Real analysis

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Remark. You are entitled to a reward of 1 point toward a Test if you are the first person to report a bona-fide mathematical mistake (i.e. not including language typos and grammatical errors.)

ABSTRACT. This is a course to study Lebesgue measure and integration theory.

- Lebesgue measure on the real numbers $\mathbb{R}$ is a nonnegative function $m : \mathcal{M} \to [0, \infty]$, where $\mathcal{M}$ is a collection of subsets of $\mathbb{R}$. Here, m "measures" how large a set A of real numbers is if $A \in \mathcal{M}$. In addition, m operates "nicely" within $\mathcal{M}$ under set operations (taking complements, countable union, and countable intersections); this also requires that $\mathcal{M}$ is closed under these set operators, i.e. $\mathcal{M}$ is a σ-algebra.
- Lebesgue integration is defined for all the Lebesgue measurable functions, which is a much larger class than the collection of Riemann integrable functions.
  - In Riemann-Lebesgue theorem, one can characterize all the Riemann integrable functions completely.
  - Lebesgue integration operates "nicely" under taking limits of functions; this also requires that limits of Lebesgue measurable functions are measurable.

Contents

The language: Sets and mappings

We speak the language of sets and mappings in mathematics.

Definition (Sets).

- Given a set A, $x \in A$ denotes that x is an element (or member, point) of A and $x \notin A$ denotes that x is not an element of A.
- We say that two sets A and B are equal, denoted by $A = B$, if they have the same elements.
- Given two sets A and B, we say that A is a subset of B, denoted by $A \subset B$ (or $A \subseteq B$), if each element of A is a member of B; we say that A is a proper subset of B, denoted by $A \subsetneq B$, if $A \subseteq B$ and $A \neq B$.

Remark. Given two sets A and B, $A = B$ if and only if (i.e. iff) $A \subset B$ and $B \subset A$.

Example.

- $\mathbb{N} = \{1, 2, 3, ...\}$ denotes the set of natural numbers. (Notice that in some other books, $\mathbb{N}$ refers to the set $\{0, 1, 2, 3, ...\}$.)
- $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$ denotes the set of integers.
- $\mathbb{Q} = \{p/q : p, q \in \mathbb{Z}, q \neq 0\}$ denotes the set of rational numbers.
- $\mathbb{R}$ denotes the set of real numbers (for which the structure will be discussed later.)
- $\mathbb{R} \setminus \mathbb{Q}$ is called the set of irrational numbers.

Definition (The empty set). The set that has no elements is called the empty set and is denoted by $\emptyset$. A set that is not equal to the empty set is called nonempty.

Definition (A singleton set). A set that is has a single element is called a singleton set.

Definition (The power set). Given a set A, the set of all the subsets of A is called the power set of A and is denoted by $\mathcal{P}(A)$.

Example. Let $A = \{1, 2, 3\}$. Then

$$\mathcal{P}(A) = \big\{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\big\}.$$

Remark. If A has N elements, then $\mathcal{P}(A)$ has 2^N elements, which is strictly greater than N. (What if A contains infinitely many elements? – See Section 1.3. Cardinality.)

Definition (Union, intersection, and complement). Let A and B be two sets.

- The union of A and B is $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
- The intersection of A and B is $A \cap B = \{x : x \in A \text{ and } x \in B\}$. We say that A and B are disjoint if $A \cap B = \emptyset$.
- The complement of A in B is $B \setminus A = \{x : x \in B \text{ and } x \notin A\}$, and is also denoted by $B \sim A$. In particular, if all the set operations are within a universal set X, then for a set $A \subset X$, we simply call $X \setminus A$ the complement of A, and is also denoted by A^c.

Remark. Given a family of sets $\mathcal{F}$, we define

$$\bigcup_{E\in\mathcal{F}} E = \{x : x \in E \text{ for some } E \in \mathcal{F}\},$$

and

$$\bigcap_{E\in\mathcal{F}} E = \{x : x \in E \text{ for all } E \in \mathcal{F}\}.$$

In particular, if $\mathcal{F} = \{E_\lambda\}_{\lambda\in\Lambda}$, where λ is called the index and Λ is called the index set, then we define

$$\bigcup_{\lambda\in\Lambda} E_\lambda = \{x : x \in E_\lambda \text{ for some } \lambda \in \Lambda\},$$

and

$$\bigcap_{\lambda\in\Lambda} E_\lambda = \{x : x \in E_\lambda \text{ for all } \lambda \in \Lambda\}.$$

For example,

$$\bigcup_{i=1}^{n} E_i = \{x : x \in E_i \text{ for some } i = 1, ..., n\},$$

and

$$\bigcap_{i=1}^{n} E_i = \{x : x \in E_\lambda \text{ for all } i = 1, ..., n\}.$$

Theorem (De Morgan's Law). *Let $\mathcal{F}$ be a family of sets. Then*

$$\left(\bigcup_{E\in\mathcal{F}} E\right)^c = \bigcap_{E\in\mathcal{F}} E^c \quad \textit{and} \quad \left(\bigcap_{E\in\mathcal{F}} E\right)^c = \bigcup_{E\in\mathcal{F}} E^c.$$

Definition (Mappings). Let A and B be two sets. A mapping f from A to B, denoted by $f : A \to B$, is a correspondence that assigns to each element of A an element of B; for $x \in A$, we denote by $f(x)$ the assigned element in B. In the case when $B = \mathbb{R}$, we call the mapping f a function.

Definition (Domain and range). Let f be a mapping from A to B. We call A the domain of f. Given a subset $E \subset A$, we define $f(E) = \{y \in B : y = f(x) \text{ for some } x \in E\}$ the image of E. We call $f(A)$ the range of f.

Definition (Inverse image). Let f be a mapping from A to B. Given a subset $E \subset B$, we define $f^{-1}(E) = \{x \in A : f(x) \in E\}$ the inverse image of E.

Notice that f^{-1} is not a mapping. In order to make f^{-1} a mapping, we need the following concepts.

Definition (Onto and one-to-one mappings). Let f be a mapping from A to B.

- We say that f is one-to-one (or injective) if $x, y \in A$ and $x \neq y$ imply $f(x) \neq f(y)$.
- We say that f is onto (or surjective) if $f(A) = B$.

Definition (Invertible mappings). Let f be a mapping from A to B. We say that f is invertible if f is both one-to-one and onto, i.e. f establishes an one-to-one correspondence (or bijection) between the sets A and B, we say that A and B are equipotent.

Let f be an inverible mapping from A to B. Given each element $y \in B$, there is exactly one element $x \in A$ such that $f(x) = y$ and we denote by $f^{-1}(y) = x$. This defines a mapping $f^{-1} : B \to A$ and we call f^{-1} the inverse of f.

CHAPTER 1

Review of *Mathematical Analysis*

1.1. The real number system

The real number system $\mathbb{R}$ is an example of a complete ordered field. We define its structure in three steps:

Step 1: $\mathbb{R}$ is a field.

Definition (Fields). A field F is a nonempty set together with two operators $+$ and $\cdot$, called addition and multiplication, which satisfy the following axioms.

(A0). The operations $+$ and $\cdot$ are binary operations, that is, if $a, b \in F$, then $a+b$ and $a \cdot b$ are uniquely determined elements of F.
(A1). Commutativity of addition: If $a, b \in F$, then $a+b = b+a$.
(A2). Associativity of addition: If $a, b, c \in F$, then $(a+b)+c = a+(b+c)$.
(A3). The additive identity: There is an element in F, denoted by 0, such that $0+a = a+0 = a$ for all $a \in F$.
(A4). The additive inverse: For each $a \in F$, there is an element $b \in F$, called the additive inverse of a and denoted by $-a$, such that $a+b=0$.
(A5). Commutativity of multiplication: If $a, b \in F$, then $a \cdot b = b \cdot a$.
(A6). Associativity of multiplication: If $a, b, c \in F$, then $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
(A7). The multiplicative identity: There is an element in F, denoted by 1, such that $1 \cdot a = a \cdot 1 = a$ for all $a \in F$.
(A8). The multiplicative inverse: For each $a \in F$ and $a \neq 0$, there is an element $b \in F$, called the multiplicative inverse of a and denoted by a^{-1} or $1/a$, such that $a \cdot b = 1$.
(A9). The distributive property: If $a, b, c \in F$, then $a \cdot (b+c) = a \cdot b + a \cdot c$.

Remark. We will simply denote $a + (-b)$ by $a - b$, the multiplication $a \cdot b$ by ab, and $a \cdot b^{-1}$ by a/b.

Question. Among $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{R} \setminus \mathbb{Q}$, which one(s) are fields, why not for $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R} \setminus \mathbb{Q}$, and why for $\mathbb{Q}$ and for $\mathbb{R}$?

After Step 1, we can do addition, subtraction (as added by the additive inverse), multiplication, and division (as multiplied by the multiplicative inverse) on $\mathbb{R}$, according to all the rules that we knew before.

Step 2: $\mathbb{R}$ is an ordered field.

Definition (Ordered field). Let F be a field. Then F is an ordered field if it satisfies the following positivity axiom.

(A10). There is a nonempty subset P of F, called the positive subset, for which
  (i). P is closed under addition: If $a, b \in P$, then $a+b \in P$.
  (ii). P is closed under multiplication: If $a, b \in P$, then $a \cdot b \in P$.
  (iii). Law of trichotomy: For each $a \in F$, exactly one of the following three alternatives is true: $a \in P$, $-a \in P$, and $a = 0$. We say a is positive if $a \in P$ and is negative if $-a \in P$.

Let F be an ordered field. Given $a, b \in F$, we write $a > b$ if $a - b \in P$; we write $a \geq b$ if $a > b$ or $a = b$. We write $a < b$ iff $b > a$ and $a \leq b$ iff $b \geq a$.

Question. Among $\mathbb{Q}$ and $\mathbb{R}$, which one(s) are ordered fields?

After Step 2, we can compare the greatness of elements in $\mathbb{R}$. We can also define the intervals in $\mathbb{R}$.

Definition (Intervals). Given two numbers $a, b \in \mathbb{R}$, we define an interval as one of the following sets.

$$\begin{aligned}
(a,b) &= \{x : a < x < b\}, & (a,b] &= \{x : a < x \leq b\},\\
[a,b) &= \{x : a \leq x < b\}, & [a,b] &= \{x : a \leq x \leq b\},\\
(-\infty,a) &= \{x : x < a\}, & (-\infty,a] &= \{x : x \leq a\},\\
(a,\infty) &= \{x : x > a\}, & [a,\infty) &= \{x : x \geq a\}.
\end{aligned}$$

Remark (About $\pm\infty$). In convention, we write $\mathbb{R} = (-\infty, \infty)$. Notice that ∞ and $-\infty$ are not elements of $\mathbb{R}$, i.e. they are not real numbers. We sometimes use the notation of extended real numbers to mean $\mathbb{R} \cup \{\pm\infty\}$. This will come handy since there are sequences of real numbers that diverges to ∞ or $-\infty$, or can be viewed as converges within the extended real numbers with limit ∞ or $-\infty$.

We will also use $-\infty < x < \infty$ to denote that x is a finite real number.

Definition (Absolute value). Let F be an ordered field. We define the absolute value $|a|$ of an element $a \in F$ by

$$|a| = \begin{cases} a & \text{if } x \geq 0, \\ -a & \text{if } x < 0. \end{cases}$$

The following inequality about absolute value in $\mathbb{R}$ is fundamental in analysis.

THEOREM 1.1 (Triangle inequality). *For any $x, y \in \mathbb{R}$, $|x + y| \leq |x| + |y|$.*

Step 3: $\mathbb{R}$ is a complete ordered field.

There are several ways to define completeness equivalently. Here it is done through least upper bounds (or greatest lower bounds).

Definition (Upper and lower bounds). Let F be an ordered field.

- A set E in F is said to be bounded above provided there is an element $b \in F$ such that $x \leq b$ for all $x \in E$; the number b is called an upper bound of E.
- A set E in F is said to be bounded below provided there is an element $b \in F$ such that $x \geq b$ for all $x \in E$; the number b is called an lower bound of E.
- A set E in F is said to be bounded if E is bounded above and is bounded below.

Remark. If a set of real numbers E is not bounded above, then we denote $\sup E = \infty$; if a set of real numbers E is not bounded below, then we denote $\inf E = -\infty$.

Definition (Completeness by least upper bound). Let F be an ordered field. We say F is complete if it satisfies the following axiom.

(A11). For any subset E of F that is bounded above, there exists $\sup E \in F$. Here, b is called the least upper bound (or supremum) of the set E (denoted by l.u.b. E or $\sup E$) if b is an upper bound of E and $b \leq c$ for any upper bound c of E.

Remark. Equivalently, we can define completeness by greatest lower bounds: Let F be an ordered field. We say F is complete if for any subset E of F that is bounded below, there exists $\inf E \in F$. Here, b is called the greatest lower bound (or infimum) of the set E (denoted by g.l.b. E or $\inf E$) if b is an lower bound of E and $b \geq c$ for any lower bound c of E.

Question. Among $\mathbb{Q}$ and $\mathbb{R}$, which one(s) are complete ordered fields, why not for $\mathbb{Q}$ and why for $\mathbb{R}$?

Question. Let $E = \{x \in \mathbb{Q} : x^2 < 2\}$. Is $\sup E \in \mathbb{Q}$? In fact, we define $\sqrt{2}$ as a "new" real number (which is irrational.)

After Step 3, we can finally do limits and therefore calculus on $\mathbb{R}$.

$\boxed{\textbf{Homework Assignment}}$.

1-1. For a nonempty set of real numbers E, prove that $\inf E = \sup E$ iff E consists of a single point.

1-2. Use the completeness axiom to prove that every nonempty set of real numbers E that is bounded below has an infimum and that

$$\inf E = -\sup\{-x : x \in E\}.$$

1.2. The natural numbers, the rational numbers, and the irrational numbers

This section involves the properties of the natural numbers, the rational numbers, and the principle of mathematical induction, which will be used in the proofs consequently.

THEOREM 1.2 (Principle of mathematical induction). *For each natural number n, let $S(n)$ be some mathematical assertion. Suppose that $S(1)$ is true and $S(k)$ is true implies that $S(k+1)$ is also true for all natural numbers (or the statement that $S(1)$,...,$S(k-1)$ are all true implies that $S(k+1)$ is true.) Then $S(n)$ is true for all natural numbers.*

THEOREM 1.3. *Every nonempty set of natural numbers has a smallest element.*

THEOREM 1.4 (Archimedean Property). *For every pair of positive real numbers a and b, there is a natural number n for which $na > b$.*

Archimeadean property is applied in nearly every proof of the convergence of sequences of real numbers.

THEOREM 1.5. *Between any two distinct real numbers, there is a rational number and an irrational number.*

Question. Prove that each real number is the supremum of a set of rational number.

PROOF. If x is rational, then $x = \sup E$ for $E = \{x\}$. If x is irrational, then there is a rational number $x_1 \in (x-1, x)$ by the above theorem; there is a rational number $x_2 \in (x - 1/2, x)$ by the above theorem again. Inductively, there is a set of rational numbers $E = \{x_1, x_2, ...\}$ such that $x_n \in (x - 1/n, x)$. One can see that $\sup E = x$. (How to check? – See the following theorem.) ☐

THEOREM 1.6.

(i). A number $a = \sup E$ for a set of real numbers E iff a is an upper bound of E and for any $\varepsilon > 0$, there exists $x(\varepsilon) \in E$ such that $x(\varepsilon) > a - \varepsilon$.

(ii). A number $a = \inf E$ for a set of real numbers E iff a is a lower bound of E and for any $\varepsilon > 0$, there exists $x(\varepsilon) \in E$ such that $x(\varepsilon) < a + \varepsilon$.

Homework Assignment.

1-3. Prove that each real number is the supremum of a set of rational numbers and also the supremum of a set of irrational numbers.

1.3. Cardinality: countable and uncountable sets

Recall that two sets A and B are equipotent if there is an one-to-one correspondence (i.e. bijection) between A and B, in this case, we say that A and B have the same cardinality, denoted by $\mathrm{Card}\,(A) = \mathrm{Card}\,(B)$. In fact,

Definition. Let A and B be two sets.

- If there is a mapping $f : A \to B$ that is onto (i.e. surjective), then we say that the cardinality of A is greater than or equal to the cardinality of B, denoted by $\mathrm{Card}\,(A) \geq \mathrm{Card}\,(B)$;
- If there is a mapping $f : A \to B$ that is one-to-one (i.e. injective), then we say that the cardinality of A is less than or equal to the cardinality of B, denoted by $\mathrm{Card}\,(A) \leq \mathrm{Card}\,(B)$;
- $\mathrm{Card}\,(A) > \mathrm{Card}\,(B)$ if $\mathrm{Card}\,(A) \geq \mathrm{Card}\,(B)$ and $\mathrm{Card}\,(A) \neq \mathrm{Card}\,(B)$;
- $\mathrm{Card}\,(A) < \mathrm{Card}\,(B)$ if $\mathrm{Card}\,(A) \leq \mathrm{Card}\,(B)$ and $\mathrm{Card}\,(A) \neq \mathrm{Card}\,(B)$.

Cardinality generalizes the simple concept of "number of elements of a set".

Definition (Finite, countable, uncountable sets).

- A set A is said to be finite if either it is empty or there is a natural number n for which E is equipotent to the set $\{1, ..., n\}$.
- A set A is said to be countably infinite if it is equipotent to $\mathbb{N}$.
- A set A is said to be countable if it is either finite or countable infinite, that is, it is equipotent with a subset of $\mathbb{N}$.
- A set A is said to be uncountable if it is not countable.

A set E is finite iff its element can be enumerated as $x_1 = f(1)$, ..., $x_n = f(n)$ by the one-to-one correspondence $f : \{1, ..., n\} \to E$; E is countably infinite iff its elements can be enumerated by $x_1 = f(1)$, ..., $x_n = f(n)$, ..., by the one-to-one correspondence $f : \mathbb{N} \to E$.

Question. Examples of countable and uncountable sets? In particular, are $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{R}\setminus\mathbb{Q}$ countable?

THEOREM 1.7. *Any subset of a countable set is countable. In particular, any set of natural numbers is countable.*

THEOREM 1.8.

(i). The Cartesian product of countable sets is countable.
(ii). The countable union of countable sets is countable.
(iii). For any set A, $\mathrm{Card}\,(\mathcal{P}(A)) > \mathrm{Card}\,(A)$. *Here, $\mathcal{P}(A)$ is the power set of A.*

Example. $\mathrm{Card}\,(\mathcal{P}(\mathbb{N})) = \mathrm{Card}\,(\mathbb{R})$.

THEOREM 1.9. *A nonempty interval of real numbers is uncountable. In particular, $\mathbb{R}$ is uncountable.*

Homework Assignment.

1-4. Let $a, b, c, d \in \mathbb{R}$, $a < b$, and $c < d$. Prove that the two intervals (a, b) and (c, d) are equipotent.

1.4. Topology of the real numbers: open, closed, compact, and Borel sets

Definition (Open sets in $\mathbb{R}$). A set U of real numbers is said to be open if for each $x \in U$, there is $\delta > 0$ such that $(x - \delta, x + \delta) \subset U$.

The empty set $\emptyset$ and the set of real numbers $\mathbb{R}$ are open.

Question. Prove that the intervals (a, b), $(-\infty, a)$, and (a, ∞) are open. These intervals are called open intervals.

PROPOSITION 1.10. *Every nonempty open set of real numbers can be written as the union of a countable and disjoint collection of open intervals.*

PROOF. Read the proof in [**Stein-Shakarchi1**, Theorem 1.3 in Chapter 1]. □

Definition (Closed sets in $\mathbb{R}$). A set V of real numbers is said to be closed if $V^c = \mathbb{R} \setminus V$ is open.

The empty set $\emptyset$ and the set of real numbers $\mathbb{R}$ are closed, because $\emptyset = \mathbb{R}^c$ and $\mathbb{R} = \emptyset^c$ are open.

Question. Prove that the intervals $[a, b]$, $(-\infty, a]$, and $[a, \infty)$ are closed. These intervals are called closed intervals.

We generalize the above discussion to

$$\mathbb{R}^d = \{(x_1, ..., x_d) : x_i \in \mathbb{R} \text{ for } i = 1, ..., d\}.$$

Definition (Norm and distance in $\mathbb{R}^d$). The norm of $x \in \mathbb{R}^d$ is denoted by $|x|$ and is defined to be the standard Euclidean norm given by

$$|x| = \sqrt{x_1^2 + x_2^2 + \cdots + x_d^2}.$$

The distance between two points $x, y \in \mathbb{R}^d$ is $|x - y|$; the diameter of a set $E \subset \mathbb{R}^d$ is defined as

$$\operatorname{diam}(E) = \sup_{x,y \in E} |x - y|.$$

The distance between two sets $E, F \subset \mathbb{R}^d$ is defined by

$$\operatorname{dist}(E, F) = \inf_{x \in E, y \in F} |x - y|.$$

Definition (Open balls in $\mathbb{R}^d$). The open ball in $\mathbb{R}^d$ centered at x and of radius r is defined by

$$B_r(x) = \{y \in \mathbb{R}^d : |y - x| < r\}.$$

Definition (Open sets and closed sets in $\mathbb{R}^d$). A set $U \subset \mathbb{R}^d$ is said to be open if for each $x \in U$, there is $\delta > 0$ such that $B_\delta(x) \subset U$. A set $V \subset \mathbb{R}^d = \mathbb{R}^d \setminus V$ is said to be closed if V^c is open.

