_\alpha A_\alpha$. Suppose that Y is not connected. Then there is a separation C and D of Y. If $p \in C$, then $A_\alpha \subset C$ for each α since A_α is connected. But this means that $\cup_\alpha A_\alpha = Y \subset C$, leading to the contradiction that $D = \emptyset$. Similarly, if $p \in D$, then $\cup_\alpha A_\alpha = Y \subset D$, leading to the contradiction that $C = \emptyset$. □

The image of a connected space under a continuous function is connected.

THEOREM 3.5. *Let X and Y be two topological spaces and $f : X \to Y$ be a continuous function. Suppose that X is connected. Then $f(X)$ is connected in Y.*

PROOF. Suppose that $Z := f(X)$ is not connected. Then there is a separation A and B of Z, from which $f^{-1}(A)$ and $f^{-1}(B)$ form a separation of X. ($f^{-1}(A)$ and $f^{-1}(B)$ are both open since f is continuous, are nonempty since A and B are nonempty, and are disjoint since A and B are disjoint.) This contradicts with the fact that X is connected. □

This theorem in particular shows that two homeomorphic topological spaces are both connected or both not connected. Since the intervals (a, b), $(-\infty, a)$, and (a, ∞) are homeomorphic to $\mathbb{R}$, these open intervals are connected. Another immediate consequence is that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}_l$ and $\mathbb{R}$ is not homeomorphic to $\mathbb{R}_f$.

THEOREM 3.6. *Let X_i be connected spaces and $X = X_1 \times \cdots \times X_n$ be equipped with the product topology. Then X is also connected.*

An immediate consequence is the $\mathbb{R}^n$ is connected.

PROOF. We prove by induction.

First, we show that $X \times Y$ is connected if both X and Y are connected. Fix $b \in Y$. Consider the set

$$T_x = (X \times \{b\}) \cup (\{x\} \times Y).$$

Here $X \times \{b\}$ is homeomorphic to X and is thus connected and $\{x\} \times Y$ is homeomorphic to Y and is thus connected. So T_x is connected by Theorem 3.4 since $(X \times \{b\}) \cap (\{x\} \times Y) = (x, b)$. It then follows that

$$X \times Y = \cup_{x \in X} T_x$$

is connected by Theorem 3.4 since

$$\cap_{x \in X} T_x = X \times \{b\}.$$

Second, by induction $X_1 \times \cdots \times X_{n-1}$ is connected. So $(X_1 \times \cdots \times X_{n-1}) \times X_n$ is connected by the first step. Then $X_1 \times \cdots \times X_n$ is homeomorphic to $(X_1 \times \cdots \times X_{n-1}) \times X_n$ and is therefore connected. □

3.1.1. Intermediate value theorem.

THEOREM 3.7. *Let X be a connected space and $f : X \to \mathbb{R}$ be continuous. Let $a, b \in X$ and r be between $f(a)$ and $f(b)$. Then there is $c \in X$ such that $f(c) = r$.*

PROOF. If $f(a) = f(b)$, then the theorem readily follows by choosing $c = a$ (or $c = b$). Without loss of generality, we may assume that $f(a) < f(b)$ so $f(a) < r < f(b)$. Consider the two sets

$$A = f(X) \cap (-\infty, r) \quad \text{and} \quad B = f(X) \cap (r, \infty).$$

Since $f(a) \in A$ and $f(b) \in B$, A and B are not empty. We also see that A and B are open in $f(X)$ (with repect to the subspace topology since they are the intersection of open intervals with $f(X)$). Now if there is no $c \in X$ such that $f(c) = r$, then $f(X) = A \cup B$, which is a separation of $f(X)$. That is, $f(X)$ is not connected, contradicting with Theorem 3.5. □

3.1.2. Path connected spaces.

Definition (Path connected spaces). Let X be a topological space and $x, y \in X$. A path in X from x to y is a continuous function $[a, b] \to X$ for some closed interval $[a, b] \in \mathbb{R}$ such that $f(a) = x$ and $f(b) = y$. We say X is path connected if every pair of points can be joined by a path in X.

Remark. We do not lose generality by assuming the closed interval $[a, b] = [0, 1]$ in the above definition. This is because if there is a continuous function $f : [a, b] \to X$ such that $f(a) = x$ and $f(b) = y$, then one can construct a continuous function $g = f \circ h : [0, 1] \to X$ such that $f(0) = x$ and $f(1) = y$. Here, $h : [0, 1] \to [a, b]$ is a continuous function.

Similar argument shows that if X is homeomorphic to a path connected space, then X is also path connected.

Example.
(1). $\mathbb{R}^n$ is path connected.
(2). The Euclidean balls $B^n(x, r) = \{y \in \mathbb{R}^n : d(x, y) < r\}$ in $\mathbb{R}^n$ are path connected.
(3). The Euclidean spheres $\mathbb{S}^{n-1}(x, r) = \{y \in \mathbb{R}^n : d(x, y) = r\}$ in $\mathbb{R}^n$ are path connected.
(4). The punctured Euclidean space $\mathbb{R}^n \setminus \{0\}$ is path connected for $n \geq 2$.

THEOREM 3.8. *Let X be a path connected space. Then X is connected.*

PROOF. Suppose that X is not connected. Then there is a separation $X = A \cup B$. Let $f : [a, b] \to X$ be a path in X. Then $f([a, b])$ is connected in X by Theorem 3.5. So either $f([a, b]) \subset A$ or $f([a, b]) \subset B$. This means there are no paths that joins points in A and points in B, contradicting with the fact that X is path connected. □

The converse of Theorem 3.8 is not true. Here is the example: Let

$$S = \{(x, \sin(1/x)) : x \in (0, 1]\}.$$

We have that S is connected since it is the image of an interval $(0, 1]$ under a continuous function $x \to (x, \sin(1/x))$. It then follows that $\overline{S}$ is connected. But $\overline{S}$ is not path connected. Notice that

$$\overline{S} = S \cup (\{0\} \times [-1, 1]).$$

We show that $(0, 0)$ and the points in S can not be joined by pathes. Suppose there is a path $f : [a, b] \to \overline{S}$ such that $f(a) = (0, 0)$ and $f(b) \in S$. Then $f^{-1}(\{0\} \times [-1, 1])$ is closed in $[a, b]$ and therefore has a largest element b. Hence, $f(b) \in \{0\} \times [-1, 1]$ and $f((b, c]) \subset S$.

Replace $[b, c]$ by $[0, 1]$ for convenience. Now continuity of $f = (x(t), y(t)) : [0, 1] \to \overline{S}$ implies that if $t \to 0$, then $x(t) \to 0$ and $y(t) \to 0$. But observe the intense oscillation of $\sin(1/x)$ when x approaches 0. One can choose $t_n \to 0$ such that $y(t_n) = (-1)^n$ and is divergent.

To find t_n, we proceed as follows. Given $n \in \mathbb{N}$, choose u with $0 < u < x(1/n)$ such that $\sin(1/u) = (-1)^n$. Then use the intermediate value theorem to find t_n with $0 < t_n < 1/n$ such that $x(t_n) = u$.

Problems.
3-1. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on a set X and $\mathcal{T}_1 \subset \mathcal{T}_2$.
(1). Suppose that $\mathcal{T}_2$ is connected. Prove that $\mathcal{T}_1$ is also connected.
(2). Suppose that $\mathcal{T}_1$ is connected. Is $\mathcal{T}_2$ connected? Prove your assertion.
3-2. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of connected subspaces of a topological space X. Suppose that $A_n \cap A_{n+1} \neq \emptyset$. Prove that $\cup_{n=1}^{\infty} A_n$ is connected.

3-3. Let X be an infinite set equipped with the finite complement topology. Prove that X is connected.

3-4. Prove that $\mathbb{R}$ is not homeomorphic to $\mathbb{R}^n$ for $n \geq 2$. (Hint: The punctured Euclidean space $\mathbb{R}^n \setminus \{p\}$ is connected for $n \geq 2$.)

3-5. Prove that there is no homeomorphism among $(0,1)$, $(0,1]$, and $[0,1]$.

3-6. Let $f : [0,1] \to [0,1]$ be a continuous function. Prove that there is a point $x \in [0,1]$ such that $f(x) = x$. Is the assertion true if one replaces $[0,1]$ by $(0,1)$ or $(0,1]$?

3.2. Compact spaces

Definition (Covers). Let X be a set. A collection $\mathcal{A}$ of subsets of X is said to be a cover of X if X is equal to the union of the elements in $\mathcal{A}$. A subcover of a cover $\mathcal{A}$ is a subcollection of $\mathcal{A}$ that also covers X.

In a topological space X, a cover is called an open cover if all its elements are open in X.

Definition (Compact spaces). Let X be a topological space. We say that X is compact if every open cover of X contains a finite subcover.

Remark (Compact subspaces). Let X be a set and $Y \subset X$. We say a collection $\mathcal{A}$ of subsets of X is a cover of Y if Y is contained in the union of elements in $\mathcal{A}$. In a topological space X, a cover of Y is called an open cover if all its elements are open in X.

Let $Y \subset X$ be equipped with the subspace topology. Then every open cover $\mathcal{A} = \{U_\alpha\}$ of Y can be written as $\mathcal{A} = \{W_\alpha \cap Y\}$ in which W_α is open in X and $Y \subset \cup_\alpha W_\alpha$. Therefore, Y is compact iff any open cover of Y in X contains a finite subcover, in which case we as usual say that Y is compact in X.

Example.

(1). Every finite topological space is compact.

(2). If X is a finite topological space, the every subspace of X is compact in X.

(3). Let X be a topological space equipped with the discrete topology. Then $Y \subset X$ is compact in X iff Y is finite.

Example.

(1). $\mathbb{R}$ is not compact, as the open cover $\{(n, n+2) : n \in \mathbb{N}\}$ contains no finite subcover.

(2). $\mathbb{N}$ is not compact in $\mathbb{R}$.

(3). The intervals (a,b), $(a,b]$, $[a,b)$, (a,∞), $[a,\infty)$, $(-\infty,a]$, and $(-\infty,a)$ are not compact in $\mathbb{R}$.

(4). $\{0\} \cup \{1/n : n \in \mathbb{N}\}$ is is compact in $\mathbb{R}$.

Theorem 3.9 (Compact sets in $\mathbb{R}$: Heine-Borel theorem). *Let $A \subset \mathbb{R}$. Then A is compact iff A is closed and bounded.*

From Heine-Borel theorem, the intervals $[a,b]$ are compact in $\mathbb{R}$. From Problem 3-7, the compact sets in $\mathbb{R}_l$ are closed and bounded. But the following theorem says more.

Theorem 3.10 (Compact sets in $\mathbb{R}_l$). *Let $A \subset \mathbb{R}_l$. If A is compact, then A is countable.*

Remark. One should caution that the countable sets are not necessarily compact, e.g. $\mathbb{N}$ is not compact in $\mathbb{R}_l$.