Question. Prove that $B_r(x)$ is open.

PROPOSITION 1.11.

(i). If $\{U_1, ..., U_n\}$ is a finite collection of open sets in $\mathbb{R}^d$, then $\cap_{j=1}^n U_j$ is open.

(ii). If $\{U_\lambda\}_{\lambda \in \Lambda}$ is a collection of open sets in $\mathbb{R}^d$, then $\cup_{\lambda \in \Lambda} U_\lambda$ is open.

Question. Let $\{U_\alpha\}_{\lambda \in \Lambda}$ be a collection of open sets in $\mathbb{R}^d$. Is $\cap_{\alpha \in I} U_\alpha$ necessarily is open? Why and why not? In particular, a countable intersection of open sets is called a G_δ sets, it is not necessarily open.

PROPOSITION 1.12.

(i). If $\{V_1, ..., V_n\}$ is a finite collection of closed sets in $\mathbb{R}^d$, then $\cup_{j=1}^n V_j$ is closed.

(ii). If $\{V_\lambda\}_{\lambda\in\Lambda}$ is a collection of closed sets in $\mathbb{R}^d$, then $\cap_{\lambda\in\Lambda} V_\lambda$ is closed.

Question. Let $\{V_\lambda\}_{\lambda\in\Lambda}$ be a collection of closed sets in $\mathbb{R}^d$. Is $\cup_{\lambda\in\Lambda} V_\lambda$ necessarily closed? Why and why not? In particular, a countable union of closed sets is called an F_σ sets, it is not necessarily closed.

Definition (Interior points). Let $E \subset \mathbb{R}^d$. A point x is said to be an interior point E if there exists $r > 0$ such that $B_r(x) \subset E$. The set of all the interior points of E is called the interior of E.

Remark. A set is open iff all its points are interior points.

Definition (Limit points, closure, and boundary). Let $E \subset \mathbb{R}^d$. A point x is said to be a point of closure of E is every $r > 0$, the ball $B_r(x)$ contains points of E. The closure $\overline{E}$ of E is the union of the E and all its limit points. The boundary ∂E of E is the set of the points which are in the closure of E but not in the interior of E.

THEOREM 1.13. *A set A of real numbers is closed iff $A = \overline{A}$.*

Definition (Rectangles and cubes). A (closed) rectangle R in $\mathbb{R}^d$ is given by the product of d one-dimensional closed and bounded intervals

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d],$$

where $a_j \leq b_j$ are real numbers, $j = 1, 2, ..., d$. In other words, we have

$$R = \{(x_1, ..., x_d) \in \mathbb{R}^d : a_j \leq x_j \leq b_j \text{ for all } j = 1, ..., d\}.$$

The volume of the rectangle R is denoted by $|R|$ and is defined to be

$$|R| = (b_1 - a_1)(b_2 - a_2) \cdots (b_d - a_d).$$

Of course, when $d = 1$ the volume equals length, and when $d = 2$ it equals area.

An open rectangle is the product of the open intervals $(a_1, b_1) \times (a_2, b_2) \times \cdots \times (a_d, b_d)$ and is the interior of a closed rectangle; a cube is a rectangle for which $b_1 - a_1 = b_2 - a_2 = \cdots = b_d - a_d$. So if $Q \subset \mathbb{R}^d$ is a cube of common side-length l, then $|Q| = l^d$.

A union of rectangles is said to be almost disjoint if the interiors of the rectangles are disjoint.

LEMMA 1.14. *If a rectangle R is the almost disjoint union of finitely many other rectangles, say $R = \cup_{k=1}^N R_k$, then*

$$|R| = \sum_{k=1}^N |R_k|.$$

If a rectangle R is a subset of the almost disjoint union of finitely many other rectangles, say $R \subset \cup_{k=1}^N R_k$, then

$$|R| \leq \sum_{k=1}^N |R_k|.$$

PROOF. Read the proof in [**Stein-Shakarchi1**, Lemmas 1.1 and 1.2 in Chapter 1]. □

PROPOSITION 1.15. *Every nonempty open set of $\mathbb{R}^d$ can be written as the union of a countable and almost disjoint collection of closed cubes.*

PROOF. Read the proof in [**Stein-Shakarchi1**, Theorem 1.4 in Chapter 1]. □

Definition (Bounded sets in $\mathbb{R}^d$). A set $E \subset \mathbb{R}^d$ is bounded if it is contained in some ball of finite radius.

Definition (Cover, open cover, finite cover). We say that a collection of sets $\{E_\lambda\}_{\lambda\in\Lambda}$ is a cover of a set A if $A \subset \cup_{\lambda\in\Lambda} E_\lambda$; if each E_λ is open, then we say $\{E_\lambda\}_{\lambda\in\Lambda}$ is an open cover of A; if Λ is finite, then we say $\{E_\lambda\}_{\lambda\in\Lambda}$ is a finite cover of A.

Definition (Compact sets). A set A is said to be compact if every open cover of A contains a finite subcover.

THEOREM 1.16 (Heine-Borel Theorem). *A set in $\mathbb{R}^d$ is compact iff it is bounded and closed.*

Question. Provide examples of sets of real numbers that are

(1) unbounded and closed but is not compact;
(2) bounded and not closed but is not compact.

THEOREM 1.17 (Nest Closed Set Theorem). *Let $\{V_n\}_{n=1}^\infty$ be a descending countable collection of nonempty closed sets of real numbers, that is, $V_1 \supset V_2 \supset V_3 \supset \cdots$. Suppose that V_1 is bounded. Then $\cap_{n=1}^\infty V_n \neq \emptyset$.*

Question. Provide an example that the above theorem fails for open sets.

Definition (σ-algebra). Given a set X, a collection $\mathcal{A}$ of subsets of X is called a σ-algebra (of subsets of X) if

(0). $\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$;
(i). if $E \in \mathcal{A}$, then $E^c \in \mathcal{A}$;
(ii). if $E_1, ..., E_n, ...$ are in $\mathcal{A}$, then $\cup_{n=1}^\infty E_n \in \mathcal{A}$.

Remark.

- By De Morgan's Law, (ii) and (iii) imply that if $E_1, ..., E_n, ...$ are in $\mathcal{A}$, then $\cap_{n=1}^\infty E_n \in \mathcal{A}$.
- Condition (0) is not necessary in the definition of σ-algebra: By (ii) and (i), $E \cup E^c = \mathbb{R} \in \mathcal{A}$; by (i), $\emptyset = \mathbb{R}^c \in \mathcal{A}$.

Question. Provide an example of a σ-algebra.

The trivial σ-algebra is $\mathcal{A} = \{\emptyset, X\}$.

Question. Given a collection of sets $\mathcal{F}$, how to construct a σ-algebra that contains $\mathcal{F}$? In particular, how to construct the smallest σ-algebra of $\mathcal{F}$?

One way is to complete $\mathcal{F}$ to be a σ-algebra by including all the possible new sets from completion and countable union; the other way is to take the intersection of all the σ-algebras that contain $\mathcal{F}$.

Definition (Borel sets). The Borel algebra $\mathcal{B}$ on $\mathbb{R}^d$ is the intersections of all the σ-algebras that contains all the open sets of real numbers. The members of the Borel algebra are called the Borel sets.

Question. Examples of the Borel sets?

All the open sets by definition; all the closed sets by (ii); all the G_δ sets, i.e. countable intersection of open sets; all the F_σ sets, i.e. countable union of closed sets, etc. The G_δ and F_σ sets play a considerable role. For example, the intervals $[a, b)$ and $(a, b]$ are both G_δ and F_σ sets. The terminology G_δ come from German "*Gebiete*" and δ refers to intersection ("*durchschnitt*"); the terminology F_σ comes from French "*Fermé*" and σ refers to union ("*summe*"). These notations are due to Hausdorff.

$\boxed{\textbf{Homework Assignment}}$.

1-5. Is $\mathbb{Q}$ open or closed?

1-6. Find two sets A and B such that $A \cap B = \emptyset$ and $\overline{A} \cap \overline{B} \neq \emptyset$.

1.5. Sequences and series of real numbers

Definition (Sequences). A sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ is a function from the natural numbers $\mathbb{N}$ to $\mathbb{R}$, defined by $a_n = f(n)$ for $n \in \mathbb{N}$. We say that a_n is the n-th term of the sequence.

We say that a sequence $\{a_n\}$ is bounded above (or below) if the set of its terms is bounded above (or below); we say that a sequence $\{a_n\}$ is bounded if it is bounded above and bounded below, or equivalently, there is a number M such that $|a_n| \leq M$ for all $n \in \mathbb{N}$.

Notice that there is no repetition or order in a set so the set of the terms of a sequence could be finite, in this case, the terms of the sequence take finite number of values, e.g. the set of the terms of a constant sequence is a singleton set.

Definition (Monotone sequences).

- We say that a sequence $\{a_n\}$ is increasing if $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$, and is strictly increasing if $a_{n+1} > a_n$ for all $n \in \mathbb{N}$.
- We say that a sequence $\{a_n\}$ is decreasing if $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$, and is strictly decreasing if $a_{n+1} < a_n$ for all $n \in \mathbb{N}$.
- We say that a sequence is monotone if it is increasing or decreasing, and is strictly monotone if it is strictly increasing or strictly decreasing.

Definition (Convergence and divergence). A sequence $\{a_n\}$ is said to be convergent if there exists $a \in \mathbb{R}$ such that for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ with

$$|a_n - a| < \varepsilon \quad \text{for all } n \geq N;$$

and we call a the limit of $\{a_n\}$ and write $\lim_{n\to\infty} a_n = a$ or $a_n \to a$.

We say that a sequence is divergent if it is not convergent.

Remark. We denote $a_n \nearrow a$ if $a_n \to a$ and $\{a_n\}$ is increasing; we denote $a_n \searrow a$ if $a_n \to a$ and $\{a_n\}$ is decreasing.

THEOREM 1.18. *Assume that* $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$. *Then*

(i). $\lim_{n\to\infty}(\alpha a_n + \beta b_n) = \alpha a + \beta b$ *for all* $\alpha, \beta \in \mathbb{R}$.

(ii). $\lim_{n\to\infty}(a_n b_n) = ab$.

(iii). $\lim_{n\to\infty} a_n/b_n = a/b$ *if* $b_n \neq 0$ *and* $b \neq 0$.

THEOREM 1.19. *Assume that* $\lim_{n\to\infty} a_n = a$ *and* $a_n \leq c$ *for all* $n \in \mathbb{N}$. *Then* $a \leq c$.

One can immediately derive that if $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, and $a_n \leq b_n$ for all $n \in \mathbb{N}$. Then $a \leq b$.

THEOREM 1.20. *A convergent sequence is bounded.*

With monotonicity, one can have the reverse.

THEOREM 1.21.

(i). A sequence that is increasing and bounded above is convergent. (And the limit is the supremum of the set of its terms.)

(ii). A sequence that is decreasing and bounded below is convergent. (And the limit is the infimum of the set of its terms.)

Question. Provide divergent sequences that monotonicity or boundedness fails in the above theorem.

Therefore, boundedness alone can not guarantee convergence. However,

THEOREM 1.22 (Bolzano-Weierstrass Theorem). *Every bounded sequence contains a convergent subsequence.*

We use $\{a_{n_k}\}$ to denote a subsequence of a sequence $\{a_n\}$, meaning that the k-th term of $\{a_{n_k}\}$ is the n_k-th term of $\{a_n\}$. We say a is a subsequential limit of $\{a_n\}$ if there is a subsequence of $\{a_n\}$ that converges to a.

THEOREM 1.23. *A sequence is convergent iff all of its subsequences are convergent with the same limit.*

Definition (Cauchy sequences). A sequence $\{a_n\}$ is said to be a Cauchy sequence if for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ with

$$|a_n - a_m| < \varepsilon \quad \text{for all } n, m \geq N.$$

THEOREM 1.24. *A sequence is convergent iff it is Cauchy.*

Definition (Convergence to infinity).

- A sequence $\{a_n\}$ is said to be convergent to ∞ (in the extended real numbers $\mathbb{R}\cup\{\pm\infty\}$) if for every $M > 0$, there is $N \in \mathbb{N}$ with
$$a_n \geq M \quad \text{for all } n \geq N;$$
and we write $\lim_{n\to\infty} a_n = \infty$.
- A sequence $\{a_n\}$ is said to be convergent to $-\infty$ (in the extended real numbers $\mathbb{R} \cup \{\pm\infty\}$) if for every $M > 0$, there is $N \in \mathbb{N}$ with
$$a_n \leq -M \quad \text{for all } n \geq N;$$
and we write $\lim_{n\to\infty} a_n = -\infty$.

One may also refer to $\lim_{n\to\infty} a_n = \infty$ as $\{a_n\}$ diverges to ∞.

Question.

(1) If $\lim_{n\to\infty} a_n = \infty$, then is $\{a_n\}$ increasing?
(2) If $\{a_n\}$ is increasing and unbounded above, then $\lim_{n\to\infty} a_n = \infty$.
(3) If $\{a_n\}$ is unbounded above, then there is a subsequence of $\{a_n\}$ that converges to ∞. – See the following discussion on lim sup and lim inf.

Definition (lim sup and lim inf). Let $\{a_n\}$ be a sequence. We define

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} \sup\{a_k : k \geq n\} \quad \text{and} \quad \liminf_{n\to\infty} a_n = \lim_{n\to\infty} \inf\{a_k : k \geq n\}.$$

lim sup and lim inf of a sequence always exists, and may be infinity.

THEOREM 1.25. *Let $\{a_n\}$ be a sequence.*

(i). $\limsup a_n$ *is the supremum of the set of the subsequential limits of $\{a_n\}$. Furthermore,* $\limsup a_n$ *is the maximum of the set of the subsequential limits of $\{a_n\}$, that is,* $\limsup a_n$ *is the greatest subsequential limit of $\{a_n\}$.*

(ii). $\liminf a_n$ *is the infimum of the set of the subsequential limits of $\{a_n\}$. Furthermore,* $\limsup a_n$ *is the minimum of the set of the subsequential limits of $\{a_n\}$, that is,* $\limsup a_n$ *is the smallest subsequential limit of $\{a_n\}$.*

Hence, a sequence $\{a_n\}$ is convergent iff $\limsup a_n = \liminf a_n$.

THEOREM 1.26. *Let $\{a_n\}$ be a sequence.*

(i). $\limsup a_n = l \in \mathbb{R}$ *iff for each $\varepsilon > 0$, there are infinitely many terms of $\{a_n\}$ for which $a_n > l - \varepsilon$ and only finitely many terms of $\{a_n\}$ for which $a_n > l + \varepsilon$.*

(ii). $\limsup a_n = \infty$ *iff $\{a_n\}$ is not bounded above.*

(iii). $\liminf a_n = l \in \mathbb{R}$ *iff for each $\varepsilon > 0$, there are infinitely many terms of $\{a_n\}$ for which $a_n < l + \varepsilon$ and only finitely many terms of $\{a_n\}$ for which $a_n < l - \varepsilon$.*

(iv). $\liminf a_n = -\infty$ *iff $\{a_n\}$ is not bounded below.*

Definition (Series). Let $\{a_n\}$ be a sequence. We define the n-th partial sum as

$$s_n = \sum_{k=1}^{n} a_k.$$

We say that the series $\sum_{k=1}^{\infty} a_k$ is summable if the sequence of partial sums $\{s_n\}$ is convergent, and write

$$\sum_{k=1}^{\infty} a_k = s \quad \text{if} \quad \lim_{n\to\infty} s_n = s.$$

PROPOSITION 1.27. *Let $\{a_n\}$ be a sequence.*

(i). The series $\sum_{k=1}^{\infty} a_k$ is summable iff for each $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that if $n \geq N$, then

$$\left|\sum_{k=n}^{n+m} a_k\right| < \varepsilon \quad \textit{for any } m \in \mathbb{N}.$$

(ii). If the series $\sum_{k=1}^{\infty} |a_k|$ is summable, then the series $\sum_{k=1}^{\infty} a_k$ is summable.

(iii). If $a_k \geq 0$ for all $k \in \mathbb{N}$, then the series $\sum_{k=1}^{\infty} a_k$ is summable iff the sequence of the partial sums is bounded.

Homework Assignment.

1-7. Prove the above proposition. Also prove that the inverse of (ii) is not true.

1.6. Continuous functions

Definition (Continuity). Let $E \subset \mathbb{R}^d$ and $f : E \to \mathbb{R}$ be a function.

- We say that f is continuous at x_0 if for each $\varepsilon > 0$, there is $\delta > 0$ such that if $x \in E$ and $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$.
- We say that f is discontinuous at $x_0 \in E$ if f is not continuous at x_0.
- We say that f is continuous on E if f is continuous at x_0 for all $x_0 \in E$.

THEOREM 1.28 (Continuity by sequences and by topology). *Let $f : E \to \mathbb{R}$ be a function.*

(i). f is continuous at $x_0 \in E$ iff for each sequence $\{x_n\} \subset E$ and $x_n \to x_0$, $f(x_n) \to f(x_0)$.

(ii). f is continuous on E iff for each open set $U \subset \mathbb{R}$, $f^{-1}(U)$ is relatively open in E, that is, $f^{-1}(U) = E \cap A$ for some open set $A \subset \mathbb{R}$.

Read the proof in [**Royden**]!

Example (Dirichlet function). Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}; \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Prove that f is discontinuous at each $x \in \mathbb{R}$.

Example (Riemann function). Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x = 0; \\ 0 & \text{if } x \text{ is irrational}; \\ \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ in the lowest terms with } q > 0. \end{cases}$$

Prove that f is discontinuous at each nonzero rational number and is continuous at 0 and at each irrational number.

Remark. Since Dirichlet and Riemann both made a wide range of contributions to mathematics, you may see many other things (e.g. functions) bearing their names elsewhere.

THEOREM 1.29 (The Intermediate Value Theorem). *Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then f attains each value that is between $f(a)$ and $f(b)$.*

Definition (Uniform continuity). Let $f : E \to \mathbb{R}$ be a function. We say that f is uniformly continuous on E if for each $\varepsilon > 0$, there is $\delta > 0$ such that if $x, y \in E$ and $|x - y| < \delta$, then $|f(x) - f(y)| < \varepsilon$.

THEOREM 1.30 (Continuous functions on compact sets). *Let $f : E \to \mathbb{R}$ be a continuous function. If E is compact, then*

(i). f is uniformly continuous;
(ii). f attains its maximum and minimum in E.

Question. Provide examples of continuous functions on noncompact sets that

(1) is not uniformly continuous;
(2) does not attain its maximum and minimum.

Homework Assignment.

1-8. Prove that a set E of real numbers is closed and bounded iff every continuous function on E takes a maximum value.

1-9. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Prove that the set of points at which f is continuous is a G_δ set, that is, it can be written as a countable intersection of open sets. (Hint: Let

$$E_n = \left\{ x : \text{there is an open interval } I \text{ containing } x \text{ such that } |f(x_1) - f(x_2)| < \frac{1}{n} \text{ when } x_1, x_2 \in I \right\}.$$

Then show that the set of points at which f is continuous is the countable intersection of E_n.)

1.7. Normed vector spaces and metric spaces

Definition (Vector spaces). A nonempty set V is a vector space (over the real numbers) if $v + w \in V$ and $cv \in V$ for all $v, w \in V$ and for all real numbers $c \in \mathbb{R}$.

Definition (Normed vector spaces). Let V be a vector space. A norm $\|\cdot\| : V \to [0, \infty)$ satisfies that

- $\|v\| = 0$ iff $v = 0$.
- $\|cv\| = |c|\|v\|$ if $c \in \mathbb{R}$ and $v \in V$.
- $\|v + w\| \le \|v\| + \|w\|$ for all $v, w \in V$.

Example. $\mathbb{R}^d$ is a normed vector space with norm $\|x\| = \sqrt{x_1^2 + \cdots + x_d^2}$ for $x = (x_1, ..., x_d) \in \mathbb{R}^d$.

Definition (Metric space). A metric space is a set X equipped with a (distance) function $d : X \times X \to [0, \infty)$ that satisfies

- $d(x, y) = 0$ iff $x = y$.
- $d(x, y) = d(y, x)$ for all $x, y \in X$.
- $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

One then sees that a normed vector space V is a metric space by defining $d(v, w) = \|v - w\|$.

Definition. A subset A of a normed vector space is said to be dense in X provided that for any $x \in X$ and any $\varepsilon > 0$, there exists $y \in A$ such that $\|x - y\| < \varepsilon$, or equivalently, there exists $\{y_n\}_{n=1}^{\infty} \subset A$ such that $\|y_n - x\| \to 0$ as $n \to \infty$.

Example. By Theorem 1.5, the rational numbers $\mathbb{Q}$ and the irrational numbers $\mathbb{R} \setminus \mathbb{Q}$ are dense in $\mathbb{R}$ equipped with the norm $\|x\|$ as the absolute value of $x \in \mathbb{R}$.

In a normed vector space X equipped with norm $\|\cdot\|$, one can define convergence and divergence with respect to the norm $\|\cdot\|$. In particular,

Definition (Cauchy sequences). Let X be a normed vector space X equipped with norm $\|\cdot\|$. We say that a sequence $\{x_n\}_{n=1}^{\infty} \subset X$ is a Cauchy sequence if for every $\varepsilon > 0$, there is $N \in \mathbb{N}$ with

$$\|x_n - x_m\| < \varepsilon \quad \text{for all } n, m \geq N.$$

Definition (Banach spaces). A normed vector space V is said to be complete if every Cauchy sequence has a limit in V, i.e. the space is "closed" under limiting operations. A complete vector space is called a Banach space.

Question. Examples of complete and incomplete normed vector spaces?

CHAPTER 2

Lebesgue measure

Our goal in this chapter to design a nonnegative function (i.e. the Lebesgue measure m) on a collection of sets of $\mathbb{R}^d$ (i.e. the Lebesgue measurable sets), that measures "how large a set in this collection is". We want the collection to contain all the rectangles and the measure of these rectangles to coincide the volume of the rectangles. We also want

- For a set $E \subset \mathbb{R}^d$ and a number $y \in \mathbb{R}$, define a translation of E, $E+y = \{x+y : x \in E\}$. Then $|R+y| = |R|$ for any rectangle R, that is, volume does not concern the position of the rectangle, only the difference of the end points (in the bounded case). Hence, for m, we should have that $m(E+y) = m(E)$ for all Lebesgue measurable sets E and real numbers y.
- If A and B are Lebesgue measurable, then $A \setminus B$ is also Lebesgue measurable; moreover, if $B \subset A$, then $m(A \setminus B) = m(A) - m(B)$.
- If $\{E_k\}_{k=1}^\infty$ is a collection of Lebesgue measurable sets, then $\cup_{k=1}^\infty E_k$ is also Lebesgue measurable; moreover, if $\{E_k\}_{k=1}^\infty$ is a disjoint collection, then the measure of the union equals the sum of the measure of E_k:
$$m\left(\bigcup_{k=1}^\infty E_k\right) = \sum_{k=1}^\infty m(E_k).$$
Of course, this condition also implies that we should have $\cup_{k=1}^\infty E_k$ to be Lebesgue measurable.

In particular, the last two conditions would require that the collection of the Lebesgue measurable sets to be closed under countable union and complement. – It should be a σ-algebra!

We follow Carathéodory construction:

(1) Define the exterior measure m^* through a natural approximation of sets by covers of cubes. (You can in fact use open or closed cubes in the cover. See Theorem 2.5.) Verify that this exterior measure satisfies several properties mentioned in the above, in particular, it coincides with the volume if restricted on rectangles, and is invariant under translation, but fails countable additivity. (Exterior measure is defined for all subsets of $\mathbb{R}^d$, and is, understandably, too general.)

(2) Fix countable additivity by introducing an additional property: We say that $E \subset \mathbb{R}^d$ is Lebesgue measurable if
$$m^*(A) = m^*(A \cap E) + m^*(A \setminus E) \quad \text{for each set } A \subset \mathbb{R}^d.$$
This condition eliminates the "bad" sets, that is, to fail the above condition, E and E^c will have to be very involved.

(3) The collection of Lebesgue measurable sets is a σ-algebra; and restricted on this σ-algebra (in which we can do countable union and complement), m^* satisfies all the required properties, is therefore called the Lebesgue measure.

(4) There are (many) non-measurable sets, but they are hard to construct.

2.1. Lebesgue exterior measure

Definition (Volume of the rectangles). A (closed) rectangle R in $\mathbb{R}^d$ is given by the product of d one-dimensional closed and bounded intervals

$$R = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_d, b_d],$$

where $a_j \leq b_j$ are real numbers, $j = 1, 2, ..., d$. The volume of the rectangle R is denoted by $|R|$ and is defined to be

$$|R| = (b_1 - a_1)(b_2 - a_2) \cdots (b_d - a_d).$$

When $b_1 - a_1 = b_2 - a_2 = \cdots = b_d - a_d$, we call R a cube.

Definition (Exterior measure). Let $E \subset \mathbb{R}^d$. We define the exterior (or outer) measure of E, denoted by $m^*(E)$, as

$$m^*(E) = \inf \left\{ \sum_{k=1}^{\infty} |Q_k| : E \subset \bigcup_{k=1}^{\infty} Q_k \right\}.$$

Here, the infimum is taken over all the collections of cubes $\{Q_k\}_{k=1}^{\infty}$ that cover E.

Remark. For any $A \subset \mathbb{R}^d$, $0 \leq m^*(A) \leq \infty$.

Remark (Monotonicity of the exterior measure). An immediate consequence of the definition of the exterior measure is that, if $A \subset B$, then $m^*(A) \leq m^*(B)$. This is called the monotonicity of the exterior measure.

Example. If $A = \{x = (x_1, ..., x_d)\}$ is a singleton set, then $m^*(A) = 0$. In fact, according to the above definition, a point is also a cube for which the side-lengths are all zero. So $m^*(A) \leq |A| = 0$.

We can prove differently. For each $\varepsilon > 0$, observe that

$$A \subset Q, \quad \text{for which } Q = \left[x_1 - \frac{\sqrt[d]{\varepsilon}}{2}, x_1 + \frac{\sqrt[d]{\varepsilon}}{2}\right] \times \cdots \times \left[x_d - \frac{\sqrt[d]{\varepsilon}}{2}, x_d + \frac{\sqrt[d]{\varepsilon}}{2}\right].$$

Hence, $m^*(A) \leq \varepsilon$ for all $\varepsilon > 0$ and therefore $m^*(A) = 0$.

Basically, we only use the fact that there exists a cube Q with arbitrarily small volume that covers a point.

Example. If A is countable, then $m^*(A) = 0$. Indeed, let $A = \{a_1, a_2, ...\}$, where $a_k = (a_{k1}, ..., a_{kd})$. Observe that for each $\varepsilon > 0$, let

$$Q_k = \left[a_{k1} - \frac{1}{2}\left(\frac{\varepsilon}{2^k}\right)^{\frac{1}{d}}, a_{k1} + \frac{1}{2}\left(\frac{\varepsilon}{2^k}\right)^{\frac{1}{d}}\right] \times \cdots \left[a_{kd} - \frac{1}{2}\left(\frac{\varepsilon}{2^k}\right)^{\frac{1}{d}}, a_{kd} + \frac{1}{2}\left(\frac{\varepsilon}{2^k}\right)^{\frac{1}{d}}\right].$$

Then

$$A \subset \bigcup_{k=1}^{\infty} Q_k \quad \text{and} \quad \sum_{k=1}^{\infty} |Q_k| = \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

Hence, $m^*(A) \leq \varepsilon$ for all $\varepsilon > 0$ and therefore $m^*(A) = 0$.

Proposition 2.1. *The exterior measure of a closed cube is its volume.*

Proof. Since a cube Q covers itself, $m^*(Q) \leq |Q|$. We then only need to prove the reverse inequality that $|Q| \leq m^*(Q)$.

Consider an arbitrary cube cover that $Q \subset \cup_{k=1}^{\infty} Q_k$. For any $\varepsilon > 0$, choose an open cube S_k which contains Q_k such that $|S_k| \leq (1+\varepsilon)|Q_k|$. Hence, $\{S_k\}_{k=1}^{\infty}$ is an open cover of Q. Since Q is compact, there exists a finite subcover $\{S_k\}_{k=1}^{N}$ such that

$$Q \subset \bigcup_{k=1}^{N} S_k.$$

Lemma 1.14 then implies that

$$|Q| \leq \sum_{k=1}^{N} |S_k| \leq (1+\varepsilon) \sum_{k=1}^{N} |Q_k| \leq (1+\varepsilon) \sum_{k=1}^{\infty} |Q_k|$$

for any $\varepsilon > 0$. Therefore,

$$|Q| \leq \sum_{k=1}^{\infty} |Q_k|$$

for any cube cover $\{Q_k\}_{k=1}^{\infty}$ of Q. Hence, taking the infimum over all such covers,

$$|Q| \leq m^*(Q).$$

□

Question. Show that $m^*(\mathbb{R}^d) = \infty$.

PROPOSITION 2.2. *The exterior measure of an open cube is its volume.*

PROOF. Let Q be an open cube. Since its closure $\overline{Q}$ covers Q, $m^*(Q) \leq |\overline{Q}| = |Q|$. We then only need to prove the reverse inequality that $|Q| \leq m^*(Q)$.

For any $\varepsilon > 0$, choose a closed cube S which is contained in Q such that $|S| \geq (1-\varepsilon)|Q|$. Hence, any cube cover $\{Q_k\}_{k=1}^{\infty}$ of Q is also a cube cover of S. By the previous proposition, $|S| = m^*(S)$ so

$$|S| = m^*(S) \leq \sum_{k=1}^{\infty} |Q_k|$$

for any cube cover $\{Q_k\}_{k=1}^{\infty}$ of Q. Hence, taking the infimum over all such covers,

$$|S| \leq m^*(Q).$$

Thus,

$$(1-\varepsilon)|Q| \leq |S| \leq m^*(Q)$$

for any $\varepsilon > 0$. Therefore, $|Q| \leq m^*(Q)$. □

PROPOSITION 2.3. *The exterior measure is invariant under translation. That is, for any set $E \subset \mathbb{R}^d$,*

$$m^*(E+y) = m^*(E).$$

PROOF. Given any collections of cubes $\{Q_k\}_{k=1}^{\infty}$ that covers E, $\{Q_k + y\}_{k=1}^{\infty}$ covers $E + y$. Since $|Q_k + y| = |Q_k|$ for all $k \in \mathbb{N}$, $m^*(E) = m^*(E+y)$. □

PROPOSITION 2.4. *The exterior measure is countably subadditive. That is, let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of sets, then*

$$m^*\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m^*(E_k).$$

PROOF. The proposition is trivially true if there exists E_k in $\{E_k\}_{k=1}^{\infty}$ such that $m^*(E_k) = \infty$. So we suppose that $m^*(E_k) < \infty$ for all $k \in \mathbb{N}$.

Let $\varepsilon > 0$. Since

$$m^*(E_k) = \inf \left\{ \sum_{j=1}^{\infty} |Q_j| : E_k \subset \bigcup_{j=1}^{\infty} Q_j \right\},$$

there exists a countable collection of cubes $\{Q_{k,j}\}_{j=1}^{\infty}$ that covers E_k and

$$\sum_{j=1}^{\infty} |Q_{k,j}| < m^*(E_k) + \frac{\varepsilon}{2^k}.$$

Now $\{Q_{k,j}\}_{k,j=1}^{\infty}$ is a countable collection of cubes that covers $\cup_{k=1}^{\infty} E_k$. Hence,

$$\begin{aligned} m^*\left(\bigcup_{k=1}^{\infty} E_k \right) &\leq \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} |Q_{k,j}| \\ &\leq \sum_{k=1}^{\infty} m^*(E_k) + \frac{\varepsilon}{2^k} \\ &= \sum_{k=1}^{\infty} m^*(E_k) + \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} \\ &= \sum_{k=1}^{\infty} m^*(E_k) + \varepsilon \end{aligned}$$

for all $\varepsilon > 0$. Hence,

$$m^*\left(\bigcup_{k=1}^{\infty} E_k \right) \leq \sum_{k=1}^{\infty} m^*(E_k).$$

□

The next theorem shows that one can in fact use open cube covers in the definition of exterior measure, too.

THEOREM 2.5. *Let $E \subset \mathbb{R}^d$. Define $\widetilde{m}^*(E)$ as*

$$\widetilde{m}^*(E) = \inf \left\{ \sum_{k=1}^{\infty} |Q_k| : E \subset \bigcup_{k=1}^{\infty} Q_k \right\}.$$

Here, the infimum is taken over all the collections of open cubes $\{Q_k\}_{k=1}^{\infty}$ that cover E. Then

$$\widetilde{m}^*(E) = m^*(E).$$

PROOF. If $\{Q_k\}_{k=1}^{\infty}$ is an open cube cover of E, then $\{\overline{Q}_k\}_{k=1}^{\infty}$ is a closed cube cover of E, and

$$\sum_{k=1}^{\infty} |Q_k| = \sum_{k=1}^{\infty} |\overline{Q}_k|.$$

This implies that

$$\widetilde{m}^*(E) \geq m^*(E).$$

Let $\varepsilon > 0$. If $\{S_k\}_{k=1}^{\infty}$ is an closed cube cover of E, then there $\{Q_k\}_{k=1}^{\infty}$ is an open cube cover of E. Here, Q_k is an open cube that covers S_k and $|Q_k| \le (1+\varepsilon)|S_k|$. Hence,

$$\sum_{k=1}^{\infty} |Q_k| \le (1+\varepsilon) \sum_{k=1}^{\infty} |S_k|.$$

This implies that for all $\varepsilon > 0$

$$\widetilde{m}^*(E) \le (1+\varepsilon) m^*(E).$$

Therefore,

$$\widetilde{m}^*(E) \le m^*(E).$$

□

$\boxed{\textbf{Homework Assignment}}$.

2-1. By using the properties of exterior measure, prove that the interval $[0, 1]$ is not countable.

2-2. Let A be the set of irrational numbers in the interval $[0, 1]$. Prove that $m^*(A) = 1$.

2-3. A subset of $\mathbb{R}^d$ is said to be a G_δ set provided it is the intersection of a countable collection of open sets. Prove that for any bounded set E, there is a G_δ set G for which

$$E \subset G \quad \text{and} \quad m^*(G) = m^*(E).$$

2-4. Prove that if $m^*(A) = 0$, then $m^*(A \cup B) = m^*(B)$.

2-5. Let A and B be bounded sets in $\mathbb{R}^d$ for which $\mathrm{dist}(A, B) > 0$. Prove that $m^*(A \cup B) = m^*(A) + m^*(B)$. (Hint: This is Observation 4 in [**Stein-Shakarchi1**, Chapter 1].)

2.2. The σ-algebra of Lebesgue measurable sets

In the previous section, we define the exterior measure and show that it has the following virtures:

(1) it is defined for all subsets of $\mathbb{R}^d$, which is a σ-algebra;
(2) the exterior measure of a closed or open cube is its volume;
(3) it is invariant under translation;
(4) it is countably subadditive.

In particular, in (4), exterior measure is only countably subadditive, but not countably additive. In fact, it fails to be finitely additive, that is, there are disjoint sets A and B such that

$$m^*(A \cup B) < m^*(A) + m^*(B).$$

In this section, we fix this problem and introduce

Definition (Measurable sets). A set $E \subset \mathbb{R}^d$ is said to be Lebesgue measurable (or simply measurable) if for any set $A \subset \mathbb{R}^d$,

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

We see that $\mathbb{R}^d$ and $\emptyset$ are measurable trivially. In fact, from the symmetry of the definition of measurable sets, E is measurable iff E^c is measurable. Indeed, this version of the definition makes one of the two requirements for the set of measurable sets to be a σ-algebra trivial. So to show that the set of measurable sets is a σ-algebra, we only need to show that it is closed under countable union, see Proposition 2.9.

Remark.

- If A or B is measurable and disjoint, say A is measurable, then
$$m^*(A \cup B) = m^*((A \cup B) \cap A) + m^*((A \cup B) \cap A^c) = m^*(A) + m^*(B).$$
So m^* becomes finitely additive.
- Since m^* is countably subadditive, we always have that
$$m^*(A) \leq m^*(A \cap E) + m^*(A \cap E^c),$$
in which A is a disjoint union of $A \cap E$ and $A \cap E^c$. Hence, E is measurable iff
$$m^*(A) \geq m^*(A \cap E) + m^*(A \cap E^c).$$

PROPOSITION 2.6. *Any set of exterior measure zero is measurable. In particular, any countable set is measurable.*

PROOF. Let E be a set of exterior measure zero. Then for any set $A \subset \mathbb{R}^d$, we have that
$$m^*(A \cap E) \leq m^*(E) = 0 \quad \text{and} \quad m^*(A \cap E^c) \leq m^*(A)$$
by monotonicity of the exterior measure. Hence,
$$m^*(A \cap E) + m^*(A \cap E^c) = 0 + m^*(A \cap E^c) = m^*(A \cap E^c) \leq m^*(A).$$
□

PROPOSITION 2.7. *The union of a finite collection of measurable sets is measurable.*

By definition, E_1 is measurable implies that E_1^c is also measurable. So if E_1 and E_2 are measurable, then $E_2 \setminus E_1 = E_2 \cap E_1^c$ is measurable by the above proposition.

PROOF. **Step 1**: We show that $E_1 \cup E_2$ is measurable if E_1 and E_2 are both measurable. It suffices to prove that for any set A,
$$m^*(A) \geq m^*(A \cap [E_1 \cup E_2]) + m^*(A \cap [E_1 \cup E_2]^c) = m^*(A \cap [E_1 \cup E_2]) + m^*(A \cap E_1^c \cap E_2^c),$$
where we use De Morgan's Law in the last equation. Since E_1 is measurable,
$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c),$$
which equals
$$m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c),$$
since E_2 is measurable. Notice that
$$A \cap [E_1 \cup E_2] = [A \cap E_1] \cup [A \cap E_1^c \cap E_2].$$
Hence, by countable subadditivity of the exterior measure,
$$m^*(A \cap [E_1 \cup E_2]) \leq m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2).$$
Therefore,
$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c \cap E_2) + m^*(A \cap E_1^c \cap E_2^c) \geq m^*(A \cap [E_1 \cup E_2]) + m^*(A \cap E_1^c \cap E_2^c),$$
and $E_1 \cup E_2$ is measurable.

Step 2: Let $E_1, ..., E_n$ be measurable. We prove that $\cup_{k=1}^n E_k$ is measurable by induction. It is trivial if $n = 1$; suppose that it is true for $n - 1$, notice that
$$\bigcup_{k=1}^n E_k = \left[\bigcup_{k=1}^{n-1} E_k\right] \cup E_n.$$
Hence, from the assumption that $\cup_{k=1}^{n-1} E_k$ is measurable, $\cup_{k=1}^n E_k$ is also measurable by Step 1. □

PROPOSITION 2.8 (Finite additivity of measurable sets). *Let A be any set and $\{E_k\}_{k=1}^n$ a finite disjoint collection of measurable sets. Then*

$$m^*\left(A\cap\left[\bigcup_{k=1}^n E_k\right]\right)=\sum_{k=1}^n m^*(A\cap E_k).$$

In particular, letting $A=\mathbb{R}^d$,

$$m^*\left(\bigcup_{k=1}^n E_k\right)=\sum_{k=1}^n m^*(E_k).$$

PROOF. We prove by induction. The theorem holds trivially when $n=1$. Suppose that it holds for $n-1$. Then

$$\begin{aligned}
& m^*\left(A\cap\left[\bigcup_{k=1}^n E_k\right]\right)\\
=\;& m^*\left(A\cap\left[\bigcup_{k=1}^n E_k\right]\cap E_n\right)+m^*\left(A\cap\left[\bigcup_{k=1}^n E_k\right]\cap E_n^c\right) \quad \text{since } E_n \text{ is measurable}\\
=\;& m^*\left(A\cap E_n\right)+m^*\left(A\cap\left[\left(\bigcup_{k=1}^n E_k\right)\cap E_n^c\right]\right)\\
=\;& m^*\left(A\cap E_n\right)+m^*\left(A\cap\left[\bigcup_{k=1}^{n-1} E_k\right]\right) \quad \text{since } E_1,...,E_n \text{ are disjoint}\\
=\;& m^*\left(A\cap E_n\right)+\sum_{k=1}^{n-1} m^*(A\cap E_k) \quad \text{by induction}\\
=\;& \sum_{k=1}^{n} m^*(A\cap E_k).
\end{aligned}$$

□

PROPOSITION 2.9. *The union of a countable collection of measurable sets is measurable.*

PROOF. Given a countable collection of measurable sets $\{A_k\}_{k=1}^\infty$, we first construct a disjoint collection of measurable sets $\{E_k\}_{k=1}^\infty$ such that $\cup_{k=1}^\infty A_k=\cup_{k=1}^\infty E_k:=E$. This can be easily done by setting

$$E_1=A_1 \quad\text{and}\quad E_k=A_k\setminus\left(\bigcup_{j=1}^{k-1}A_j\right) \text{ for } k\ge 2.$$

To show E is measurable, it suffices to show that for any set A,

$$m^*(A)\ge m^*(A\cap E)+m^*(A\cap E^c).$$

Denote $F_n=\cup_{k=1}^n E_k$. By Proposition 2.7, F_n is measurable, so

$$m^*(A)=m^*(A\cap F_n)+m^*(A\cap F_n^c)\ge m^*(A\cap F_n)+m^*(A\cap E^c),$$

since $F_n\subset E$ for all $n\in\mathbb{N}$.

By Proposition 2.8,

$$m^*(A\cap F_n)=m^*\left(A\cap\left[\bigcup_{k=1}^n E_k\right]\right)=\sum_{k=1}^n m^*(A\cap E_k),$$

so

$$m^*(A) \geq \sum_{k=1}^{n} m^*(A \cap E_k) + m^*(A \cap E^c),$$

for all $n \in \mathbb{N}$. Now the left-hand-side of the above inequality is independent of n, taking $n \to \infty$ in the right-hand-side, we have that

$$m^*(A) \geq \sum_{k=1}^{\infty} m^*(A \cap E_k) + m^*(A \cap E^c) \geq m^*(A \cap E) + m^*(A \cap E^c),$$

in which we use the countable subadditivity of exterior measure:

$$m^*(A \cap E) = m^*\left(A \cap \left[\bigcup_{k=1}^{\infty} E_k\right]\right) \leq \sum_{k=1}^{\infty} m^*(A \cap E_k).$$

□

Now we showed that the set of measurable sets $\mathcal{M}$ is closed under complement (by definition) and under countable union (by Proposition 2.9) so $\mathcal{M}$ is a σ-algebra.