Proof. Fix $a \in A$. Notice that

$$(-\infty, \infty) \subset \left(\bigcup_{n=1}^{\infty} \left(-\infty, a - \frac{1}{n} \right) \right) \cup [a, \infty).$$

So

$$\left\{\bigcup_{n=1}^{\infty}\left(-\infty, a-\frac{1}{n}\right)\right\} \cup \{[a, \infty)\}$$

covers A. Hence, there is $n_a \in \mathbb{N}$ such that

$$A \subset \left(\bigcup_{n=1}^{n_a}\left(-\infty, a-\frac{1}{n}\right)\right) \cup [a, \infty) = \left(-\infty, a-\frac{1}{n_a}\right) \cup [a, \infty).$$

This means that

$$\left(a-\frac{1}{n_a}, a\right) \cap A = \emptyset.$$

Let $b \in A$ and $b \neq a$. We claim that

$$\left(a-\frac{1}{n_a}, a\right) \cap \left(b-\frac{1}{n_b}, b\right) = \emptyset.$$

If the above intersection is nonempty, then without loss of generality, assume $a < b$. It then follows that $a \in (b - 1/n_b, b)$, but this is impossible since $a \in A$ and $(b - 1/n_b, b) \cap A = \emptyset$.

Choose $q_a \in \mathbb{Q} \cap (a - 1/n_a, a)$. Then $q_a \neq q_b$ if $a \neq b$. So the map $a \to q_a$ is injective and hence A is countable. □

THEOREM 3.11. *Let X be a compact space and $Y \subset X$ be closed. Then Y is also compact.*

PROOF. Let $\mathcal{A} = \{U_\alpha\}$ be an open cover of Y in X, in which U_α is open in X for each α. Notice that $\{U_\alpha\} \cup \{X \setminus Y\}$ is an open cover of X, since $X \setminus Y$ is open in X. Because X is compact, there is a finite subcover of $\{U_\alpha\} \cup \{X \setminus Y\}$, denoted as

$$\{U_{\alpha_1}, ..., U_{\alpha_n}\} \cup \{X \setminus Y\}.$$

Then $\{U_{\alpha_1}, ..., U_{\alpha_n}\}$ is a finite cover of Y, proving that Y is compact. □

THEOREM 3.12. *Let X be a Hausdorff space and $Y \subset X$ be compact. Then Y is closed.*

PROOF. We show that $X \setminus Y$ is open in X. That is, for $x \in X \setminus Y$, there is a neighborhood W of x such that $W \subset X \setminus Y$, i.e. $W \cap Y = \emptyset$.

For each $y \in Y$, since $y \neq x$, there are neighborhood U_y of y and W_y of x such that $U_y \cap W_y = \emptyset$. Notice that $\{U_y : y \in Y\}$ is an open cover of Y. There is a finite subcover since Y is compact, denoted as

$$\{U_{y_1}, ..., U_{y_n}\}.$$

Set

$$W = \cap_{i=1}^n W_{y_i}.$$

Then W is open. We show that $W \cap Y = \emptyset$. This is because for each $z \in W$, $z \in W_{y_i}$ for each $i = 1, ..., n$. Hence, $z \notin U_{y_i}$ for each $i = 1, ..., n$ so $z \notin \cup_{i=1}^n U_{y_i}$. But $Y \subset \cup_{i=1}^n U_{y_i}$. It then follows that $z \notin Y$. □

THEOREM 3.13. *Let X and Y be two topological spaces and $f : X \to Y$ be a continuous function. Suppose that X is compact. Then $f(X)$ is compact in Y.*

PROOF. Let $\mathcal{A} = \{U_\alpha\}$ be an open cover of $f(X)$ in Y. Since f is continuous, $f^{-1}(U_\alpha)$ is open for each α. It then follows that $\{f^{-1}(U_\alpha)\}$ is an open cover of X. There is a finite subcover because X is compact, denoted as

$$\{f^{-1}(U_{\alpha_1}), ..., f^{-1}(U_{\alpha_n})\}.$$

Then

$$\{U_{\alpha_1}, ..., U_{\alpha_n}\}$$

is a finite subcover of $f(X)$. Therefore, $f(X)$ is compact in Y. □

This theorem in particular shows that two homeomorphic topological spaces are both compact or both not compact. Therefore, open intervals are not homeomorphic to closed and bounded intervals in $\mathbb{R}$.

THEOREM 3.14. *Let X_i be compact spaces and $X = X_1 \times \cdots \times X_n$ be equipped with the product topology. Then X is also compact.*

An immediate consequence is that $A \subset \mathbb{R}^n$ is compact iff A is closed and bounded.

3.2.1. Maximum value theorem.

THEOREM 3.15. *Let X be a compact space and $f : X \to \mathbb{R}$ be continuous. Then there are $a, b \in X$ such that $f(a) \leq f(x) \leq f(b)$ for all $x \in X$.*

PROOF. Let $A = f(X) \subset \mathbb{R}$. It suffices to show that there are largest element and smallest element in A.

Suppose that there is no largest element in A. Then $\{(-\infty, a) : a \in A\}$ is an open cover of A. (This is because if there is $M \in A$ that is not contained in $(-\infty, a)$ for each $a \in A$, then $M \geq a$ for each $a \in A$, implying that M is the maximum in A.) Since A is compact, there is a finite open cover of A, denoted as

$$\{(-\infty, a_1), ..., (-\infty, a_n)\}.$$

Let $a = \max\{a_1, ..., a_n\}$. Then $a \notin (-\infty, a_i)$ for all $i = 1, ..., n$. But $a \in A$, leading to a contradiction to the fact that $\{(-\infty, a_1), ..., (-\infty, a_n)\}$ covers A.