Next we show that $\mathcal{M}$ contains the Borel algebra $\mathcal{B}$, that is, all the Borel sets are measurable. It suffices to prove that open sets are measurable. Then the σ-algebra $\mathcal{M}$ contains all the open sets, therefore contains the Borel algebra $\mathcal{B}$ (which is the intersection of all the σ-algebras that contain open sets.) From Proposition 1.15 that any open set can be written as a countable union of almost disjoint closed cubes, we only need to show that closed cubes are measurable.

Proposition 2.10. *Every closed cube is measurable.*

Proof. Let S be a closed cube and A be any set. We need to show that

$$m^*(A) \geq m^*(A_1) + m^*(A_2),$$

where $A_1 = A \cap S$ and $A_2 = A \cap S^c$. It suffices to prove that

$$\sum_{k=1}^{\infty} |Q_k| \geq m^*(A_1) + m^*(A_2)$$

for all cube cover $\{Q_k\}_{k=1}^{\infty}$ of A. Now denote $R_k' = Q_k \cap S$ and $R_k'' = Q_k \cap S^c$. Then

$$|Q_k| = |R_k'| + |R_k''|.$$

By breaking R_k' and R_k'' into countable union of almost disjoint closed cubes and possibly renumbering, we can assume that R_k' and R_k'' are closed cubes. Hence, $\{R_k'\}_{k=1}^{\infty}$ and $\{R_k''\}_{k=1}^{\infty}$ are cube covers of A_1 and A_2, respectively. Hence,

$$\sum_{k=1}^{\infty} |R_k'| \geq m^*(A_1) \quad \text{and} \quad \sum_{k=1}^{\infty} |R_k''| \geq m^*(A_2).$$

Therefore,

$$\sum_{k=1}^{\infty} |Q_k| = \sum_{k=1}^{\infty} |R_k'| + \sum_{k=1}^{\infty} |R_k''| \geq m^*(A_1) + m^*(A_2).$$

□

Theorem 2.11. *The collection $\mathcal{M}$ of measurable sets is a σ-algebra that contains all the Borel sets.*

Proposition 2.12. *The translate of a measurable set is measurable.*

Proof. Let E be a measurable set. Let $A \subset \mathbb{R}^d$ and $y \in \mathbb{R}^d$. Then

$$\begin{aligned} m^*(A) &= m^*(A-y) \quad \text{since } m^* \text{ is translation-invariant,} \\ &= m^*([A-y]\cap E) + m^*([A-y]\cap E^c) \quad \text{since } E \text{ is measurable,} \\ &= m^*\big(([A-y]\cap E)+y\big) + m^*\big(([A-y]\cap E^c)+y\big) \quad \text{since } m^* \text{ is translation-invariant,} \\ &= m^*(A\cap[E+y]) + m^*(A\cap[E+y]^c). \end{aligned}$$

Hence, $E+y$ is measurable. □

Homework Assignment.

2-6. Prove that if a σ-algebra of subsets of $\mathbb{R}$ contains intervals of the form (a,∞), then it contains all intervals.

2-7. Prove that if a set $E \subset \mathbb{R}^d$ has positive exterior measure, then there is a bounded subset of E that also has positive exterior measure.

2.3. Exterior and interior approximation of Lebesgue measurable sets

Observe that if $A \subset B$ and A is measurable with finite exterior measure, then

$$m^*(B) = m^*(B\cap A) + m^*(B\cap A^c) = m*(A) + m^*(B\setminus A),$$

which implies that

$$m^*(B\setminus A) = m^*(B) - m^*(A).$$

Such property is called the excision property of measurable sets.

The following theorem asserts that any measurable set can be approximated by open set from outside and by closed set from inside.

Theorem 2.13 (Approximate measurable sets by open, G_δ, closed, and F_σ sets). *Let $E \subset \mathbb{R}^d$. Then the following five statements are equivalent.*

(0). *E is measurable.*
(i). *For each $\varepsilon > 0$, there is an open set U containing E such that $m^*(U\setminus E) < \varepsilon$.*
(ii). *There is a G_δ set G containing E for which $m^*(G\setminus E) = 0$.*
(iii). *For each $\varepsilon > 0$, there is a closed set V contained in E such that $m^*(E\setminus V) < \varepsilon$.*
(iv). *There is an F_σ set F contained in E for which $m^*(E\setminus F) = 0$.*

Proof. (i) $\Leftrightarrow$ (iii) and (ii) $\Leftrightarrow$ (iv) follow De Morgan's Law. So it suffices to show that

$$(0) \Rightarrow (\text{i}) \Rightarrow (\text{ii}) \Rightarrow (0).$$

(0) $\Rightarrow$ (i). Let E be measurable.

Step 1: Assume that $m^*(E) < \infty$. For any $\varepsilon > 0$, by Theorem 2.5, there exists an open cube cover $\{Q_k\}_{k=1}^\infty$ such that

$$\sum_{k=1}^\infty |Q_k| < m^*(E) + \varepsilon.$$

Let $U = \cup_{k=1}^\infty Q_k$. Then U is open. In addition, from countable subadditivity of the exterior measure, we have that

$$m^*(U) \le \sum_{k=1}^\infty m^*(Q_k) = \sum_{k=1}^\infty |Q_k| < m^*(E) + \varepsilon,$$

so

$$m^*(U) - m^*(E) < \varepsilon.$$

But $m^*(U) = m^*(E) + m^*(U \setminus E)$ by the excision property since E is measurable. Hence,

$$m^*(U \setminus E) = m^*(U) - m^*(E) < \varepsilon.$$

Step 2: Assume that $m^*(E) = \infty$. Let

$$Q_k = [-k,k] \times \cdots \times [-k,k] \quad \text{for } k \in \mathbb{N}.$$

Then

$$E = \bigcup_{k=1}^{\infty} E_k, \quad \text{where } E_k = E \cap Q_k,\ k \in \mathbb{N}.$$

Since $m^*(E_k) < \infty$, there is an open set $U_k \supset E_k$ such that

$$m^*(U_k \setminus E_k) < \frac{\varepsilon}{2^k},$$

by Step 1. Let $U = \cup_{k=1}^{\infty} U_k$. Then U is open and contains E. Now

$$U \setminus E = \left[\bigcup_{k=1}^{\infty} U_k\right] \setminus E = \left[\bigcup_{k=1}^{\infty} (U_k \setminus E)\right] \subset \left[\bigcup_{k=1}^{\infty} (U_k \setminus E_k)\right].$$

Hence, by countable subadditivity of the exterior measure,

$$m^*(U \setminus E) \le \sum_{k=1}^{\infty} m^*(U_k \setminus E_k) \le \sum_{k=1}^{\infty} \frac{\varepsilon}{2^k} = \varepsilon.$$

(i) $\Rightarrow$ (ii). By the assumption, for any $k \in \mathbb{N}$, there exists an open set $U_k \supset E$ such that $m^*(U_k \setminus E) < 1/k$. Let $G = \cap_{k=1}^{\infty} U_k$. Then G is a G_δ set. Moreover, $G \setminus E \subset U_k \setminus E$ for all $k \in \mathbb{N}$, so

$$m^*(G \setminus E) \le m^*(U_k \setminus E) < \frac{1}{k}$$

for all $k \in \mathbb{N}$. Hence, $m^*(G \setminus E) = 0$.

(ii) $\Rightarrow$ (0). By the assumption, there is a G_δ set G such that $G \supset E$ and $m^*(G \setminus E) = 0$. Hence $G \setminus E$ is measurable. Since G is also measurable, $E = G \setminus (G \setminus E)$ is measurable. □

Remark. In (i), one can similarly show that for any set $E \subset \mathbb{R}^d$, measurable or not, for any $\varepsilon > 0$ there is an open set U such that $m^*(U) < m^*(E) + \varepsilon$. But it does not imply that $m^*(U \setminus E) < \varepsilon$ unless the excision property holds, which requires that E is measurable.

THEOREM 2.14 (Approximate measurable sets by open cubes). *Let $E \subset \mathbb{R}^d$ be measurable and $m^*(E) < \infty$. Then for each $\varepsilon > 0$, there is a finite disjoint collection of open cubes $\{Q_k\}_{k=1}^n$ for which*

$$m^*(E \setminus U) + m^*(U \setminus E) < \varepsilon,$$

where $U = \cup_{k=1}^n Q_k$.

PROOF. By (i) of Theorem 2.13, for any $\varepsilon > 0$ there is an open set $W \supset E$ such that $m^*(W \setminus E) < \varepsilon/2$.

By Lemma 2.5, the open set W is a countable union of almost disjoint closed cubes $\{S_k\}_{k=1}^{\infty}$:

$$W = \bigcup_{k=1}^{\infty} S_k.$$

By finite additivity of the exterior measure of the cubes (which are measurable),

$$\sum_{k=1}^{n} |S_k| = \sum_{k=1}^{n} m^*(S_k) = m^*\left(\bigcup_{k=1}^{n} S_k\right) \le m^*(W) = m^*(E) + m^*(W \setminus E) < \infty.$$

for all $n \in \mathbb{N}$. But $|S_k| > 0$, then the series $\sum_{k=1}^{\infty} |S_k|$ is summable. Hence, there is $n \in \mathbb{N}$ such that

$$\sum_{k=n+1}^{\infty} |S_k| < \frac{\varepsilon}{2}.$$

Denote Q_k as the open cube as the interior of S_k. Let

$$V = \bigcup_{k=1}^{n} S_k \quad \text{and} \quad U = \bigcup_{k=1}^{n} Q_k.$$

Notice that

$$m^*(S_k \setminus Q_k) = 0,$$

we have that

$$m^*(V \setminus U) \leq \sum_{k=1}^{n} (S_k \setminus Q_k) = 0.$$

We next show that U satisfies the required condition in the theorem.

Firstly, $U \setminus E \subset W \setminus E$ since $U \subset W$. So by monotonicity of the exterior measure,

$$m^*(U \setminus E) \leq m^*(W \setminus E) < \frac{\varepsilon}{2}.$$

Secondly, since

$$E \setminus U \subset W \setminus U \subset (V \setminus U) \cup \left(\bigcup_{k=n+1}^{\infty} S_k \right).$$

So by monotonicity of the exterior measure again,

$$m^*(E \setminus U) \leq m^*(V \setminus U) + m^* \left(\bigcup_{k=n+1}^{\infty} S_k \right) \leq \sum_{k=n+1}^{\infty} |S_k| < \frac{\varepsilon}{2}.$$

The theorem is thus completed. □

Homework Assignment.

2-8. Prove that a set $E \subset \mathbb{R}^d$ is measurable iff for each $\varepsilon > 0$, there is a closed set V and open set U for which $V \subset E \subset U$ and $m^*(U \setminus V) < \varepsilon$.

2-9. Let $E \subset \mathbb{R}$ have finite exterior measure. Prove that E is measurable iff for each open and bounded interval (a, b),

$$b - a = m^*((a, b) \cap E) + m^*((a, b) \setminus E).$$

2.4. Countable additivity and continuity of the Lebesgue measure

Definition (Lebesgue measure). The restriction of the exterior measure to the σ-algebra of measurable sets $\mathcal{M}$ is called Lebesgue measure. It is denoted by m, and $m(E) = m^*(E)$ if $E \in \mathcal{M}$.

PROPOSITION 2.15 (Countable additivity of Lebesgue measure). *If $\{E_k\}_{k=1}^{\infty}$ is a countable disjoint collection of measurable sets, then $\cup_{k=1}^{\infty} E_k$ is measurable and*

$$m \left(\bigcup_{k=1}^{\infty} E_k \right) = \sum_{k=1}^{\infty} m(E_k).$$

Proof. By Proposition 2.9, $\cup_{k=1}^{\infty} E_k$ is measurable; and by the countable subadditivity of exterior measure in Proposition 2.4,

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m(E_k).$$

So it suffices to show that

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) \geq \sum_{k=1}^{\infty} m(E_k).$$

Since for any $n \in \mathbb{N}$, $\cup_{k=1}^{n} E_k \subset \cup_{k=1}^{\infty} E_k$, by monotonicity of the exterior measure,

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) \geq m\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} m(E_k),$$

by Proposition 2.8. Since the left-hand-side is independent of n, taking $n \to \infty$ on the right-hand-side, we prove the proposition. □

Theorem 2.16. *The Lebesgue measure m on the σ-algebra of Lebesgue measurable sets $\mathcal{M}$ satisfies that*

(i). $m(Q) = |Q|$ for any open or closed cube Q,
(ii). $m(A) \leq m(B)$ if $A, B \in \mathcal{M}$ and $A \subset B$,
(iii). $m(E + y) = m(E)$ for any $E \in \mathcal{M}$ and $y \in \mathbb{R}^d$,
(iv). m is countably additive.

Theorem 2.17 (Continuity of measure).

(i). Let $\{A_k\}_{k=1}^{\infty}$ be an ascending collection of measurable sets, i.e. $A_k \subset A_{k+1}$ for all $k \in \mathbb{N}$. Then

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k).$$

(ii). Let $\{B_k\}_{k=1}^{\infty}$ be a descending collection of measurable sets, i.e. $B_{k+1} \subset B_k$ for all $k \in \mathbb{N}$. Assume that $m(B_1) < \infty$. Then

$$m\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} m(B_k).$$

Proof. (i). If there is $k_0 \in \mathbb{N}$ such that $m(A_{k_0}) = \infty$, then $m(\cup_{k=1}^{\infty} A_k) \geq m(A_{k_0}) = \infty$ by monotonicity of the exterior measure and so the theorem is proved.

If $m(A_k) < \infty$ for all $k \in \mathbb{N}$, then define $A_0 = \emptyset$ and $E_k = A_k \setminus A_{k-1}$ for $k \in \mathbb{N}$. We have that $\cup_{k=1}^{\infty} A_k = \cup_{k=1}^{\infty} E_k$.

Since $\{A_k\}_{k=1}^{\infty}$ is an ascending collection of measurable sets, $\{E_k\}_{k=1}^{\infty}$ be a disjoint collection of measurable sets. By countable additivity of measure, we have that

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k) = \sum_{k=1}^{\infty} m(A_k \setminus A_{k-1}) = \sum_{k=1}^{\infty} [m(A_k) - m(A_{k-1})] = \lim_{k\to\infty} m(A_k).$$

Hence,

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k).$$

(ii). Define $A_k = B_1 \setminus B_k$ for $k \in \mathbb{N}$. Since $\{B_k\}_{k=1}^{\infty}$ is a descending collection of measurable sets, $\{A_k\}_{k=1}^{\infty}$ is an ascending collection of measurable sets. So by (i),

$$m\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k),$$

in which the left-hand-side equals

$$m\left(\bigcup_{k=1}^{\infty}(B_1 \setminus B_k)\right) = m\left(B_1 \setminus \left[\bigcap_{k=1}^{\infty} B_k\right]\right) = m(B_1) - m\left(\bigcap_{k=1}^{\infty} B_k\right)$$

by De Morgan's Law, while the right-hand-side equals

$$\lim_{k\to\infty} m(A_k) = \lim_{k\to\infty} m(B_1 \setminus B_k) = m(B_1) - \lim_{k\to\infty} m(B_k).$$

The theorem is therefore completed.

□

Definition (Almost everywhere). We say that a property hold almost everywhere (or for almost all $x \in \mathbb{R}^d$) if the property holds for all $x \in E$ with $m(\mathbb{R}^d \setminus E) = 0$, i.e. the set of points for which the property fails is of measure zero.

Remark (Borel-Cantelli Lemma). Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of measurable sets for which $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then almost all $x \in \mathbb{R}^d$ belong to at most finitely many E_k's.

Proof. For each $n \in \mathbb{N}$, by countable subadditivity of measure,

$$m\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} m(E_k) < \infty.$$

Hence, by continuity of measure,

$$m\left(\bigcap_{n=1}^{\infty}\left[\bigcup_{k=n}^{\infty} E_k\right]\right) = \lim_{n\to\infty} m\left(\bigcup_{k=n}^{\infty} E_k\right) \leq \lim_{n\to\infty} \sum_{k=n}^{\infty} m(E_k) = 0.$$

Therefore, letting

$$A = \bigcap_{n=1}^{\infty}\left[\bigcup_{k=n}^{\infty} E_k\right],$$

$m(A) = 0$, and $x \notin A$ implies that x belongs to finitely many E_k's. □

Homework Assignment.

2-10. Prove that if $E \subset \mathbb{R}^d$ has finite measure and $\varepsilon > 0$, then E is the disjoint union of a finite number of measurable sets, each of which has measure at most ε.

2-11. Prove that if E_1 and E_2 are measurable in $\mathbb{R}^d$, then

$$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2).$$

2-12. Prove (by example) that the assumption that $m(B_1) < \infty$ is necessary in part (ii) of the theorem regarding continuity of measure.

2-13. Prove the continuity of measure together with finite additivity of measure implies countable additivity of measure.

2.5. Nonmeasurable sets

Given any measurable set E with positive measure, we construct a nonmeasurable set that is contained in E.

Step 1: Let $E = [0, 1]$. We divide $[0, 1]$ into classes by rational equivalence. That is, we say that $x \sim y$ for $x, y \in [0, 1]$ if $x - y \in \mathbb{Q}$. It is an equivalence relation since it is

- reflective: $x \sim x$,
- symmetric: $x \sim y$ implies that $y \sim x$,
- transitive: $x \sim y$ and $y \sim z$ imply $x \sim z$.

So $[0, 1] = \cup_{\lambda\in\Lambda} A_\lambda$, where A_λ is an equivalent class. It is easy to see that A_λ are disjoint.

Step 2: We choose an element in each A_λ and form a choice set $\mathcal{N}$. Its (seemingly obvious) well-definedness rests in the Axiom of Choice. Next we show that $\mathcal{N}$ is not measurable by contradiction.

Suppose that $\mathcal{N}$ is measurable. Enumerate $\mathbb{Q} \cap [-1, 1]$ as $\{r_j\}_{j=1}^\infty$. Then $\{\mathcal{N} + r_j\}_{j=1}^\infty$ is a collection of disjoint measurable sets with the same measure.

By countable additivity of m,

$$\bigcup_{j=1}^{\infty} (\mathcal{N} + r_j) \subset [-1, 2]$$

implies that

$$m\left(\bigcup_{j=1}^{\infty} (\mathcal{N} + r_j)\right) = \sum_{j=1}^{\infty} m(\mathcal{N} + r_j) = \sum_{j=1}^{\infty} m(\mathcal{N}) \le m([-1, 2]) = 3,$$

So $m(\mathcal{N}) = 0$. On the other hand,

$$[0, 1] \subset \bigcup_{j=1}^{\infty} (\mathcal{N} + r_j)$$

implies that

$$1 = m([0, 1]) \le \sum_{j=1}^{\infty} m(\mathcal{N} + r_j) = 0,$$

which is a contradiction.

Step 3: This construction can be generalized to all measurable sets with positive measure.

Remark. The existence of the nonmeasurable sets proves that

$$m^*(A \cup B) < m^*(A) + m^*(B)$$

for some disjoint sets A and B.

$\boxed{\textbf{Homework Assignment}}$.

2-14. Let E be a nonmeasurable set of finite exterior measure. Prove that there is a G_δ set G that contains E for which

$$m^*(E) = m^*(G), \quad \text{while } m^*(G \setminus E) > 0.$$

2.6. The Cantor set and the Cantor-Lebesgue function

In this section, we construct the Cantor set, which is of measure zero but is uncountable. We will also define the Cantor-Lebesgue function that is used to prove the existence of measurable but not Borel sets.

We first construct the Cantor set through the "open middle one-third removal" procedure. Construct inductively as follows.

- $C_0 = [0,1]$, $m(C_0) = 1$.
- $C_1 = [0,1/3] \cup [2/3,1]$, which is obtained by removing the middle one-third open interval in the interval of C_1, $m(C_1) = 2/3$.
- $C_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1]$, which is obtained by removing the middle one-third open intervals in the intervals of C_2, $m(C_2) = 4/9 = (2/3)^2$.
- $\cdots$.
- C_n is a union of 2^n closed intervals, each of which has length $(1/3)^n$. So $m(C_n) = (2/3)^n$.

Define the Cantor set $C = \cap_{n=1}^{\infty} C_n$. It is easy to see that C is closed, measurable, and $m(C) = \lim_{n\to\infty} m(C_n) = 0$ by continuity of measure.

We can also define the Cantor set by ternary expansion of the real numbers. That is, for each $x \in [0,1]$, $x = 0.a_1a_2a_3\cdots$ with $a_j = 0,1,2$. Then

$$C = \{x = 0.a_1a_2a_3\cdots : a_j = 0,2\}.$$

To prove that C is uncountable, we argue by contradiction. Suppose that we can enumerate $C = \{x_j\}_{j=1}^{\infty}$. Then

$$a_j = 0.a_{1j}a_{2j}a_{3j}\cdots \quad \text{with } a_{ij} = 0,2.$$

Let $y = 0.b_1b_2b_3\cdots$ such that $b_j = 0$ if $a_{jj} = 2$ and $b_j = 2$ if $a_{jj} = 0$. Then it is easy to see that $y \in [0,1]$ but $y \neq x_j$ for all $j \in \mathbb{N}$.

Remark. One can similarly argue that $[0,1]$ is uncountable. In fact, one can show that $\mathrm{Card}\,(C) = \mathrm{Card}\,([0,1]) = \mathrm{Card}\,(\mathbb{R})$.

Definition (Cantor-Lebesgue function). One can define the Cantor-Lebesgue function on $[0,1]$ as follows.

- Let $x \in C$. Then $x = 0.a_1a_2a_3\cdots$ in the ternary expansion, where $a_j = 0,2$. Define
$$\varphi(x) = 0.b_1b_2b_3\cdots \quad \text{in the binary expansion,}$$
in which $b_j = 0$ if $a_j = 0$ and $b_j = 1$ if $a_j = 2$.
- Let $x \in [0,1] \setminus C$. Then $x \in (a,b)$ for some open interval (a,b). (In fact, x is from one of the open intervals that we removed in the construction of the Cantor set.) Observe that $a, b \in C$ and $\varphi(a) = \varphi(b)$. Define
$$\varphi(x) = \varphi(a).$$

Observe that $[0,1] \setminus C$ is open and can be written as a countable union of disjoint open intervals, indeed, these intervals are the ones removed from the construction. One can easily see that the Cantor-Lebesgue function is constant on each of these intervals. Therefore, $\varphi : [0,1] \to [0,1]$ is continuous and $\varphi' = 0$ on $[0,1]\setminus C$. This means that φ is differentiable almost everywhere on $[0,1]$ since $m(C) = 0$.

Now define $f(x) = \varphi(x) + x$ for $x \in [0,1]$. Then f is continuous and strictly increasing on $[0,1]$. Precisely,

- $f : [0,1] \to [0,2]$ is onto;
- $m(f(C)) = m(f([0,1] \setminus C)) = 1$, where C is the Cantor set.

Now we use f to prove the existence of measurable but not Borel sets. Pick $B \subset f(C)$ be non-measurable, since $m(f(C)) > 0$. Then $A = f^{-1}(B) \subset C$ is measurable since $m(A) = 0$; but A is not Borel since otherwise $f(A)$ would be Borel and therefore measurable.

Homework Assignment.

2-15. Prove that a strictly increasing function that is defined on an interval has a continuous inverse.

2-16. Let f be a continuous function and B be a Borel set. Prove that $f^{-1}(B)$ is a Borel set.

2-17. Use the preceding two problems to prove that a continuous strictly increasing function that is defined on an interval maps Borel sets to Borel sets.

CHAPTER 3

Lebesgue measurable functions

3.1. Sums, products, and compositions

In this section, we define the measurable functions and prove their basic properties via the fact that Lebesgue measure operates nicely under taking complements, unions, and intersections, in the σ-algebra of measurable sets.

Definition (Lebesgue measurable functions). Let $E \subset \mathbb{R}^d$ be a measurable set. A function $f : E \to \mathbb{R}$ is said to be Lebesgue measurable, or simply measurable, if $f^{-1}(U) = \{x \in E : f(x) \in U\}$ is measurable for each open set $U \subset \mathbb{R}$.

Question. Any examples of measurable functions and non-measurable functions?

Answer. Let E be a set of real numbers. Define the characteristic function of E as

$$\chi_E(x) = \begin{cases} 1 & \text{if } x \in E; \\ 0 & \text{if } x \notin E. \end{cases}$$

It is evident that E is measurable iff χ_E is measurable.

Remark. Recall that $f : E \to \mathbb{R}$ is continuous if $f^{-1}(U)$ is open for each open set $U \subset \mathbb{R}$. Therefore, a continuous function is automatically measurable.

We frequently take inverse images of sets so we recall that

(1) $f^{-1}(A \setminus B) = f^{-1}(A) \setminus f^{-1}(B)$,
(2) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$,
(3) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$.

Proposition 3.1. *Let $E \subset \mathbb{R}^d$ be a measurable set and $f : E \to \mathbb{R}$. Then the following statements are equivalent.*

(0). *f is measurable.*
(i). *For each $c \in \mathbb{R}$, the set $f^{-1}\big((c,\infty)\big) = \{x \in E : f(x) > c\}$ is measurable.*
(ii). *For each $c \in \mathbb{R}$, the set $f^{-1}\big([c,\infty)\big) = \{x \in E : f(x) \geq c\}$ is measurable.*
(iii). *For each $c \in \mathbb{R}$, the set $f^{-1}\big((-\infty,c)\big) = \{x \in E : f(x) < c\}$ is measurable.*
(iv). *For each $c \in \mathbb{R}$, the set $f^{-1}\big((-\infty,c]\big) = \{x \in E : f(x) \leq c\}$ is measurable.*
(v). *For each closed set V in $\mathbb{R}$, $f^{-1}(V)$ is measurable.*

Remark. Let $f : E \to \mathbb{R} \cup \{\pm\infty\}$. We define that

$$\{x \in E : f(x) = \infty\} = \bigcap_{k=1}^{\infty} \{x \in E : f(x) > k\}$$

and

$$\{x \in E : f(x) = -\infty\} = \bigcap_{k=1}^{\infty} \{x \in E : f(x) < -k\}$$

We can then define a function $f : E \to \mathbb{R} \cup \{\pm\infty\}$ is measurable if any of (i)-(iv) is valid.

PROOF. (i) $\Leftrightarrow$ (iv), (ii) $\Leftrightarrow$ (iii), and (0) $\Leftrightarrow$ (v) because the collection of measurable sets is closed under taking complement. So it suffices to show that

$$(0) \Rightarrow (\mathrm{i}) \Rightarrow (\mathrm{ii}) \Rightarrow (0).$$

(0) $\Rightarrow$ (i) is trivial.

(i) $\Rightarrow$ (ii). Notice that

$$\{x \in E : f(x) \geq c\} = \bigcap_{k=1}^{\infty} \left\{x \in E : f(x) > c - \frac{1}{k}\right\},$$

which is measurable since every set in the countable intersection is measurable from (i).

(ii) $\Rightarrow$ (0). See Proposition 3.2. Notice that

$$\{x \in E : f(x) > c\} = \bigcup_{k=1}^{\infty} \left\{x \in E : f(x) \geq c + \frac{1}{k}\right\},$$

which is measurable since every set in the countable union is measurable from (ii). This shows that (i) is true.

Similarly,

$$\{x \in E : f(x) < c\} = \bigcup_{k=1}^{\infty} \left\{x \in E : f(x) \leq c - \frac{1}{k}\right\},$$

which is measurable since every set in the countable union is measurable from (ii). This shows that (iii) is true.

Since every open set U can be written as a countable union of bounded open intervals $\cup_{k=1}^{\infty} I_k$. Here, $I_k = (a_k, b_k) = (-\infty, b_k) \cap (a_k, \infty)$. Then

$$\begin{aligned} f^{-1}(U) = f^{-1}\left(\bigcup_{k=1}^{\infty} I_k\right) &= \bigcup_{k=1}^{\infty} f^{-1}(I_k) = \bigcup_{k=1}^{\infty} f^{-1}\big((-\infty, b_k) \cap (a_k, \infty)\big) \\ &= \bigcup_{k=1}^{\infty} f^{-1}\big((-\infty, b_k)\big) \cap f^{-1}\big((a_k, \infty)\big), \end{aligned}$$

which is measurable from (i) and (iii) (both follows (ii)). $\square$

Remark.

- The above theorem simply follows the fact that f^{-1} is interchangeable with taking complement, union, and intersection, and that the collection of measurable sets is closed under such operations, i.e. it is a σ-algebra.
- One can also define that f is measurable iff f^{-1} maps Borel sets to measurable sets (see Problem 3-5).
- Each of the above statement implies that for each $c \in \mathbb{R}$, the set $f^{-1}(\{c\}) = \{x \in E : f(x) = c\}$ is measurable. This follows that

  $$f^{-1}(\{c\}) = f^{-1}\big((-\infty, c] \cap [c, \infty)\big) = \{x \in E : f(x) \leq c\} \cap \{x \in E : f(x) \geq c\}.$$

  However, $f^{-1}(c)$ is a measurable set for all $c \in \mathbb{R}$ does not guarantee that f is measurable, see Problem 3-3.
- One can use any of (0)-(v) as the definition of measurable functions.

We know that if a set A is measurable and B is measure-zero, then $A \setminus B$ and $A \cup B$ are both measurable and have the same measure as A. Part (i) in the next proposition asserts a similar property for measurable functions.

PROPOSITION 3.2. *Let $f : E \to \mathbb{R}$ be a function on a measurable set E.*

(i). If f is measurable on E and $f = g$ a.e. on E, then g is measurable on E.

(ii). For a measurable subset D of E, f is measurable on E iff the restriction of f to D and $E \setminus D$ are measurable.

PROOF. (i). Since $f = g$ a.e. on E, there is a measure-zero set $A \subset E$ such that $f = g$ on $E \setminus A$. For every $c \in \mathbb{R}$,

$$\{x \in E : g(x) > c\} = \{x \in E \setminus A : g(x) > c\} \cup \{x \in A : g(x) > c\},$$

in which $\{x \in E \setminus A : g(x) > c\} = \{x \in E \setminus A : f(x) > c\}$ is measurable since f is measurable, and $\{x \in A : g(x) > c\} \subset A$ is measurable since $m(\{x \in A : g(x) > c\}) \le m(A) = 0$. Hence, $\{x \in E : g(x) > c\}$ is measurable so g is measurable.

(ii). Necessity: For every $c \in \mathbb{R}$, $\{x \in E : f(x) > c\}$ is measurable. Hence,

$$\{x \in D : f(x) > c\} = \{x \in E : f(x) > c\} \cap D$$

and

$$\{x \in E \setminus D : f(x) > c\} = \{x \in E : f(x) > c\} \setminus D$$

are both measurable. So $f|_D$ and $f|_{E\setminus D}$ are both measurable.

Sufficiency: Notice that for every $c \in \mathbb{R}$, $\{x \in D : f(x) > c\}$ and $\{x \in E \setminus D : f(x) > c\}$ are both measurable since $f|_D$ and $f|_{E\setminus D}$ are measurable. Then

$$\{x \in E : f(x) > c\} = \{x \in D : f(x) > c\} \cup \{x \in E \setminus D : f(x) > c\},$$

is measurable. So f is measurable. $\square$

Question. Is the above proposition still valid if the measurable functions are replaced by continuous functions?

THEOREM 3.3. *Let f and g be measurable functions on E that are finite a.e. on E.*

(i). For any $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g$ is measurable on E.

(ii). fg is measurable.

PROOF. (i). **Step 1**: If f is measurable, we prove that αf is measurable for $\alpha \in \mathbb{R}$. If $\alpha = 0$, then $\alpha f = 0$ is measurable. For every $c \in \mathbb{R}$, notice that

$$\{x \in E : \alpha f(x) > c\} = \{x \in E : f(x) > c/\alpha\} \quad \text{if } \alpha > 0,$$

and

$$\{x \in E : \alpha f(x) > c\} = \{x \in E : f(x) < c/\alpha\} \quad \text{if } \alpha < 0,$$

which are both measurable since f is measurable. So αf is measurable.

Step 2: It suffices to prove that $f + g$ is measurable if f and g are measurable. For every $c \in \mathbb{R}$, we want to break $\{x \in E : f(x) + g(x) < c\}$ to two (groups of) measurable sets, each of which relates to f and g. It is evident that

$$\{x \in E : f(x) + g(x) < c\} = \bigcup_{r\in\mathbb{R}} \{x \in E : f(x) < r\} \cap \{x \in E : g(x) < c - r\}.$$

Here, $\supset$ is trivial; $\subset$ is because for every x that $f(x)+g(x) < c$, $f(x) < c-g(x)$ then there exists $r \in \mathbb{R}$ such that $f(x) < r < c - g(x)$. But here the union is uncountable so we can not take advantage of the σ-algebra of measurable sets. On the other hand, since the rational numbers $\mathbb{Q}$ is dense in $\mathbb{R}$, we also have that

$$\{x \in E : f(x) + g(x) < c\} = \bigcup_{q\in\mathbb{Q}} \{x \in E : f(x) < q\} \cap \{x \in E : g(x) < c - q\}.$$

Here, $\supset$ is trivial; $\subset$ is because for every x that $f(x) + g(x) < c$, $f(x) < c - g(x)$ then there exists $q \in \mathbb{Q}$ such that $f(x) < q < c - g(x)$.

Now the sets in the countable union are measurable since f and g are measurable. Hence, $\{x \in E : f(x) + g(x) < c\}$ is measurable so $f + g$ is measurable.

(ii). Since

$$fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right],$$

it suffices to prove that f^2 is measurable given that f is measurable. For every $c \in \mathbb{R}$, if $c \leq 0$, then $\{x \in E : f^2(x) \geq c\} = E$; if $c > 0$, then

$$\{x \in E : f(x)^2 \geq c\} = \{x \in E : f(x) \geq \sqrt{c}\} \cup \{x \in E : f(x) \leq -\sqrt{c}\}$$

is measurable. Hence, f^2 is measurable. □

Proposition 3.4. *Let $g : E \to \mathbb{R}$ be measurable and $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then $f \circ g : E \to \mathbb{R}$ is measurable.*

Proof. Notice that $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$. For any open set $U \in \mathbb{R}$, $f^{-1}(U)$ is open since f is continuous; $g^{-1}(f^{-1}(U))$ is measurable since g is measurable. Hence $f \circ g$ is measurable. □

Remark.

- Let $g : E \to \mathbb{R}$ be measurable and $f : \mathbb{R} \to \mathbb{R}$ be continuous. Then $g \circ f : E \to \mathbb{R}$ is not necessarily measurable.
- By the above example, the composition of two measurable functions is not necessarily measurable.

Proposition 3.5. *Let $\{f_k\}_{k=1}^n$ be a family of measurable function on E. Then the functions $\max\{f_1, ..., f_n\}$ and $\min\{f_1, ..., f_n\}$ are measurable on E.*

Proof. For any $c \in \mathbb{R}$, notice that

$$\{x \in E : \max\{f_1, ..., f_n\} > c\} = \bigcup_{k=1}^n \{x \in E : f_k(x) > c\}$$

and

$$\{x \in E : \min\{f_1, ..., f_n\} > c\} = \bigcap_{k=1}^n \{x \in E : f_k(x) > c\}$$

□

Remark. Let $\{f_n\}_{n=1}^\infty$ be a family of measurable function on E. Then one can prove similarly that the functions $\sup_n\{f_n(x)\}$ and $\inf_n\{f_n(x)\}$ are measurable on E. In addition,

$$\limsup_{n\to\infty} f_n(x) = \inf_n\{\sup_{k\geq n} f_k(x)\} \quad \text{and} \quad \liminf_{n\to\infty} f_n(x) = \sup_n\{\inf_{k\geq n} f_k(x)\}$$

are both measurable.

Definition. Let $f : E \to \mathbb{R}$. Define

- $|f|(x) = \max\{f(x), -f(x)\}$,
- $f^+(x) = \max\{f(x), 0\}$,
- $f^-(x) = \max\{-f(x), 0\}$.

Then f^+ and f^- are both nonnegative, and $f = f^+ - f^-$.

By Proposition 3.5, if f is measurable, then $|f|$, f^+, and f^- are all measurable.

Homework Assignment.

3-1. Suppose f and g are continuous functions on $[a, b]$. Prove that if $f = g$ a.e. on $[a, b]$, then, in fact, $f = g$ on $[a, b]$. Is a similar assertion true if $[a, b]$ is replaced by a general measurable set E? Prove your assertion.

3-2. Let D and E be measurable sets in $\mathbb{R}^d$ and f a function with domain $D \cup E$. We prove that f is measurable iff its restrictions to D and E are measurable. Is the same true if "measurable" is replaced by "continuous"? Prove your assertion.

3-3. Suppose f is a real-valued function on $\mathbb{R}$ such that $f^{-1}(\{c\})$ is measurable for each number c. If f necessarily measurable? Prove your assertion.

3-4. Let f be a function with measurable domain $D \subset \mathbb{R}^d$. Prove that f is measurable iff the function g defined on $\mathbb{R}$ by $g(x) = f(x)$ for $x \in D$ and $g(x) = 0$ for $x \notin D$ is measurable.

3-5. Let the function f be defined on a measurable set E. Prove that f is measurable iff for each Borel set A, $f^{-1}(A)$ is measurable.

3-6. Suppose $\{f_n\}$ be a sequence of measurable functions defined on a measurable set E. Define E_0 to be the set of points x in E at which $\{f_n(x)\}$ converges. Is the set E_0 measurable? Prove your assertion.

3.2. Sequential pointwise limits and simple approximation

Definition. Let $\{f_n\}_{n=1}^{\infty}$ be a family of measurable function on E in $\mathbb{R}^d$. Let f be a function on $A \subset E$.

- We say that $\{f_n\}$ converges to f pointwise on A provided that
$$\lim_{n\to\infty} f_n(x) = f(x) \quad \text{for all } x \in A.$$
- We say that $\{f_n\}$ converges to f pointwise a.e. on A provided that $\{f_n\}$ converges to f pointwise on $A \setminus B$ with $m(B) = 0$.
- We say that $\{f_n\}$ converges to f uniformly on A provided that for each $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that
$$|f_n(x) - f(x)| < \varepsilon \quad \text{for all } n \geq N \text{ and } x \in A.$$

PROPOSITION 3.6. *Let $\{f_n\}_{n=1}^{\infty}$ be a family of measurable functions that converges to f pointwise a.e. on E. Then f is measurable.*

PROOF. Since $\{f_n\}_{n=1}^{\infty}$ be a family of measurable functions that converges to f pointwise a.e. on E, there is $A \subset E$ such that $f_n \to f$ on A and $m(E \setminus A) = 0$. Then $f(x) = \lim_{n\to\infty} f_n(x)$ is measurable on A and hence measurable on E. $\square$

Definition (Simple functions). A function φ on a measurable set E is called simple if it is measurable and takes only a finite number of values. We can then write
$$\varphi = \sum_{k=1}^{n} c_k \cdot \chi_{E_k} \quad \text{where } E_k = \{x \in E : \varphi(x) = c_k\} \text{ and } c_k\text{'s are distinct.}$$
Such expression φ is called the canonical representation of the simple function φ. In the canonical representation of a simple function, c_k's are distinct and E_k's are pairwise disjoint.

THEOREM (The Simple Approximation Lemma). *Let f be measurable on E. Assume that f is bounded on E, that is, $|f| < M$ for some $M \in \mathbb{R}$. Then for each $\varepsilon > 0$, there are simple functions φ_ε and ψ_ε defined on E such that*
$$\varphi_\varepsilon \leq f \leq \psi_\varepsilon \quad \text{and} \quad 0 \leq \psi_\varepsilon - \varphi_\varepsilon < \varepsilon \text{ on } E.$$

PROOF. Since f is bounded, there is an open bounded interval (c, d) such that $f(E) \subset (c, d)$. For each $\varepsilon > 0$, divide $[c, d)$ into

$$c = y_0 < y_1 \cdots < y_{n-1} < y_n = d$$

such that $y_k - y_{k-1} < \varepsilon$ for all $k = 1, ..., n$. Denote $I_k = [y_{k-1}, y_k)$ and $E_k = f^{-1}(I_k)$ for $k = 1, ..., n$. Then E_k is measurable since f is measurable. We also see that E_k's are pairwisely disjoint.

Define the simple functions φ_ε and ψ_ε as

$$\varphi_\varepsilon = \sum_{k=1}^{n} y_{k-1}\chi_{E_k} \quad \text{and} \quad \psi_\varepsilon = \sum_{k=1}^{n} y_k \chi_{E_k}.$$

For each $x \in E$, $f(x) \in f(E) \subset (c, d)$, there is $k = 1, .., n$ such that $f(x) \in I_k$. So

$$y_{k-1} = \varphi_\varepsilon(x) \leq f(x) < y_k = \psi_\varepsilon(x).$$

But $y_k - y_{k-1} < \varepsilon$ therefore $|\varphi_\varepsilon - \psi_\varepsilon| < \varepsilon$. □

THEOREM (The Simple Approximation Theorem). *A function $f : E \to \mathbb{R} \cup \{\pm\infty\}$ on a measurable set E is measurable iff there is a sequence of simple functions $\{\varphi_n\}$ that converges to f pointwise on E and $|\varphi_n| \leq |f|$ for all $n \in \mathbb{N}$. In addition, if $f \geq 0$, then we may choose $\{\varphi_n\}$ to be increasing.*

PROOF. Sufficiency is proved by Proposition 3.6.

Necessity: It suffices to show for nonnegative functions. It is because for every function f, $f = f^+ - f^-$. Then there exist $\{\varphi_{n,+}\}$ and $\{\varphi_{n,-}\}$ such that $\varphi_{n,+} \to f^+$ and $\varphi_{n,+} \leq f^+$, $\varphi_{n,-} \to f^-$ and $\varphi_{n,-} \leq f^-$. Hence,

$$\varphi_n = \varphi_{n,+} - \varphi_{n,-} \to f^+ - f^- = f \quad \text{and} \quad |\varphi_n| = \max\{\varphi_{n,+}, \varphi_{n,-}\} \leq \max\{f^+, f^-\} = |f|.$$

Now assume that $f \geq 0$. Define $E_n = \{x \in E : f(x) \leq n\}$. Then f is bounded by n on E_n. By the Simple Approximation Lemma, for every $n \in \mathbb{N}$, there exists simple functions φ_n and ψ_n such that

$$\varphi_n \leq f \leq \psi_n \quad \text{and} \quad 0 \leq \psi_n - \varphi_n < \frac{1}{n} \text{ on } E_n.$$

Extend φ_n to E by setting $\varphi_n(x) = n$ if $x \in E \setminus E_n$ (or equivalently, if $f(x) > n$). The function φ_n is a simple function defined on E and $0 \leq \varphi_n \leq f$ on E.