Similarly, there is a smallest element of A. □

3.2.2. Uniform continuity theorem.

THEOREM 3.16. *Let X be a compact metric space and Y be a metric space. Suppose that $f : X \to Y$ be continuous. Then f is uniformly continuous.*

PROOF. Let d_X and d_Y be the metrics on X and on Y, respectively. Let $\varepsilon > 0$. Since f is continuous, there is $\delta_z > 0$ for each $z \in X$ such that $d_Y(f(z), f(y)) < \varepsilon/2$ if $d_X(z, y) < \delta_z$ and $y \in X$. Now $\{B_{d_X}(z, \delta_z/2) : z \in X\}$ covers X, there is a finite open cover of X since X is compact, denoted as

$$\{B_{d_X}(z_1, \delta_{z_1}/2), ..., B_{d_X}(z_n, \delta_{z_n}/2)\}.$$

Let $\delta = \min\{\delta_{z_1}/2, ..., \delta_{z_n}/2\}$. Let $x, y \in X$ such that $d_X(x, y) < \delta$. There is z_i for some $i = 1, ..., n$ such that $x \in B_{d_X}(z_i, \delta_{z_i}/2)$. It then follows that $d_Y(f(x), f(z)) < \varepsilon/2$.

By triangle inequality,

$$d_X(z, y) \leq d_X(z, x) + d_X(x, y) < \frac{\delta_{z_i}}{2} + \delta < \delta_{z_i}.$$

Hence, $d_Y(f(z), f(y)) < \varepsilon/2$. By triangle inequality again,

$$d_Y(f(x), f(y)) \leq d_Y(f(x), f(z)) + d_Y(f(z), f(y)) < \varepsilon.$$

□

Problems.

3-7. Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on a set X and $\mathcal{T}_1 \subset \mathcal{T}_2$.

(1). Suppose that X equipped with $\mathcal{T}_2$ is compact. Prove that X equipped with $\mathcal{T}_1$ is also compact.

(2). Suppose that X equipped with $\mathcal{T}_1$ is compact. Is X equipped with $\mathcal{T}_2$ compact? Prove your assertion.

3-8. Prove that finite union of compact sets is compact.

3-9. Let X be a metric space and $Y \subset X$. Prove that if Y is compact in X, then Y is closed and bounded in X. (We say that Y is bounded if there exist $x \in X$ and $R > 0$ such that $Y \subset B(x, R)$.)

3-10. Let X be a metric space and $Y \subset X$ is closed and bounded. Is Y compact? Prove your assertion.

3-11. Let $\mathbb{R}$ be equipped with finite complement topology. Prove that every subset of $\mathbb{R}$ is compact.

3-12. Let $\mathbb{R}$ be equipped with the countable complement topology, i.e. $U \subset \mathbb{R}$ is open if $\mathbb{R} \setminus U$ is countable or $\mathbb{R}$. Is $[0, 1]$ compact? Prove you assertion.

CHAPTER 4

The metrization theorems

We learn from the previous discussions that metric spaces possess several nice properties than general topological spaces, see e.g. §2.4.1. It is then desirable to determine whether a given topology can be induced by a metric, in which case we say that the topological space is metrizable. We introduce Urysohn metrization theorem (which provides sufficient condition for metrizability) and Nagata-Smirnov[ii] metrization theorem (which provides sufficient and necessary condition for metrizability), without proof. We use $\mathbb{R}$ and $\mathbb{R}_l$ as primary examples of metrizable and non-metrizable spaces.

THEOREM 4.1 (Urysohn metrization theorem). *Every Hausdorff regular space with a countable basis is metrizable.*

Example.
(1). $\mathbb{R}$ equipped with the standard topology has a countable basis.
(2). $\mathbb{R}_l$ equipped with the lower limit topology does not have a countable basis. See Problem 4-1.

The following definition entails the different degrees of separability.

Definition (Hausdorff, regular, and normal spaces). Let X be a topological space.
(i). We say X is Hausdorff if for every $x, y \in X$ and $x \neq y$, there are open sets $U \ni x$ and $W \ni y$ such that $U \cap W = \emptyset$.
(ii). We say X is regular if for every $x \in X$ and $B \subset X$ closed and $B \not\ni x$, there are open sets $U \ni x$ and $W \supset B$ such that $U \cap W = \emptyset$.
(iii). We say X is normal if for every closed sets $A, B \subset X$ and $A \cap B = \emptyset$, there are open sets $U \supset A$ and $W \supset B$ such that $U \cap W = \emptyset$.

Remark. Regular spaces are not necessarily Hausdorff. For example, let $X = \{a, b\}$ be equipped with the indiscrete topology $\{\emptyset, \{a, b\}\}$. Then X is regular but not Hausdorff (i.e. a and b can not be separated by open sets.) However, if all the singleton sets are closed, then normal⇒regular⇒Hausdorff.

Example.
(1). $\mathbb{R}$ is Hausdorff, regular, and normal.
(2). $\mathbb{R}_l$ is Hausdorff, regular, and normal. See Problem 4-2.

PROPOSITION 4.2. *Every metric space is normal.*

Remark. Since singleton sets are closed in the metric space, an immediate consequence is that every metric space is Hausdorff and regular.