We next show that $\varphi_n(x) \to f(x)$ for all $x \in E$.

Case 1: If $f(x) < \infty$, then there is $N \in \mathbb{N}$ such that $f(x) < N$. So $x \in E_n$ for all $n \geq N$. Then

$$0 \leq f(x) - \varphi_n(x) < \frac{1}{n} \quad \text{for all } n \geq N.$$

Hence, $\varphi_n(x) \to f(x)$.

Case 2: If $f(x) = \infty$, then $\varphi_n(x) = n$ for all $n \in \mathbb{N}$. Hence, $\varphi_n(x) \to \infty = f(x)$.

Replacing each φ_n by $\max\{\varphi_1, ..., \varphi_n\}$, we can assume φ_n is increasing. □

Homework Assignment.

3-7. Let f be a bounded measurable function on E. Prove that there are sequences of simple functions on E, $\{\varphi_n\}$ and $\{\psi_n\}$, such that $\{\varphi_n\}$ is increasing and $\{\psi_n\}$ is decreasing and each of these sequences converges to f uniformly on E.

3-8. Let f be a measurable function on E that is finite a.e. on E and $m(E) < \infty$. For each $\varepsilon > 0$, prove that there is a measurable set F contained in E such that f is bounded on F and $m(E \setminus F) < \varepsilon$.

3-9. Let A and B be any sets. Prove that

$$\chi_{A\cap B} = \chi_A \cdot \chi_B,$$
$$\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B,$$
$$\chi_{A^c} = 1 - \chi_A.$$

3-10. Let I be an interval and $f : I \to \mathbb{R}$ be increasing. Prove that f is measurable. (Hint: You can show that $f_n(x) = f(x)+x/n$ is measurable for all $n \in \mathbb{N}$ then take the limit as $n \to \infty$.)

3.3. Littlewood's three principles, Egoroff's Theorem, and Lusin's Theorem

Littlewood's three principles:

(1) Every measurable set is *nearly* a finite union of intervals, see Theorem 2.14.
(2) Every measurable function is *nearly* continuous, see Lusin's Theorem.
(3) Every pointwise convergence of a sequence of measurable functions is *nearly* uniform, see Egoroff's Theorem.

THEOREM (Egoroff's Theorem). *Suppose that $m(E) < \infty$. Let $\{f_n\}$ be a sequence of measurable functions on E such that $f_n \to f$ pointwise on E for the function $f : E \to \mathbb{R}$. Then for every $\varepsilon > 0$, there exists a closed set $F \subset E$ such that $f_n \to f$ uniformly on F and $m(E\setminus F) < \varepsilon$.*

LEMMA 3.7. *Under the assumptions of the Egoroff's Theorem, for each $\eta > 0$ and $\delta > 0$, there exists a measurable set $A \subset E$ and $N \in \mathbb{N}$ such that $|f_n - f| < \eta$ on A for all $n \geq N$ and $m(E \setminus A) < \delta$.*

PROOF. Since $|f_k - f|$ is measurable, $\{x \in E : |f(x) - f_k(x)| < \eta\}$ is measurable and so

$$E_n = \{x \in E : |f(x) - f_k(x)| < \eta \text{ for all } k \geq n\} = \bigcap_{k=n}^{\infty} \{x \in E : |f(x) - f_k(x)| < \eta\}$$

is measurable. Notice that if $\lim_{k\to\infty} f_k(x) = f(x)$, then $x \in E_n$ for some n, that is, $x \in \cup_{n=1}^{\infty} E_n$. Hence, $m(\cup_{n=1}^{\infty} E_n) = m(E)$ since $f_n \to f$ pointwise on E. But $\{E_n\}_{n=1}^{\infty}$ is ascending then

$$m\left(\bigcup_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} m(E_n).$$

That is, $\lim_{n\to\infty} m(E_n) = m(E) < \infty$. Thus for each $\delta > 0$ there is $N \in \mathbb{N}$ such that $m(E_n) > m(E) - \delta$ for all $n \geq N$. (Here, we use the condition that $m(E) < \infty$. See Problem 3-12.)

Let $A = E_N$. Then $m(E \setminus A) = m(E) - m(E_N) < \delta$ and $|f_n(x) - f(x)| < \eta$ for all $x \in A$ and $n \geq N$. $\square$

PROOF OF EGOROFF'S THEOREM. Given any $\varepsilon > 0$, by the previous lemma, for each $m \in \mathbb{N}$, there exists $A_m \subset E$ and $N(m) \in \mathbb{N}$ such that $|f_n - f| < 1/m$ on A_m for all $n \geq N(m)$ and $m(E \setminus A_m) < \varepsilon/2^{m+1}$.

Let $A = \cap_{m=1}^{\infty} A_m$. Then $A \subset E$ and

$$m(E \setminus A) = m\left(E \setminus \left[\bigcap_{m=1}^{\infty} A_n\right]\right) = m\left(\bigcup_{m=1}^{\infty} E \setminus A_n\right) \leq \sum_{m=1}^{\infty} m(E \setminus A_m) = \frac{\varepsilon}{2}.$$

Also, $f_n \to f$ uniformly on A. This is because for each $\eta > 0$, there is $m \in \mathbb{N}$ such that $1/m < \eta$. Hence, $|f_n - f| < 1/m < \eta$ on $A \subset A_m$ for all $n \geq N(m)$.

Since A is measurable, there exists a closed set $F \subset A$ such that $m(A\setminus F) < \varepsilon/2$ by Theorem 2.13 in Section 2.4. Now $f_n \to f$ uniformly on F and $m(E\setminus F) = m(E\setminus A)+m(A\setminus F) < \varepsilon$. $\square$

THEOREM (Lusin's Theorem). *Let $f : E \to \mathbb{R}$ be measurable. Then for every $\varepsilon > 0$, there exist a continuous function g on a closed set $F \subset E$ such that $g = f$ on F and $m(E \setminus F) < \varepsilon$.*

PROPOSITION 3.8. *Let f be a simple function defined on E. Then for each $\varepsilon > 0$, there is a continuous function g on a closed set $F \subset E$ such that $g = f$ on F and $m(E \setminus F) < \varepsilon$.*

PROOF. Let f take on the distinct values $c_1, ..., c_n$ on the disjoint measurable sets $E_1, ...E_n$, respectively. Given any $\varepsilon > 0$, By Theorem 2.13 in Section 2.4, there are closed sets $F_1, ..., F_n$ such that $F_j \subset E_j$ and $m(E_j \setminus F_j) < \varepsilon/n$ for all $j = 1, ..., n$.

Let $F = \cup_{j=1}^n F_j$. Then

$$m(E \setminus F) \leq \sum_{j=1}^{n} m(E_j \setminus F_j) = \varepsilon.$$

Define $g = \sum_{j=1}^n c_j \chi_{F_j}$ on F. Then $g = f$ on F. We claim that g is continuous on F. For any $x \in F_i \subset F$, there exists an open ball $B \ni x$ and $B \cap F_j = \emptyset$ for all $j \neq i$. (If not, then x would be a limit point of F_j for some $j \neq i$ and $x \in F_j$ since F_j is closed. But this is impossible since $F_j \cap F_i = \emptyset$.) Now g is constant on $B \cap F_i$ then g is continuous at x. □

PROOF OF LUSIN'S THEOREM.

Step 1: Assume that $m(E) < \infty$. Given any $n \in \mathbb{N}$, by the Simple Approximation theorem, there is a simple function $f_n \to f$ pointwise on E. Given any $\varepsilon > 0$, by Egoroff's Theorem, there exists $A \subset E$ such that $f_n \to f$ uniformly on $A \subset E$ and $m(E \setminus A) < \varepsilon/2$.

By the previous proposition, there is $g_n = f_n$ on a closed set $F_n \subset A$ such that g_n is continuous on F_n and $m(A \setminus F_n) < 1/2^n$. Let $N \in \mathbb{N}$ be chosen so that

$$\sum_{n=N}^{\infty} \frac{1}{2^n} < \frac{\varepsilon}{2}.$$

Let $F = \cap_{n=N}^{\infty} F_n$. Then F is closed and $g_n \to g = f$ uniformly on F since $f_n = g_n$ converges uniformly to f on $A \cap F$. So g is continuous on F and $g = f$ on F. Notice that

$$m(E \setminus F) \leq m(E \setminus A) + m(A \setminus F) < \varepsilon.$$

Step 2. Assume that $m(E) \leq \infty$. Then $E = \cup_{m=1}^{\infty} E_m$ such that $E_m = E \cap [Q_{m+1} \setminus Q_m]$, where $Q_m = [-m, m] \times \cdots [-m, m]$ for $m \in \mathbb{N}$. By Step 1, there are a continuous function g_m on a closed set $F_m \subset E_m$ such that $g_m = f$ on F_m and $m(E_m \setminus F_m) < \varepsilon/2^{m+1}$.

Let $F' = \cup_{m=1}^{\infty} F_m$. ($F'$ may not be closed.) Then

$$m(E \setminus F') \leq \sum_{m=1}^{\infty} m(E_m \setminus F_m) < \sum_{m=1}^{\infty} \frac{\varepsilon}{2^{m+1}} = \frac{\varepsilon}{2}.$$

Define

$$g = \sum_{m=1}^{\infty} \chi_{F_m} \cdot g_m.$$

Then g is continuous on F' and $g = f$ on F'. Now choose a closed subset $F \subset F'$ such that $m(F' \setminus F) < \varepsilon/2$. Thus

$$m(E \setminus F) \leq m(E \setminus F') + m(F' \setminus F) < \varepsilon.$$

□

$\boxed{\textbf{Homework Assignment}}$.

3-11. Suppose f is a function that is continuous on a closed set F of real numbers. Prove that f has a continuous extension to all of $\mathbb{R}$.

3-12. Prove that the conclusion of Egoroff's Theorem can fail if we drop the assumption that the domain has finite measure.

CHAPTER 4

Lebesgue integration

4.1. The Riemann integration and Riemann-Lebesgue Theorem

Definition (Partitions). A partition $P = \{x_0, x_1, ..., x_n\}$ of the interval $[a, b]$ is a finite set of numbers $x_0, x_1, ..., x_n$ such that

$$a = x_0 < x_1 < \cdots < x_n = b.$$

- The norm (or mesh, gap) of a partition $P = \{x_0, x_1, ..., x_n\}$, denoted by $\|P\|$, is $\max\{\Delta x_i : i = 1, ..., n\}$, in which $\Delta x_i = x_i - x_{i-1}$.
- Let P and Q be partitions of $[a, b]$. We say that Q is a refinement of P if $P \subset Q$.

Let $f : [a, b] \to \mathbb{R}$ be bounded and $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[a, b]$. Define the lower Riemann sum with respect to the partition P as

$$\underline{S}(f, P) = \sum_{i=1}^{n} m_i \Delta x_i, \quad \text{where } m_i = \inf_{x \in [x_{i-1}, x_i]} f(x),$$

and the upper Riemann sum with respect to the partition P as

$$\overline{S}(f, P) = \sum_{i=1}^{n} M_i \Delta x_i, \quad \text{where } M_i = \sup_{x \in [x_{i-1}, x_i]} f(x).$$

Remark. Let $f : [a, b] \to \mathbb{R}$ be bounded and P and Q be two partitions of $[a, b]$. Then

- $m(b - a) \le \underline{S}(f, P) \le \overline{S}(f, P) \le M(b - a)$, where $m = \inf_{x \in [a,b]} f(x)$ and $M = \sup_{x \in [a,b]} f(x)$.
- If Q is a refinement of P, then $\underline{S}(f, P) \le \underline{S}(f, Q)$ and $\overline{S}(f, P) \ge \overline{S}(f, Q)$.
- $\underline{S}(f, P) \le \underline{S}(f, P \cup Q) \le \overline{S}(f, P \cup Q) \le \overline{S}(f, Q)$.

Definition (Lower and upper Riemann sums). Let $f : [a, b] \to \mathbb{R}$ be bounded. Write

$$\underline{S}(f) = \sup\{\underline{S}(f, P) : P \text{ is a partition of } [a, b]\},$$

and

$$\overline{S}(f) = \inf\{\overline{S}(f, P) : P \text{ is a partition of } [a, b]\}.$$

Since $\underline{S}(f, P) \le \overline{S}(f, Q)$ for any two partitions P and Q of $[a, b]$, $\underline{S}(f) \le \overline{S}(f)$.

Definition (Riemann integrable functions). Let $f : [a, b] \to \mathbb{R}$ be bounded. We say that f is Riemann integrable on $[a, b]$ if $\underline{S}(f) = \overline{S}(f)$; in this case, we say that the Riemann integral of f on $[a, b]$ is $I := \underline{S}(f) = \overline{S}(f)$, denoted as

$$\int_a^b f\, dx = I.$$

Example. Let $f : [0,2] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \cap [0,2]; \\ 1 & \text{if } x \in (\mathbb{R} \setminus \mathbb{Q}) \cap [0,2]. \end{cases}$$

Then $\underline{S}(f,P) = 0$ and $\overline{S}(f,P) = 2$ for any partition P of $[0,2]$, so $\underline{S}(f) = 0 \neq 2 = \overline{S}(f)$. Hence, f is not Riemann integrable on $[0,2]$. Furthermore, f is discontinuous at all $x \in [0,2]$ and the set the all the points at which f is not continuous is of measure $m([0,2]) = 2$. See the Riemann-Lebesgue theorem, in which we can completely characterize the Riemann integrable functions by Lebesgue measure theory.

PROPOSITION 4.1. *Let $f : [a,b] \to \mathbb{R}$ be bounded. Then f is Riemann integrable on $[a,b]$ iff for every $\varepsilon > 0$, there exists a partition $P = P(\varepsilon)$ of $[a,b]$ such that*

$$\overline{S}(f,P) - \underline{S}(f,P) < \varepsilon.$$

PROOF. Sufficiency: For every $\varepsilon > 0$, there exists a partition $P = P(\varepsilon)$ of $[a,b]$ such that

$$\overline{S}(f,P) - \underline{S}(f,P) < \varepsilon.$$

Then $0 \leq \overline{S}(f) - \underline{S}(f) \leq \overline{S}(f,P) - \underline{S}(f,P) < \varepsilon$. So $\overline{S}(f) = \underline{S}(f)$ and f is Riemann integrable.

Necessity: Let f be Riemann integrable. Then $\overline{S}(f) = \underline{S}(f) = I$. For any $\varepsilon > 0$, since $\underline{S}(f) = \sup\{\underline{S}(f,P) : P \text{ is a partition of } [a,b]\}$, there is a partition P_1 of $[a,b]$ such that

$$I - \frac{\varepsilon}{2} < \underline{S}(f,P_1) \leq \underline{S}(f) = I;$$

since $\overline{S}(f) = \inf\{\overline{S}(f,P) : P \text{ is a partition of } [a,b]\}$, there is a partition P_2 of $[a,b]$ such that

$$I = \overline{S}(f) \leq \overline{S}(f,P_2) \leq I + \frac{\varepsilon}{2}.$$

That is,

$$I - \frac{\varepsilon}{2} < \underline{S}(f,P_1) \leq I \leq \overline{S}(f,P_2) \leq I + \frac{\varepsilon}{2}.$$

Let $P = P_1 \cup P_2$. Then P is a refinement of P_1 and of P_2. So

$$\underline{S}(f,P_1) \leq \underline{S}(f,P) \leq I \quad \text{and} \quad I \leq \overline{S}(f,P) \leq \overline{S}(f,P_2).$$

Therefore,

$$I - \frac{\varepsilon}{2} < \underline{S}(f,P) \leq I \leq \overline{S}(f,P) \leq I + \frac{\varepsilon}{2},$$

and

$$\overline{S}(f,P) - \underline{S}(f,P) < \varepsilon.$$

□

THEOREM 4.2 (Riemann-Lebesgue Theorem). *Let $f : [a,b] \to \mathbb{R}$ be bounded. Then f is Riemann integrable on $[a,b]$ iff f is continuous a.e. on $[a,b]$.*

Remark. Notice that to completely characterize the Riemann integrablitity, one has to use Lebesgue measure theory, which is not "present" in the Riemann integration at all.

Definition (Oscillation). Let f be a function with domain D. The oscillation of a function f on a set A is defined as

$$\operatorname{osc}(f,A) := \sup_{x_1,x_2 \in A \cap D} \{|f(x_1) - f(x_2)|\}.$$

The oscillation of a function f at a point x is defined as

$$\operatorname{osc}(f,x) := \lim_{h\downarrow 0} \sup_{x_1,x_2\in(x-h,x+h)\cap D} \{|f(x_1)-f(x_2)|\}.$$

THEOREM 4.3. *Let f be a function with domain D. Then f is continuous at $x \in D$ iff* $\operatorname{osc}(f,x) = 0$.

PROOF. Sufficiency: Suppose that $\operatorname{osc}(f,x) = 0$. For every $\varepsilon > 0$, since

$$\operatorname{osc}(f,x) = \lim_{h\downarrow 0} \sup_{x_1,x_2\in(x-h,x+h)\cap D} \{|f(x_1)-f(x_2)|\} = 0,$$

there exists $\delta > 0$ such that

$$\sup_{x_1,x_2\in(x-h,x+h)\cap D} \{|f(x_1)-f(x_2)|\} < \varepsilon$$

if $0 < h < \delta$. Hence, for any $x_1 \in (x-\delta, x+\delta)\cap D$, $|f(x_1) - f(x)| < \varepsilon$ and so f is continuous at x.

Necessity: Suppose that f is continuous at $x \in D$. For every $\varepsilon > 0$, then there exists $\delta > 0$ such that $|f(x_1) - f(x)| < \varepsilon/2$ if $x_1 \in (x-\delta, x+\delta)\cap D$. Hence, if $0 < h < \delta$, then for all $x_1, x_2 \in (x-h, x+h)\cap D$, we have that

$$|f(x_1)-f(x_2)| \le |f(x_1)-f(x)| + |f(x)-f(x_2)| < \varepsilon.$$

That is, if $h < \delta$, then

$$\sup_{x_1,x_2\in(x-h,x+h)\cap D} \{|f(x_1)-f(x_2)|\} \le \varepsilon.$$

So

$$\operatorname{osc}(f,x) = \lim_{h\downarrow 0} \sup_{x_1,x_2\in(x-h,x+h)\cap D} \{|f(x_1)-f(x_2)|\} = 0.$$

□

PROOF OF THE RIEMANN-LEBESGUE THEOREM.

Necessity: Suppose that f is Riemann integrable on $[a,b]$. Let A be the set of points at where f is not continuous. By the previous theorem,

$$A = \{x \in [a,b] : \operatorname{osc}(f,x) > 0\}.$$

Notice that

$$A = \bigcup_{k=1}^{\infty} A_k, \quad \text{where } A_k = \left\{x \in [a,b] : \operatorname{osc}(f,x) > \frac{1}{k}\right\}.$$

By subadditivity of measure, to prove $m(A) = 0$, it suffices to prove that $m(A_k) = 0$ for all $k \in \mathbb{N}$.

Now fix $k \in \mathbb{N}$. For every $\varepsilon > 0$, since f is Riemann integrable, there exists a partition $P = \{x_0, x_1, ..., x_n\}$ of $[a,b]$ for which

$$\overline{S}(f,P) - \underline{S}(f,P) < \frac{\varepsilon}{k}.$$

Among all the interval $I_i := [x_{i-1}, x_i]$ for $i = 1, ..., n$, select the intervals that if $I_i \cap A_k \neq \emptyset$ and denote them as I_{i_j}, $j = 1, ..., m$. For each I_{i_j}, there is $\bar{x} \in [x_{i-1}, x_i] \cap A_k$ and thus

$$M_i - m_i = \sup_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x_i]} f(x) \ge \operatorname{osc}(f,\bar{x}) > \frac{1}{k}.$$

Hence,

$$\frac{\varepsilon}{k} > \overline{S}(f,P) - \underline{S}(f,P) \geq \sum_{j=1}^{m}(M_{i_j} - m_{i_j})\Delta x_{i_j} \geq \frac{1}{k}\sum_{j=1}^{m}\Delta x_{i_j}.$$

Thus,

$$\sum_{j=1}^{m}\Delta x_{i_j} < \varepsilon.$$

Since $\{I_{i_j}\}_{j=1}^m$ is an interval cover of A_k,

$$m^*(A_k) \leq \sum_{j=1}^{m}\Delta x_{i_j} < \varepsilon,$$

for all $\varepsilon > 0$. So $m(A_k) = 0$ for all $k \in \mathbb{N}$ and therefore $m(A) = 0$.

Sufficiency: Suppose that f is continuous a.e. on $[a,b]$. Let A be the set of points at where f is not continuous. By the previous theorem,

$$A = \{x \in [a,b] : \operatorname{osc}(f,x) > 0\}.$$

Then $m(A) = 0$. It then follow that for any $s > 0$,

$$m(A_s) = 0, \quad \text{where } A_s = \{x \in [a,b] : \operatorname{osc}(f,x) \geq s\},$$

since $A_s \subset A$. Denote

$$M = \sup_{x\in[a,b]} f(x) \quad \text{and} \quad m = \inf_{x\in[a,b]} f(x).$$

For every $\varepsilon > 0$, let $s = \varepsilon/(2(b-a))$. Since $m(A_s) = 0$, there is an open interval cover $\{I_i\}_{i=1}^{\infty}$ such that

$$\sum_{i=1}^{\infty}|I_i| < \frac{\varepsilon}{2(M-m)}.$$

One can show that A_s is compact so there is a finite subcover of $\{I_i\}_{i=1}^{\infty}$, called $\{I_i\}_{i=1}^{N}$.

Now for any

$$x \in [a,b] \setminus \left(\bigcup_{i=1}^{N} I_i\right) \subset [a,b] \setminus A_s,$$

we have that

$$\lim_{h\downarrow 0} \sup_{x_1,x_2\in(x-h,x+h)\cap[a,b]} \{|f(x_1) - f(x_2)|\} = \operatorname{osc}(f,x) < s = \frac{\varepsilon}{2(b-a)}.$$

Hence, there exists $\delta_x > 0$ such that

$$\sup_{x_1,x_2\in(x-\delta_x,x+\delta_x)\cap[a,b]} \{|f(x_1) - f(x_2)|\} < \frac{\varepsilon}{2(b-a)}.$$

Notice that

$$\{(x-\delta_x, x+\delta_x)\}_{x\in[a,b]\setminus A_s} \text{ covers } [a,b] \setminus \left(\bigcup_{i=1}^{N} I_i\right),$$

which is compact. Then there are finite points $x_1, ..., x_K$ in $[a,b] \setminus A_s$ such that

$$[a,b] \setminus \left(\bigcup_{i=1}^{N} I_i\right) \subset \bigcup_{j=1}^{K}(x_j - \delta_{x_j}, x_j + \delta_{x_j}).$$

Hence we have constructed a finite cover of $[a,b]$:

$$\{(x_1-\delta_{x_1},x_1+\delta_{x_1}),...,(x_K-\delta_{x_K},x_K+\delta_{x_K}),I_1,...,I_N\}.$$

We are now ready to construct a partition P of $[a,b]$ such that

$$\overline{S}(f,P)-\underline{S}(f,P)<\varepsilon,$$

so f is Riemann integrable. Indeed, let $P=\{x_0,...,x_n\}$ be a partition of $[a,b]$ such that any interval $[x_{i-1},x_i]$ is contained in one interval of the above finite cover. We then divide the intervals $[x_{i-1},x_i]$, $i=1,...,n$ into two classes: $\mathcal{C}_1$ consists of all the intervals that are contained in some I_i, and $\mathcal{C}_2$ consists of all others. Then

$$\begin{aligned}
&\overline{S}(f,P)-\underline{S}(f,P)\\
= &\sum_{i=1}^{n}M_i\Delta x_i-\sum_{i=1}^{n}m_i\Delta x_i\\
= &\sum_{\mathcal{C}_1}(M_i-m_i)\Delta x_i+\sum_{\mathcal{C}_2}(M_i-m_i)\Delta x_i\\
\le &(M-m)\sum_{\mathcal{C}_1}\Delta x_i+\sum_{\mathcal{C}_2}\frac{\varepsilon}{2(b-a)}\Delta x_i\\
\le &(M-m)\sum_{i=1}^{N}\ell(I_i)+\frac{\varepsilon}{2(b-a)}\sum_{\mathcal{C}_2}\Delta x_i\\
\le &(M-m)\cdot\frac{\varepsilon}{2(M-m)}+\frac{\varepsilon}{2(b-a)}\cdot(b-a)\\
< &\varepsilon.
\end{aligned}$$

□

Homework Assignment.

4-1. Let $f:[a,b]\to\mathbb{R}$ be continuous. Prove that f is Riemann integrable over $[a,b]$.

4-2. Let $\{f_n\}$ be a sequence of bounded functions that converges uniformly to f on $[a,b]$. Suppose that f_n is Riemann integrable over $[a,b]$ for all $n\in\mathbb{N}$. Prove that f is also Riemann integrable over $[a,b]$. Is it true that

$$\lim_{n\to\infty}\int_a^b f_n\,dx=\int_a^b f\,dx?$$

Prove your assertion.

4.2. The Lebesgue integration theory

The Lebesgue integral is defined in a step-by-step fashion, proceeding successively to larger families of functions.

(1) Simple functions on a set of finite measure.
(2) Bounded measurable functions on a set of finite measure.
(3) Nonnegative functions.
(4) Lebesgue integrable functions.

Since the Riemann integral is in Stage 2, we review its property in the Lebesgue integration framework then.

At each stage we see that the (Lebesgue) integral satisfies the following properties.

(i). Linearity: Let $\alpha, \beta \in \mathbb{R}$ and f and g be integrable. Then

$$\int \alpha f + \beta g = \alpha \int f + \beta \int g.$$

(ii). Additivity: Let f be integrable on two disjoint measurable sets E and F. Then

$$\int_{E \cup F} f = \int_E f + \int_F f.$$

(iii). Monotonicity: Let f and g be two integrable functions that $f \leq g$. Then

$$\int f \leq \int g.$$

(iv). The triangle inequality: Let f be integrable. Then

$$\left| \int f \right| \leq \int |f|.$$

We also prove appropriate convergence theorems that amount to interchanging the integral with limits.

4.2.1. Simple functions on a set of finite measure.

Recall that a function φ is called simple if it is measurable and takes only a finite number of values. We can then write

$$\varphi = \sum_{k=1}^{n} c_k \cdot \chi_{E_k} \quad \text{where } E_k = \{x \in E : \varphi(x) = c_k\}.$$

Such expression φ is called the canonical representation of the simple function φ. In the canonical representation, c_k's are distinct and E_k's are disjoint.

Definition (Lebesgue integral of simple function on a set of finite measure). Let φ be a simple function on a measurable set $E \subset \mathbb{R}^d$ with the canonical representation

$$\varphi = \sum_{k=1}^{n} c_k \cdot \chi_{E_k}, \quad \text{where } E_k = \{x \in E : \varphi(x) = c_k\}.$$

Suppose that $m(E) < \infty$. We define the Lebesgue integral of φ over E

$$\int_E \varphi \, dx = \sum_{k=1}^{n} c_k \cdot m(E_k).$$

Given any measurable set F, $E_k \cap F$ is measurable for all $k = 1, ..., n$. We define the Lebesgue integral of φ over F as

$$\int_F \varphi \, dx = \sum_{k=1}^{n} c_k \cdot m(E_k \cap F).$$

In particular,

$$\int_{\mathbb{R}} \varphi \, dx = \sum_{k=1}^{n} c_k \cdot m(E_k) = \int_E \varphi \, dx.$$

Remark.

- If $\varphi = c \cdot \chi_E$, where $m(E) < \infty$, then $\int_E \varphi \, dx = cm(E)$.

- To emphasize the choice of Lebesgue measure m in the above definition of integral, one sometimes writes
$$\int_E \varphi \, dm(x).$$
Also, we simply denote $\int f$ for $\int_{\mathbb{R}^d} f \, dx$.
- If φ is a simple function on a set of finite measure E and F be a measurable set, then $\varphi \cdot \chi_F$ is also simple, and
$$\int \varphi \cdot \chi_F \, dx = \int_F \varphi \, dx.$$

PROPOSITION 4.4. *Let φ and ψ be simple functions, E and F be two measurable sets with finite measure.*

(0). *Independence of representation: If*
$$\varphi = \sum_{j=1}^{m} d_j \cdot \chi_{F_j}, \quad \text{then} \int \varphi = \sum_{j=1}^{m} d_j \cdot m(F_j).$$

(i). *Linearity: Let $a, b \in \mathbb{R}$. Then*
$$\int a\varphi + b\psi = a \int \varphi + b \int \psi.$$

(ii). *Additivity: If E and F are disjoint, then*
$$\int_{E \cup F} \varphi = \int_E \varphi + \int_F \varphi.$$

(iii). *Monotonicity: If $\varphi \leq \psi$, then*
$$\int \varphi \leq \int \psi.$$

(iv). *Triangle inequality:*
$$\left| \int \varphi \right| \leq \int |\varphi|,$$
in which $|\varphi|$ is also a simple functions.

PROOF.

(0). It is evident that φ takes finite number of values on the measurable set $E := \cup_{j=1}^{m} F_j$. Let
$$\varphi = \sum_{i=1}^{n} c_i \cdot \chi_{E_i}$$
be the canonical representations of φ. Here, E is a disjoint union of $\{E_i\}_{i=1}^n$ and $c_1, ..., c_n$ are distinct.

Step 1: Suppose that $\{F_j\}_{j=1}^m$ is a disjoint collection. Denote
$$E_{ij} = E_i \cap F_j.$$
Then for each $j = 1, ..., m$, F_j is a disjoint union of $\{E_{ij}\}_{i=1}^n$; for each $i = 1, ..., n$, E_i is a disjoint union of $\{E_{ij}\}_{j=1}^m$. Moreover, for $x \in E_i$, there is a unique $j = 1, ..., m$ such that $x \in F_j$. So $\varphi(x) = d_j = c_i$ when $x \in E_{ij}$. Compute that
$$\sum_{j=1}^{m} d_j \cdot m(F_j) \;\; = \;\; \sum_{j=1}^{m} d_j \sum_{i=1}^{n} m(E_{ij})$$

$$\begin{aligned}
&= \sum_{i=1}^{n}\sum_{j=1}^{m} d_j \cdot m(E_{ij}) \\
&= \sum_{i=1}^{n} c_i \sum_{j=1}^{m} m(E_{ij}) \\
&= \sum_{i=1}^{n} c_i \cdot m(E_i) \\
&= \int \varphi.
\end{aligned}$$

Step 2: If $\{F_j\}_{j=1}^m$ is not disjoint, then let $N = 2^m - 1$ and consider the collection of the sets $\{B_k\}_{k=1}^N$ that consists of the sets of the following form except $\cap_{j=1}^m F_j^c$.

$$A_1 \cap A_2 \cap \cdots \cap A_{m-1} \cap A_m, \quad \text{where } A_j = F_j \text{ or } F_j^c,\ j = 1, ..., m.$$

Hence, $\{B_k\}_{k=1}^N$ is a disjoint collection. Moreover,

$$\bigcup_{k=1}^{N} B_k = \bigcup_{j=1}^{m} F_j \quad \text{and} \quad F_j = \bigcup_{B_k \subset F_j} B_k.$$

Observe that if $x \in B_k$, then

$$\varphi(x) = \sum_{F_j \supset B_k} d_j := b_k.$$

Hence,

$$\varphi = \sum_{k=1}^{N} b_k \cdot \chi_{B_k}.$$

Thus by Step 1 that

$$\int \varphi = \sum_{k=1}^{N} b_k \cdot m(B_k).$$

So

$$\begin{aligned}
\int \varphi &= \sum_{k=1}^{N} b_k \cdot m(B_k) \\
&= \sum_{k=1}^{N} \left(\sum_{F_j \supset B_k} d_j \right) \cdot m(B_k) \\
&= \sum_{j=1}^{m} d_j \sum_{B_k \subset F_j} m(B_k) \\
&= \sum_{j=1}^{m} d_j \cdot m(F_j).
\end{aligned}$$

(i). Let

$$\varphi = \sum_{i=1}^{n} c_i \cdot \chi_{E_i} \quad \text{and} \quad \psi = \sum_{j=1}^{m} d_j \cdot \chi_{F_j},$$

in their canonical representations. Here, E is a disjoint union of $\{E_i\}_{i=1}^n$ and is also a disjoint union of $\{F_j\}_{j=1}^m$. Then by (0), we use $\{E_i \cap F_j\}_{i=1,\dots,n;j=1,\dots,m}$ as a representation of E and have that

$$\begin{aligned}
\int a\varphi + b\psi &= \sum_{i=1}^n \sum_{j=1}^m (ac_i + bd_j) m(E_i \cap F_j) \\
&= a\sum_{i=1}^n c_i \sum_{j=1}^m m(E_i \cap F_j) + b\sum_{j=1}^m d_j \sum_{i=1}^n m(E_i \cap F_j) \\
&= a\sum_{i=1}^n c_i m(E_i) + b\sum_{j=1}^m d_j m(F_j) \\
&= a\int \varphi + b\int \psi.
\end{aligned}$$

(ii). Notice that if E and F are disjoint, then

$$\chi_{E\cup F} = \chi_E + \chi_F.$$

By the linearity of integration,

$$\int_{E\cup F} \varphi = \int \varphi \cdot \chi_{E\cup F} = \int \varphi \cdot (\chi_E + \chi_F) = \int \varphi \cdot \chi_E + \int \varphi \cdot \chi_F = \int_E \varphi + \int_F \varphi.$$

(iii). Notice that $\psi - \varphi$ is a nonnegative simple function, it is evident by the definition that $\int(\psi - \varphi) \geq 0$. Then by linearity of integration,

$$\int \psi - \int \varphi = \int (\psi - \varphi) \geq 0.$$

(iv). Write φ in the canonical form

$$\varphi = \sum_{k=1}^n c_k \cdot \chi_{E_k}, \quad \text{where } E_k = \{x \in E : \varphi(x) = c_k\}.$$

Then

$$|\varphi| = \sum_{k=1}^n |c_k| \cdot \chi_{E_k}.$$

(Notice that the above representation of $|\varphi|$ may not be canonical.) By (0),

$$\left|\int \varphi\, dx\right| = \left|\sum_{j=1}^n c_k \cdot m(E_k)\right| \leq \sum_{k=1}^n |c_k| \cdot m(E_k) = \int |\varphi|\, dx.$$

□

4.2.2. Bounded measurable functions on a set of finite measure, including Riemann integrable functions.

Definition (Lebesgue integral of bounded measurable functions on a set of finite measure). Let $f : E \to \mathbb{R}$ be a bounded and measurable function on a set E with finite measure. Define the lower Lebesgue integral

$$L(f) = \sup\left\{ \int_E \varphi \,\middle|\, \varphi \text{ is simple and } \varphi \leq f \text{ on } E \right\},$$

and the upper Lebesgue integral

$$U(f) = \inf \left\{ \int_E \varphi \,\middle|\, \varphi \text{ is simple and } \varphi \geq f \text{ on } E \right\}.$$

We say that f is (Lebesgue) integrable if $L(f) = U(f) = I \in \mathbb{R}$, in this case, we say that the (Lebesgue) integral of f over E is I and denote as

$$\int_E f \, dx = I.$$

Given any measurable set $F \subset E$, define the (Lebesgue) integral of f over F by

$$\int_F f \, dx = \int_E f \cdot \chi_F \, dx.$$

The definition of Lebesgue integrals follows a very similar fashion as the one used in the Riemann integrals. We next characterize the integrable functions.

PROPOSITION 4.5. *Let $f : E \to \mathbb{R}$ be a bounded and measurable function on a set E with finite measure. Then f is integrable.*

PROOF. By Simple Function Approximation lemma, there are two sequences of simple functions $\{\varphi_n\}$ and $\{\psi_n\}$ that

$$\varphi_n \leq f \leq \psi_n \quad \text{and} \quad \psi_n - \varphi_n < \frac{1}{n}.$$

So $\{\varphi_n\}$ and $\{\psi_n\}$ both converge to f uniformly on E. By monotonicity of integration for simple functions,

$$\int_E \psi_n - \int_E \varphi_n \leq \int_E \frac{1}{n} = \frac{1}{n} m(E).$$

But

$$\int_E \varphi_n \leq L(f) \leq U(f) \leq \int_E \psi_n.$$

Hence, for all $n \in \mathbb{N}$,

$$U(f) - L(f) \leq \int_E \psi_n - \int_E \varphi_n \leq \frac{1}{n} m(E).$$

So $L(f) = U(f)$ and f is integrable. $\square$

Return to Riemann integrable functions.

THEOREM 4.6. *Let f be bounded on $[a, b]$. Suppose that f is Riemann integrable and $\int_a^b f = I$. Then f is (Lebesgue) integrable and $\int_{[a,b]} f = I$.*

PROOF. Since f is Riemann integrable, given any $\varepsilon > 0$, there exists a partition $P = \{x_0, ..., x_n\}$ of $[a, b]$ such that

$$I - \varepsilon \leq \underline{S}(f, P) \leq I \leq \overline{S}(f, P) \leq I + \varepsilon,$$

by Proposition 4.1.

Write

$$\varphi(x) = \sum_{i=1}^{n-1} \left(\inf_{x \in [x_{i-1}, x_i]} f(x) \right) \cdot \chi_{[x_{i-1}, x_i)}(x) + \left(\inf_{x \in [x_{n-1}, x_n]} f(x) \right) \cdot \chi_{[x_{n-1}, x_n]}(x),$$

and

$$\psi(x) = \sum_{i=1}^{n-1} \left(\sup_{x \in [x_{i-1}, x_i]} f(x) \right) \cdot \chi_{[x_{i-1}, x_i)}(x) + \left(\sup_{x \in [x_{n-1}, x_n]} f(x) \right) \cdot \chi_{[x_{n-1}, x_n]}(x).$$

Then φ and ψ are both simple functions on $[a, b]$, and $\varphi \le f \le \psi$. Notice that

$$\underline{S}(f, P) = \int_{[a,b]} \varphi \le L(f) \quad \text{and} \quad \overline{S}(f, P) = \int_{[a,b]} \psi \ge U(f).$$

Hence,

$$I - \varepsilon \le \underline{S}(f, P) \le L(f) \le U(f) \le \overline{S}(f, P) < I + \varepsilon,$$

and therefore f is Lebesgue integrable and $L(f) = U(f) = I$. □

Remark (Step functions). A function φ is called a step function if

$$\varphi(x) = \sum_{k=1}^{n} c_k \cdot \chi_{R_k}, \quad \text{where } R_k\text{'s are intervals.}$$

It is evident that step functions are simple functions. In the above proof, we use the Lebesgue integrals of the step functions to define the Riemann integral.

Notice that not all simple functions are step functions, e.g.

$$\chi_{\mathbb{Q}} - \chi_{\mathbb{R} \setminus \mathbb{Q}}.$$

From the properties of integration on simple functions, we have that

PROPOSITION 4.7. *Let f and g be two bounded and measurable functions on a set E with finite measure.*

(i). *Linearity: Let $a, b \in \mathbb{R}$. Then $af + bg$ is bounded and measurable on E and*

$$\int_E af + bg = a \int_E f + b \int_E g.$$

(ii). *Additivity: Let E and F two disjoint measurable sets with finite measure. If $f : E \cup F \to \mathbb{R}$ is bounded and measurable function, then*

$$\int_{E \cup F} f = \int_E f + \int_F f.$$

(iii). *Monotonicity: If $f \le g$, then*

$$\int_E f \le \int_E g.$$

(iv). *Triangle inequality:*

$$\left| \int_E f \right| \le \int_E |f|,$$

in which $|f|$ is also bounded and measurable on E.

THEOREM 4.8 (Bounded convergence theorem). *Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of measurable functions on a set E with finite measure. Suppose that $\{f_n\}$ converges to f pointwise a.e. on E and $|f_n| < M$ for all $n \in \mathbb{N}$ with some $M \in \mathbb{R}$. Then*

$$\lim_{n \to \infty} \int_E |f_n - f| = 0,$$

and consequently by the triangle inequality,

$$\lim_{n\to\infty}\int_E f_n = \int_E f.$$

PROOF. Since $m(E) < \infty$, given any $\varepsilon > 0$, by Egoroff's Theorem, there is a closed set $F \subset E$ such that $f_n \to f$ uniformly on F and $m(E \setminus F) < \varepsilon$. (In fact, we only need F to be measurable here.) Then there exists $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in F$ and $n \geq N$. Notice that f is measurable and $|f| \leq M$ a.e. on E. Compute that

$$\begin{aligned}\int_E |f_n - f| &\leq \int_F |f_n - f| + \int_{E\setminus F} |f_n - f| \\ &\leq \int_F \varepsilon + \int_{E\setminus F} 2M \\ &\leq \varepsilon m(F) + 2Mm(E \setminus F) \\ &\leq \varepsilon\,[m(E) + 2M] \quad \text{for all } n \geq N.\end{aligned}$$

□

In particular, the bounded convergence theorem implies that if a sequence of uniformly bounded simple functions $\varphi_n \to f$ pointwise a.e. on E with $m(E) < \infty$, then $\int_E f = \lim \int \varphi_n$.

4.2.3. Nonnegative functions.

Definition (Support of a function). The support of a function $g : D \to \mathbb{R} \cup \{\pm\infty\}$ is defined to be the set of all points where g does not vanish,

$$\operatorname{supp} g = \{x \in D : g(x) \neq 0\}.$$

Observe that if $g : E \to \mathbb{R} \cup \{\pm\infty\}$ is a measurable function on a measurable set E, then $\operatorname{supp} g = \{x \in E : g(x) > 0\} \cup \{x \in E : g(x) < 0\}$ is a measurable set.

Definition (Lebesgue integral of bounded and measurable functions supported on a set of finite measure). Let $g : E \to \mathbb{R}$ be a bounded and measurable function on a measurable set E. Suppose that $m(\operatorname{supp} g) < \infty$. Then define the Lebesgue integral of g on E as

$$\int_E g\,dx = \int_{\operatorname{supp} g} g\,dx.$$

Here, the right-hand side integral is defined in the previous stage.

Definition (Lebesgue integral of nonnegative and measurable functions). Let $f : E \to \mathbb{R} \cup \{\infty\}$ be a nonnegative and measurable function on a measurable set E. Define the (Lebesgue) integral of f over E by

$$\int_E f(x)\,dx = \sup_g \left\{ \int_E g(x)\,dx \right\},$$

where the supremum is taken over all measurable functions g such that $0 \leq g \leq f$ on E, and g is bounded and supported on a set of finite measure.