PROOF. Let A and B be two disjoint closed sets in a metric space X with metric d. For each $a \in A$, since $a \in B^c$ which is open, there is $\varepsilon_a > 0$ such that $B(a, \varepsilon_a) \subset B^c$. Similarly, for

[ii]This is Yuriĭ Smirnov (1921-2017), not the Fields medalist Stanislav Smirnov (1970-).

each $b \in B$, there is $\varepsilon_b > 0$ such that $B(b, \varepsilon_b) \subset A^c$. Denote

$$U = \bigcup_{a \in A} B(a, \varepsilon_a/2) \quad \text{and} \quad W = \bigcup_{b \in B} B(b, \varepsilon_b/2).$$

Then $U \supset A$ and is open, $W \supset B$ and is open.

We next show that $U \cap W = \emptyset$. Suppose that there are $a \in A$ and $b \in B$ such that $c \in B(a, \varepsilon_a/2) \cap B(b, \varepsilon_b/2)$. Then triangle inequality implies that

$$d(a, b) \le d(a, c) + d(c, b) < \frac{\varepsilon_a + \varepsilon_b}{2}.$$

If $\varepsilon_a \le \varepsilon_b$, then $d(a, b) < \varepsilon_b$. It then follows that $a \in B(b, \varepsilon_b)$, which is not possible since $a \in A$ and $B(b, \varepsilon_b) \subset A^c$. Similarly, if $\varepsilon_a > \varepsilon_b$, then $d(a, b) < \varepsilon_a$. It then follows that $b \in B(a, \varepsilon_a)$, which is not possible since $b \in B$ and $B(a, \varepsilon_a) \subset B^c$. □

The three conditions in Urysohn metrization theorem to imply metrizability: Hausdorff, regular, and with a countable basis are not all necessary. In particular, Problem 2-19 shows that every metric space is Hausdorff; Proposition 4.2 shows that every metric space is regular; the following theorem shows that having countable basis is not necessary.

Theorem 4.3 (Nagata-Smirnov metrization theorem). *A space is metrizable iff it is Hausdorff regular and has a basis that is countably locally finite.*

Definition (Local finiteness). Let X be a topological space. A collection $\mathcal{A}$ of subsets of X is said to be locally finite in X if every point of X has a neighborhood that intersects only finitely many elements of $\mathcal{A}$. A collection $\mathcal{C}$ of subsets of X is said to be countably locally finite if $\mathcal{C}$ can be written as a countable union of locally finite collections.

Example.
(1). The collection $\{(n, n+2) : n \in \mathbb{N}\}$ is locally finite in $\mathbb{R}$.
(2). The collection $\{(0, 1/n) : n \in \mathbb{N}\}$ is not locally finite in $\mathbb{R}$.
(3). The collection $\{(0, 1/n) : n \in \mathbb{N}\}$ is locally finite in $(0, 1)$.

Remark. It is straightforward to see that a countable collection is countably locally finite, when written in a union of singleton sets, but the reverse is not true.

Example. Let $X = \{0, 1\}^\omega$ be equipped with the discrete topology. The elements of X are sequences of numbers $(x_1, x_2, ...)$, in which $x_i = 0, 1$. Then X is a metric metric with metric, e.g.

$$d(x, y) = \sup_{i \in \mathbb{N}} \{|x_i - y_i|\}$$

for $x = (x_1, x_2, ...)$ and $y = (y_1, y_2, ...)$.

Notice that X does not have a countable basis. But consider the collection

$$\mathcal{A}_n = \left\{ \prod_{i=1}^{\infty} X_i : X_i = \{0, 1\} \text{ for } i = 1, ..., n \text{ and } X_i = \{0\} \text{ or } \{1\} \text{ for } i \ge n+1 \right\}.$$

Then each singleton set $\{x\}$ as a neighborhood of x intersect only finitely many elements of $\mathcal{A}_n$. Finally, notice that

$$\mathcal{B} = \bigcup_{n=1}^{\infty} \mathcal{A}_n$$

is a countable locally finite basis of X.

Problems.

4-1. Prove that $\mathbb{R}_l$ does not have a countable basis.

4-2. Prove that $\mathbb{R}_l$ is normal (and therefore is Hausdorff and regular since singleton sets in $\mathbb{R}_l$ are closed).

4-3. Prove that $\mathbb{R}_l$ does not have a countably locally finite basis. (Hint: Using Nagata-Smirnov metrization theorem, it suffices to show that $\mathbb{R}_l$ is not metrizable since $\mathbb{R}_l$ is Hausdorff and normal. Notice that $\overline{\mathbb{Q}} = \mathbb{R}$ in the lower limit topology, that is, there is a countable dense set $\mathbb{Q}$. Suppose that $\mathbb{R}_l$ is metrizable, then there is a countable basis. But this is not possible according to 4-1.)

CHAPTER 5

The fundamental groups

5.1. Homotopy of paths

Let X be a topological space. Recall that a path from the initial point x_0 to the final point x_1 for $x_0, x_1 \in X$ is a continuous function $f : I = [0, 1] \to X$ such that $f(0) = x_0$ and $f(1) = x_1$. We say X is path connected if any two points in X can be joined by a path. We also know that path connected spaces are connected, but the reverse is not true. See §3.1.2.

Definition (Homotopic paths). Let X be a topological space and $f, g : I \to X$ be two paths from the initial point x_0 to the final point x_1 for $x_0, x_1 \in X$. We say that f and g are path homotopic if there is a continuous function $F : I \times I \to X$ such that for all $(t, s) \in I \times I$,

(i). $F(t, 0) = f(t)$ and $F(t, 1) = g(t)$, i.e. the initial path at $s = 0$ is f and the final path at $s = 1$ is g,

(ii). $F(0, s) = x_0$ and $F(1, s) = x_1$, i.e. the initial point at $t = 0$ is x_0 and the final point at $t = 1$ is x_1 on all the paths for $s \in I$.

We denote as $f \simeq_p g$ if f and g are path homotopic. We call F a path homotopy between f and g.

Remark. In the above definition, Condition (i) says that F represents a continuous way of deforming the path f to the path g; while Condition (ii) says that the end points x_0 and x_1 remain fixed during the deformation.

The above path homotopy is in fact a special case of homotopy between continuous functions.

Definition (Homotopy). Let X and Y be topological spaces and $f, g : X \to Y$ be two continuous functions. We say that f and g are homotopic if there is a function $F : X \times I \to Y$ such that for all $x \in X$,

$$F(x, 0) = f(x) \quad \text{and} \quad F(x, 1) = g(x),$$

and denoted as $f \simeq g$. We call F a homotopy between f and g.

Remark. Similarly as in the path homotopy, a continuous function f is homotopic to another continuous function g if f can be deformed continuously to g. Notice however there is no restrictions of the end points as in the path homotopies.

LEMMA 5.1. *Let X be a topological space. The path homotopy $\simeq_p$ is an equivalence relation in the paths.*

PROOF.

(i). [Reflexivity] Let $f : I \to X$ be a path from x_0 to x_1. We show that $f \simeq_p f$. Observe that $F(t, s) = f(t)$ for all $(t, s) \in I \times I$ is a path homotopy between f and f.

(ii). [Symmetry] Let $f, g : I \to X$ be two paths from x_0 to x_1 and $F : I \times I \to X$ be a path homotopy between f and g. Observe that $G(t, s) = F(t, 1 - s)$ for all $(t, s) \in I \times I$ is a homotopy between g and f.

(iii). [Transitity] Let $f, g, h : I \to X$ be three paths from x_0 to x_1. Let $F : I \times I \to X$ be a path homotopy between f and g and $G : I \times I \to X$ be a path homotopy between g and h. Define

$$H(t,s) = \begin{cases} F(t,2s) & \text{for } s \in [0,1/2], \\ G(t,2s-1) & \text{for } s \in [1/2,1]. \end{cases}$$

is a path homotopy between f and h. The continuity of H follows the Pasting Lemma in Theorem 2.16.

□

Let X, Y be topological spaces. A similar argument shows that homotopy is an equivalence relation in the set of continuous functions from X to Y.

Definition. Let f be a path in a topological space X. We denote $[f]$ the equivalence class under the path homotopy relation $\simeq_p$.

Definition (Identity paths). Let X be a topological space and $x \in X$. We define the identity path $e_x : I \to X$ from x to x as the constant function $e_x(t) = x$ for all $t \in I$.

PROPOSITION 5.2. *Let X be a topological space and $f : I \to X$ be a path from x_0 to x_1. Then f, e_{x_0}, and e_{x_1} are homotopic.*

PROOF. Observe that $F : I \times I \to X$ defined by $F(t,s) = f(s)$ is a homotopy between e_{x_0} and e_{x_1}. That is,

$$F(t,0) = f(0) = x_0 = e_{x_0}(t) \quad \text{and} \quad F(t,1) = f(1) = x_1 = e_{x_1}(t).$$

We see that F is continuous since $F = f \circ \pi_2$, in which $\pi_2 : I \times I \to I$ is the projection map by $\pi_2(t,s) = s$.

Define $G(t,s) = f(st)$. Then G is a homotopy between e_{x_0} and f. That is,

$$G(t,0) = f(0) = x_0 = e_{x_0}(t) \quad \text{and} \quad G(t,1) = f(t).$$

We see that G is continuous since $G = f \circ h$, in which $h : I \times I \to I$ is defined $h(t,s) = st$. □

Example.

(1). Let $x_0, x_1 \in \mathbb{R}^2$. Then all the paths from x_0 to x_1 are equivalent. In particular, all the paths from x to x are equivalent to the identity path e_x.

(2). Let $X = \mathbb{R}^2 \setminus \{p\}$ with $p = (0,0)$ be the punctured plane. Let $f(t) = (\cos \pi t, \sin \pi t)$, $g(t) = (\cos \pi t, 2 \sin \pi t)$, and $h(t) = (\cos \pi t, -\sin \pi t)$ be three paths from $(1,0)$ to $(-1,0)$. Then $f \simeq_p g$ (with the path homotopy $F : I \times I \to X$ defined by $F(t,s) = (\cos \pi t, (s+1) \sin \pi t)$) but $f \not\simeq_p h$.

Definition (Products of paths). Let X be a topological space and $x_0, x_1, x_2 \in X$. Let f be a path from x_0 to x_1 and g be a path from x_1 to x_2. We define the product $f * g : I \to X$ of f and g to be the path from x_0 to x_2 given by the equations

$$f * g(t) = \begin{cases} f(2t) & \text{for } t \in [0,1/2], \\ g(2t-1) & \text{for } t \in [1/2,1]. \end{cases}$$

The continuity of $f * g$ follows the Pasting Lemma in Theorem 2.16.

LEMMA 5.3. *Let X be a topological space and $x_0, x_1, x_2 \in X$. Let f be a path from x_0 to x_1 and g be a path from x_1 to x_2. Then $[f] * [g] = [f * g]$. That is, if $\tilde{f} \simeq_p f$ and $\tilde{g} \simeq_p g$, then $\tilde{f} * \tilde{g} \simeq_p f * g$.*

PROOF. Let f and $\tilde{f}$ be two paths from x_0 to x_1 and $F : I \times I \to X$ be a path homotopy between f and $\tilde{f}$. That is,

$$F(t,0) = f(t), \quad F(t,1) = \tilde{f}(t), \quad F(0,s) = x_0, \quad F(1,s) = x_1.$$

Let g and $\tilde{g}$ be two paths from x_1 to x_2 and $G : I \times I \to X$ be a path homotopy between g and $\tilde{g}$. That is,

$$G(t,0) = g(t), \quad G(t,1) = \tilde{g}(t), \quad G(0,s) = x_1, \quad G(1,s) = x_2.$$

Define $H : I \times I \to X$ by

$$H(t,s) = \begin{cases} F(2t,s) & \text{for } t \in [0,1/2], \\ G(2t-1,s) & \text{for } t \in [1/2,1]. \end{cases}$$

Then H is a path homotopy between $f * g$ and $\tilde{f} * \tilde{g}$. That is,

$$H(t,0) = f * g(t), \quad H(t,1) = \tilde{f} * \tilde{g}(t), \quad H(0,s) = x_0, \quad H(1,s) = x_2.$$

The continuity of H is implied by the Pasting Lemma in Theorem 2.16. □

THEOREM 5.4. *Let X be a topological space. Let f, g, h be paths in X.*

(i). [Left and right identities] If f is a path from x_0 to x_1, then

$$[e_{x_0}] * [f] = [f] \quad \text{and} \quad [f] * [e_{x_1}] = [f].$$

(ii). [Inverse] Let f be a path from x_0 to x_1. Define the inverse path by $f^{-1}(t) = f(1-t)$ as a path from x_1 to x_0. Then $[f^{-1}] = [f]^{-1}$,

$$[f] * [f^{-1}] = [e_{x_0}], \quad \text{and} \quad [f^{-1}] * [f] = [e_{x_1}].$$

(iii). [Associativity] If $([f] * [g]) * [h]$ is well defined, then $[f] * ([g] * [h])$ is also well defined and equals $([f] * [g]) * [h]$.

Problems.

5-1. Let X, Y, Z be topological spaces. Suppose that $f, \tilde{f} : X \to Y$ are two homotopic functions and $g, \tilde{g} : Y \to Z$ are two homotopic functions. Prove that $g \circ f, \tilde{g} \circ \tilde{f} : X \to Z$ are homotopic.

5-2. Let X be a path connected space. Prove that all continuous functions from I to X are homotopic.

5.2. The fundamental groups

First recall the concept of groups.

Definition (Groups). A group is a set G equipped with an operation $\cdot$ that satisfies the following group laws.

(0). For all $a, b \in G$, $a \cdot b \in G$.
(i). (Identity element) There is $e \in G$ such that $a \cdot e = e \cdot a = a$ for all $a \in G$.
(ii). (Inverses) For all $a \in G$, there is $b \in G$ such that $a \cdot b = b \cdot a = e$. We call b the inverse of a and denote as a^{-1}.
(iii). (Associativity) For all $a, b, c \in G$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Example.

(1). The singleton set $G = \{e\}$ that contains the identity element is a group.
(2). $\mathbb{Z}$ is a group with addition as the group operation, i.e. $a \cdot b = a + b$ for $a, b \in \mathbb{Z}$. The identity element is 0 and the inverse of a is $-a$.

(3). $\mathbb{Z}$ equipped with multiplication as the operation is not a group.
(4). $\mathbb{Q}$ is a group with addition as the group operation. The identity element is 0 and the inverse of $a \in \mathbb{Q}$ is $-a$.
(5). $\mathbb{Q} \setminus \{0\}$ is a group with multiplication as the group operation. The identity element is 1 and the inverse of $a \in \mathbb{Q} \setminus \{0\}$ is $1/a$.
(6). $\mathbb{Z}^2$ is a group with the group operation defined by $(a_1, b_1) \cdot (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$ for $(a_1, b_1), (a_2, b_2) \in \mathbb{Z}^2$. The identity element is $(0, 0)$ and the inverse of (a, b) is $(-a, -b)$.

Definition (Fundamental groups). Let X be a topological space and $x_0 \in X$. A path $f : I \to X$ such that $f(0) = f(1) = x_0$ is called a loop based at x_0. The set of path homotopy classes of loops based at x_0, with the operation $*$, is called the fundamental group of X relative to the base point x_0. It is denoted by $\pi_1(X, x_0)$.

Remark. Let X be a topological space and $x_0 \in X$. Then

$$\pi_1(X, x_0) = \{[f] : f \text{ is a loop based at } x_0\}$$

is a group such that

(0). For two equivalence classes of loops $[f]$ and $[g]$ based at x_0, $[f] * [g]$ is also an equivalence class of loops based at x_0.
(i). (Identity element) For all equivalence classes of loops $[f]$ based at x_0, $[f] * [e_{x_0}] = [e_{x_0}] * [f] = [f]$.
(ii). (Inverses) For each equivalence class of loops $[f]$ based at x_0, there is an equivalence class of loops $[f^{-1}]$ based at x_0 such that $[f] * [f^{-1}] = [f^{-1}] * [f] = [e_{x_0}]$.
(iii). (Associativity) For equivalence classes of loops $[f], [g], [h]$ based at x_0, $([f] * [g]) * [h] = [f] * ([g] * [h])$.

Example.
(1). $\pi_1(\mathbb{R}^n, x_0)$ is trivial for all $x_0 \in \mathbb{R}^n$, $n \geq 1$. That is, $\pi_1(\mathbb{R}^n, x_0) = \{[e_{x_0}]\}$, i.e. all loops based at x_0 are path homotopic to the identity loop e_{x_0}.
(2). $\pi_1(\mathbb{S}^n, x_0)$ is trivial for all $x_0 \in \mathbb{S}^n$, $n \geq 2$. Here $\mathbb{S}^n$ is the unit sphere in $\mathbb{R}^{n+1}$. That is, $\pi_1(\mathbb{S}^n, x_0) = \{[e_{x_0}]\}$, i.e. all loops based at x_0 are path homotopic to the identity loop e_{x_0}.

Therefore, the fundamental groups in these two examples are the same as the singleton group $\{e\}$.

Definition (Simply connected spaces). Let X be a path connected space. We say X is simply connected if $\pi_1(X, x_0) = \{[e_{x_0}]\}$ for all $x_0 \in X$.

Example.
(1). $\mathbb{R}^n$, $n \geq 1$, are simply connected.
(2). $\mathbb{S}^n$, $n \geq 2$, are simply connected.

To describe more complicated fundamental groups, we need to introduce

Definition (Group isomorphism). Let G, H be two groups. Let $f : G \to H$ be a bijective function. We say that f is an isomorphism if $f(a \cdot b) = f(a) \cdot f(b)$ for all $a, b \in G$. In this case, G and H are called isomorphic.

Remark. The isomorphism between two isomorphic groups is not necessarily unique. For example, let $G = \mathbb{Z}$ be a group equipped with addition as the group operation. Then both $f : G \to G$ defined by $f(a) = a$ and $g : G \to G$ defined by $g(a) = -a$ are isomorphisms between G and itself.

THEOREM 5.5. *Let X be a topological space and α be a path from x_0 to x_1 for $x_0, x_1 \in X$. Then there is a group isomorphism $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ defined by*

$$\hat{\alpha}([f]) = [\alpha^{-1}] * [f] * [\alpha].$$

PROOF. Let f and g be two loops based at x_0. Then

$$\begin{aligned} \hat{\alpha}([f]) * \hat{\alpha}([g]) &= ([\alpha^{-1}] * [f] * [\alpha]) * ([\alpha^{-1}] * [g] * [\alpha]) \\ &= [\alpha^{-1}] * [f] * [\alpha] * [\alpha^{-1}] * [g] * [\alpha] \\ &= [\alpha^{-1}] * [f] * [g] * [\alpha] \\ &= [\alpha^{-1}] * ([f] * [g]) * [\alpha] \\ &= \hat{\alpha}([f] * [g]). \end{aligned}$$

This shows that $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ preserves the group operation.

To show $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ is a bijection, it suffices to show that $\hat{\alpha}$ has an inverse. Denote $\beta = \alpha^{-1}$. We show that $\hat{\beta} \circ \hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_0)$ and $\hat{\alpha} \circ \hat{\beta} : \pi_1(X, x_1) \to \pi_1(X, x_1)$ are both identity maps. Compute that

$$\begin{aligned} \hat{\beta} \circ \hat{\alpha}([f]) &= \hat{\beta}([\alpha^{-1}] * [f] * [\alpha]) \\ &= [\beta^{-1}] * ([\alpha^{-1}] * [f] * [\alpha]) * [\beta] \\ &= [\beta^{-1}] * [\alpha^{-1}] * [f] * [\alpha] * [\beta] \\ &= [\alpha] * [\alpha^{-1}] * [f] * [\alpha] * [\alpha^{-1}] \\ &= ([\alpha] * [\alpha^{-1}]) * [f] * ([\alpha] * [\alpha^{-1}]) \\ &= [e_{x_0}] * [f] * [e_{x_0}] \\ &= [f]. \end{aligned}$$

This shows that $\hat{\beta} \circ \hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_0)$ is an identity maps. Similar argument shows that $\hat{\alpha} \circ \hat{\beta} : \pi_1(X, x_1) \to \pi_1(X, x_1)$ is an identity map. Therefore, $\hat{\alpha} : \pi_1(X, x_0) \to \pi_1(X, x_1)$ is a bijection. ☐

PROPOSITION 5.6. *Let X be a path connected space and $x_0, x_1 \in X$. Then $\hat{\alpha} = \hat{\beta}$ for all paths α and β from x_0 to x_1 iff $\pi_1(X, x_0)$ is abelian.*

Example.

(1). $\pi_1(\mathbb{S}^1, x_0)$ is isomorphic to $\mathbb{Z}$.

(2). Let $X = \mathbb{R}^2 \setminus \{p\}$ with $p \in \mathbb{R}^2$ be the punctured plane. Then $\pi_1(X, x_0)$ is isomorphic to $\mathbb{Z}$.

(3). $\pi_1(\mathbb{R}^{n+1} \setminus \{p\}, x_0)$ is isomorphic to $\pi_1(\mathbb{S}^n, y_0)$ for $n \geq 1$.

(4). $\pi_1(\mathbb{T}^2, x_0)$ is isomorphic to $\mathbb{Z}^2$. Here, $\mathbb{T}^2$ is the 2-dim torus.

Problems.

5-3. Let X be a topological space and $x_0, x_1, x_2 \in X$. Let α be a path from x_0 to x_1 and β be a path from x_1 to x_2. Denote $\gamma = \alpha * \beta$. Prove that $\hat{\gamma} = \hat{\beta} \circ \hat{\alpha}$.

5-4. Prove Proposition 5.6.

Bibliography

[Munkres] Munkres, James, *Topology: a first course*. Prentice-Hall, Inc., Englewood Cliffs, NJ, 1975. xvi+413 pp.