Given any measurable set $F \subset E$, define the (Lebesgue) integral of f over F by

$$\int_F f\,dx = \int_E f \cdot \chi_F\,dx.$$

Notice that $\int_E f\,dx \leq \infty$. If $\int_E f\,dx < \infty$, then we say that f is (Lebesgue) integrable over E.

From the properties of integration of bounded and measurable functions on sets of finite measure, we have that

PROPOSITION 4.9. *Let f and g be two nonnegative and measurable functions on a measurable set E.*

(i). *Linearity: Let a and b be two nonnegative real numbers. Then*

$$\int_E af + bg = a\int_E f + b\int_E g.$$

(ii). *Additivity: Let E and F two disjoint measurable sets. If $f : E\cup F \to \mathbb{R}$ is a nonnegative and measurable function, then*

$$\int_{E\cup F} f = \int_E f + \int_F f.$$

(iii). *Monotonicity: If $f \leq g$, then*

$$\int_E f \leq \int_E g.$$

(iv). *If g is integrable and $0 \leq f \leq g$, then f is integrable.*

(v). *If f is integrable, then $f < \infty$ a.e. on E.*

PROOF. (i). We verify through two inequalities. It suffices to prove the case when $a = b = 1$. Firstly we prove that

$$\int_E f + \int_E g \leq \int_E f + g.$$

Let $0 \leq h_1 \leq f$ and $0 \leq h_2 \leq g$ be two bounded and measurable functions supported on sets of finite measures. Then $h = h_1 + h_2$ is a bounded and measurable function supported on a set of finite measure. Since

$$\int_E h_1 + \int_E h_2 = \int_E h_1 + h_2,$$

we have that

$$\sup_{h_1}\left\{\int_E h_1\right\} + \sup_{h_2}\left\{\int_E h_2\right\} = \sup_{h=h_1+h_2}\left\{\int_E h\right\} \leq \sup_{h}\left\{\int_E h\right\},$$

where the last supremum is taken over all measurable functions h such that $0 \leq h \leq f + g$ on E, and h is bounded and supported on a set of finite measure. Hence, $\int_E f + \int_E g \leq \int_E f + g$.

Then we prove that

$$\int_E f + g \leq \int_E f + \int_E g.$$

Let $0 \leq h \leq f + g$ be a bounded and measurable function supported on a set of finite measure. Write $h_1 = \min\{f, h\}$ and $h_2 = h - h_1$. Then $0 \leq h_1 \leq f$ and $0 \leq h_2 \leq g$ are two bounded and measurable function supported on a set of finite measure. (Here, if $h_1 = f$, then $h_2 = h - h_1 \leq f + g - f = g$; if $h_1 = h$, then $h_2 = h - h_1 = 0$.) Since

$$\int_E h_1 + h_2 = \int_E h_1 + \int_E h_2,$$

we have that

$$\sup_{h}\left\{\int_E h\right\} = \sup_{h_1=\min\{f,h\}}\left\{\int_E h_1\right\} + \sup_{h_2=h-\min\{f,h\}}\left\{\int_E h_2\right\} \leq \sup_{h_1}\left\{\int_E h_1\right\} + \sup_{h_2}\left\{\int_E h_2\right\}.$$

Hence, $\int_E f + g \leq \int_E f + \int_E g$.

(v). Let $E_k = \{x \in E : f(x) \geq k\}$. Then

$$E_\infty = \{x \in E : f(x) = \infty\} = \cap_{k=1}^{\infty} E_k.$$

Observe that

$$\int_E f \geq \int_E f \cdot \chi_{E_k} = \int_{E_k} f \geq \int_{E_k} k = km(E_k).$$

Then

$$m(E_\infty) \leq m(E_k) \leq \frac{1}{k} \int_E f \quad \text{for all } k \in \mathbb{N}.$$

Hence, $m(E_\infty) = 0$. □

LEMMA 4.10 (Fatou's Lemma). *Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of nonnegative and measurable functions on a measurable set E. If $f_n \to f$ pointwise a.e. on E, then*

$$\int_E f \leq \liminf_{n\to\infty} \int_E f_n.$$

PROOF. Let $0 \leq g \leq f$ be a bounded and measurable functions supported on a set F of finite measure. Write $g_n = \min\{g, f_n\}$. Then g_n is a bounded and measurable functions supported on F. Notice that $g_n \to g$ pointwise a.e. on F. (If $g(x) = f(x)$, then $f_n(x) \to f(x) = g(x)$; if $g(x) < f(x)$, then $f_n(x) \geq g(x)$ for sufficiently large n and so $g_n(x) = g(x)$.) By bounded convergence theorem in Theorem 4.8,

$$\lim_{n\to\infty} \int_F g_n = \int_F g.$$

By construction, we also have that $g_n \leq f_n$, so $\int_F g_n \leq \int_F f_n \leq \int_E f_n$. Therefore,

$$\int_F g = \lim_{n\to\infty} \int_F g_n \leq \liminf_{n\to\infty} \int_E f_n.$$

Taking supremum of g on the left-hand side finishes the proof. □

COROLLARY 4.11. *Let $f : E \to \mathbb{R}$ be a nonnegative and measurable function on a measurable set E. Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of nonnegative and measurable functions on E. If $f_n \leq f$ and $f_n \to f$ pointwise a.e. on E, then*

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

PROOF. Since $f_n \leq f$ pointwise a.e. on E, $\int_E f_n \leq \int_E f$ for all $n \in \mathbb{N}$. Therefore,

$$\limsup_{n\to\infty} \int_E f_n \leq \int_E f.$$

Combining with Fatou's Lemma finishes the proof. □

A special case of the above corollary is recorded as

THEOREM 4.12 (Monotone convergence theorem). *Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of nonnegative and measurable functions on a measurable set E. If $f_n \nearrow f$ pointwise a.e. on E, then*

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

4.2.4. Lebesgue integrable functions.

Recall that any function f can be written as $f = f^+ - f^-$, in which $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$ are nonnegative. Moreover, $f : E \to \mathbb{R}$ on a measurable set E is measurable iff both f^+ and f^- are measurable.

Definition (Lebesgue integral). Let $f : E \to \mathbb{R}$ be measurable on a measurable set E. We say that f is (Lebesgue) integrable over E if $|f|$ is (Lebesgue) integrable over E, that it, $\int_E |f|\, dx < \infty$.

Since $0 \le f_\pm \le |f|$, f^+ and f^- are integrable, we then define the (Lebesgue) integral of f over E as

$$\int_E f\, dx = \int_E f^+\, dx - \int_E f^-\, dx.$$

Given any measurable set $F \subset E$, define the (Lebesgue) integral of f over F by

$$\int_F f\, dx = \int_E f \cdot \chi_F\, dx.$$

We are now ready to prove a cornerstone of the Lebesgue integration theory, the dominated convergence theorem. It can be viewed as a culmination of our efforts, and is a general statement about the interplay between limits and integrals.

THEOREM 4.13 (Dominated convergence theorem). *Suppose that $\{f_n\}_{n=1}^\infty$ is a sequence of measurable functions on a measurable set E. If $|f_n| \le g$ for some integrable function g and $f_n \to f$ pointwise a.e. on E, then*

$$\lim_{n\to\infty} \int_E |f_n - f| = 0,$$

and consequently by the triangle inequality,

$$\lim_{n\to\infty} \int_E f_n = \int_E f.$$

PROOF. Let $E_k = \{x \in E : |x| \le k, g(x) \le k\}$. Since $g_k = g \cdot \chi_{E_k} \nearrow g$ pointwise on E,

$$\lim_{k\to\infty} \int_E g_k = \lim_{k\to\infty} \int_{E_k} g = \int_E g,$$

by monotone convergence theorem in Theorem 4.12. Hence, for any $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that

$$\int_{E \setminus E_k} g = \int_E g - \int_{E_k} g < \varepsilon \quad \text{for all } k \ge K.$$

Notice that $\{f_n \cdot \chi_{E_K}\}_{n=1}^\infty$ is a sequence of bounded and measurable functions on E. By bounded convergence theorem in Theorem 4.8, we have that

$$\lim_{n\to\infty} \int_{E_K} |f_n - f| = 0.$$

There exists $N \in \mathbb{N}$ such that

$$\int_{E_K} |f_n - f| < \varepsilon \quad \text{for all } n \ge N.$$

Hence, for all $n \ge N$,

$$\int_E |f_n - f| = \int_{E_K} |f_n - f| + \int_{E\setminus E_K} |f_n - f| \le \int_{E_K} |f_n - f| + 2\int_{E\setminus E_K} g \le \varepsilon + 2\varepsilon = 3\varepsilon.$$

$\square$

Homework Assignment.

4-3. Let $f : E \to \mathbb{R}$ be a bounded and measurable function on a measurable set E. Suppose that $m(E) = 0$. Prove that $\int_E f = 0$.

4-4. Let f and g be two bounded and measurable functions on a set E of finite measure. Suppose that $f = g$ a.e. on E. Prove that $\int_E f = \int_E g$.

4-5. Does the bounded convergence theorem in Theorem 4.8 hold for the Riemann integral? That is, let $\{f_n\}_{n=1}^{\infty}$ be a sequence of Riemann integrable functions on $[a, b]$. Suppose that $\{f_n\}$ converges to f pointwise a.e. on E and $|f_n| < M$ for all $n \in \mathbb{N}$ with some $M \in \mathbb{R}$. Then f is also Riemann integrable on $[a, b]$ and

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

4-6. Does the bounded convergence theorem in Theorem 4.8 hold if we drop the assumption that $\{|f_n|\}$ is uniformly bounded? Prove your assertion.

4-7. Let $f \equiv \infty$ on a set E with measure zero. Prove that $\int_E f = 0$.

4-8. Let $\{f_n\}$ be a sequence of nonnegative and measurable functions that converges to f pointwise a.e. on E. Suppose that $\int_E f_n \le M$ for some $M \in \mathbb{R}$ and all $n \in \mathbb{N}$. Prove that $\int_E f \le M$.

4-9. Let $\{f_n\}$ be a sequence of nonnegative and integrable functions that converges to f pointwise a.e. on $\mathbb{R}$. Suppose that f is integrable and

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n = \int_{\mathbb{R}} f.$$

Prove that

$$\lim_{n\to\infty} \int_E f_n = \int_E f \quad \text{for any measurable set } E.$$

4-10. Does the monotone convergence theorem in Theorem 4.12 hold for decreasing sequences of functions? Prove your assertion.

4-11. Prove the following generalized Fatou's Lemma: Suppose that $\{f_n\}_{n=1}^{\infty}$ is a sequence of nonnegative and measurable functions on a measurable set E. Then

$$\int_E \liminf_{n\to\infty} f_n \le \liminf_{n\to\infty} \int_E f_n.$$

Provide an example for which the equation fails in the above inequality.

4-12. Let $\{f_n\}$ be a sequence of nonnegative and measurable functions on a measurable set E. Prove that

(1)

$$\int_E \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int_E f_n.$$

(2) If

$$\sum_{n=1}^{\infty} \int_E f_n < \infty,$$

then

$$\sum_{n=1}^{\infty} f_n(x) \text{ converges pointwise a.e. on } E.$$

4-13. (Tchebychev inequality) Suppose that f is a nonnegative and measurable function on a measurable set E. Prove that for all $\lambda > 0$,

$$m\left(\{x \in E : f(x) > \lambda\}\right) \leq \frac{1}{\lambda} \int_E f.$$

4.3. The L^1 space of integrable functions

Definition (L^1 space). Let E be a measurable set with positive measure. We define $L^1(E)$ as the space of all integrable functions over E with the norm

$$\|f\|_{L^1(E)} := \int_E |f(x)|\, dx.$$

Note that $\|f\|_{L^1(E)} = 0$ if $f = 0$ a.e. on E, and $\|f\|_{L^1(E)} = \|g\|_{L^1(E)}$ if $f = g$ a.e. on E. These properties of the norm reflects the practice that we have already adopted not to distinguish two functions which agree almost everywhere. With this in mind, we take the precise definition of $L^1(E)$ to be the space of equivalence classes of integrable functions, where we define two functions to be equivalent if they agree almost everywhere. It is easy to verify that it is indeed an equivalence.

Often, however, it is convenient to retain the (imprecise) terminology that an element $f \in L^1(E)$ is an integrable function, even though it is only an equivalence class of such functions. The norm $\|f\|_{L^1(E)}$ is well-defined by the choice of any integrable function in its equivalence class. Moreover, $L^1(E)$ inherits the property that is a vector space. This and other straightforward facts are summarized in the following proposition.

PROPOSITION 4.14. *Let E be a measurable space with positive measure.*
(i). $L^1(E)$ is a normed vector space equipped with the norm $\|f\|_{L^1(E)}$ for $f \in L^1(E)$.
(ii). $L^1(E)$ is a metric space equipped with the metric $d(f,g) = \|f-g\|_{L^1(E)}$ for $f, g \in L^1(E)$.

THEOREM 4.15 (Riesz-Fischer Theorem). *$L^1(E)$ is a Banach space.*

Remark (Convergence in norm). It suffices to prove that $\{f_n\}_{n=1}^{\infty}$ has a limit $f \in L^1(E)$ if $\|f_n - f_m\|_{L^1(E)} \to 0$ as $n, m \to \infty$. That is, we need to show that $\|f_n - f\|_{L^1(E)} \to 0$. Here, if $\|f_n - f\|_{L^1(E)} \to 0$, then we say that $\{f_n\}$ converges to f (in norm) in $L^1(E)$.

However, note that $f_n \to f$ in norm does not imply that $f_n \to f$ pointwise.

PROOF. By the above remark, it is not necessary that f_n converges pointwise so that we have a limiting function to study directly. So our plan of the proof is to extract a subsequence of $\{f_n\}$ that converges to a function f pointwise a.e. on E, and also in norm.

Indeed, consider a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ of $\{f_n\}_{n=1}^{\infty}$ with the following property:

$$\left\|f_{n_{k+1}} - f_{n_k}\right\|_{L^1(E)} \leq \frac{1}{2^k} \quad \text{for all } k \in \mathbb{N}.$$

The existence of such a subsequence is guaranteed by the fact that $\|f_n - f_m\|_{L^1(E)} < \varepsilon$ whenever $n, m \geq N(\varepsilon)$, so that it suffices to take $n_k = N(2^{-k})$.

We now consider the series

$$g_m(x) = |f_{n_1}(x)| + \sum_{k=1}^{m-1} \left|f_{n_{k+1}}(x) - f_{n_k}(x)\right|.$$

Hence, $g(x) = \lim_{m\to\infty} g_m(x)$ is measurable on E. (The convergence is evident since every term in the summation is nonnegative.) Note that

$$\int_E g \le \int_E |f_{n_1}| + \sum_{k=1}^{\infty} \int_E \left|f_{n_{k+1}}(x) - f_{n_k}(x)\right| \le \int_E |f_{n_1}| + \sum_{k=1}^{\infty} \frac{1}{2^k} < \infty.$$

By monotone convergence theorem, g is integrable over E. Therefore, $g < \infty$ a.e. on E and the series

$$\sum_{k=1}^{\infty} \left|f_{n_{k+1}}(x) - f_{n_k}(x)\right| \quad \text{converges a.e. on } E.$$

Then

$$\sum_{k=1}^{\infty} \left[f_{n_{k+1}}(x) - f_{n_k}(x)\right] \quad \text{converges a.e. on } E.$$

Now let

$$h_m(x) = f_{n_1}(x) + \sum_{k=1}^{m-1} \left[f_{n_{k+1}}(x) - f_{n_k}(x)\right],$$

and

$$f(x) = f_{n_1}(x) + \sum_{k=1}^{\infty} \left[f_{n_{k+1}}(x) - f_{n_k}(x)\right].$$

Thus, f is integrable over E since $|f| < g$ and g is integrable. Note that $\lim_{m\to\infty} h_m(x) = f(x)$ a.e. on E and $h_m = f_{n_m}$ so

$$\lim_{k\to\infty} f_{n_k}(x) = f(x) \quad \text{a.e. on } E.$$

Since $|f_{n_k}| < g$ and g is integrable, by dominated convergence theorem, $\|f_{n_k} - f\|_{L^1(E)} \to 0$ as $k \to \infty$.

Next we show that $\|f_n - f\|_{L^1(E)} \to 0$ as $n \to \infty$. Given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $\|f_n - f_m\|_{L^1(E)} < \varepsilon$ if $n, m \ge N$. Choose n_k so that $n_k > N$ and $\|f_{n_k} - f\|_{L^1(E)} < \varepsilon$. Hence,

$$\|f_n - f\|_{L^1(E)} \le \|f_n - f_{n_k}\|_{L^1(E)} + \|f_{n_k} - f\|_{L^1(E)} < 2\varepsilon.$$

□

Corollary 4.16. *If $\{f_n\}_{n=1}^{\infty}$ converges to f in $L^1(E)$, then there is a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ such that $f_{n_k} \to f$ a.e. on E.*

4.3.1. The space $L^1(\mathbb{R})$.

Theorem 4.17. *The following families of functions are dense in $L^1(\mathbb{R})$.*

(i). The simple functions.
(ii). The step functions.
(iii). The continuous functions of compact support.

Proof. A subset A is dense in $L^1(\mathbb{R})$ iff for any $f \in L^1(\mathbb{R})$ and $\varepsilon > 0$, there is a function $g \in A$ such that $\|f - g\|_{L^1(\mathbb{R})} < \varepsilon$, or equivalently, there is a sequence $\{g_n\}_{n=1}^{\infty} \subset A$ such that $\|g_n - f\|_{L^1(\mathbb{R})} \to 0$ as $n \to \infty$.

It suffices to prove for nonnegative $f \in L^1(\mathbb{R})$. (Since any function $f \in L^1(\mathbb{R})$ can be written as $f = f^+ - f^-$ and if there are $g_1, g_2 \in A$ such that $\|f^+ - g_1\|_{L^1(\mathbb{R})} < \varepsilon$ and $\|f^- - g_2\|_{L^1(\mathbb{R})} < \varepsilon$, then $\|(f^+ - f^-) - (g_1 - g_2)\|_{L^1(\mathbb{R})} \le 2\varepsilon$.)

(i). For any integrable function $f \in L^1(\mathbb{R})$, there is a sequence of simple functions $\varphi_n \nearrow f$ pointwise on $\mathbb{R}$. By monotone convergence theorem in Theorem 4.8, $\|\varphi_n - f\|_{L^1(\mathbb{R})} \to 0$ as $n \to \infty$.

(ii). A step function is a linear combination of characteristic functions of intervals. By (i), for any $\varepsilon > 0$, there is a simple function

$$\varphi(x) = \sum_{k=1}^{n} c_k \cdot \chi_{E_k}, \quad \text{in which } c_k\text{'s are positive and distinct,}$$

such that $\|f - \varphi\|_{L^1(\mathbb{R})} < \varepsilon$. Since $\|f\|_{L^1(\mathbb{R})} < \infty$, $\|\varphi\|_{L^1(\mathbb{R})} < \infty$ and $m(E_k) < \infty$.

By Theorem 2.14, there is a finite union of disjoint intervals $U_k = \cup_j I_{k,j}$ such that

$$m\left(E_k \setminus U_k\right) + m\left(U_k \setminus E_k\right) < \frac{\varepsilon}{nc_k}.$$

Hence,

$$m(E_k) - m(U_k) \leq m(E_k \setminus U_k) < \frac{\varepsilon}{nc_k} \quad \text{and} \quad m(U_k) - m(E_k) \leq m(U_k \setminus E_k) < \frac{\varepsilon}{nc_k}.$$

Therefore,

$$|c_k m(E_k) - c_k m(U_k)| < \frac{\varepsilon}{n}.$$

Let

$$g(x) = \sum_{k=1}^{n} c_k \cdot \chi_{U_k} = \sum_{k=1}^{n} \sum_{j} c_k \cdot \chi_{I_{k,j}}.$$

Then

$$\|g - \varphi\|_{L^1(\mathbb{R})} \leq \sum_{k=1}^{n} |c_k m(E_k) - c_k m(U_k)| < \varepsilon,$$

and

$$\|g - f\|_{L^1(\mathbb{R})} \leq \|g - \varphi\|_{L^1(\mathbb{R})} + \|f - \varphi\|_{L^1(\mathbb{R})} < 2\varepsilon.$$

(iii). Similar to (ii), it suffice to prove the case when f is a characteristic of an interval $[a, b]$. In this case, define

$$g(x) = \begin{cases} 1 & \text{if } a \leq x \leq b, \\ 0 & \text{if } x \leq a - \varepsilon \text{ or } x \geq b + \varepsilon, \end{cases}$$

and with g linear on the intervals $[a - \varepsilon, a]$ and $[b, b + \varepsilon]$. Then $\|g - f\|_{L^1(\mathbb{R})} < 2\varepsilon$. ☐

Theorem 4.18 (Invariance properties). *Let $f \in L^1(\mathbb{R})$.*

(i). Translation invariance: Given $h \in \mathbb{R}$, $\|f_h\|_{L^1(\mathbb{R})} = \|f\|_{L(\mathbb{R})}$, where $f_h(x) := f(x - h)$.

(ii). Reflection invariance: $\|f(-x)\|_{L^1(\mathbb{R})} = \|f(x)\|_{L^1(\mathbb{R})}$.

(iii). Dilation invariance: Given $\delta > 0$, $\delta\|f(\delta x)\|_{L^1(\mathbb{R})} = \|f(x)\|_{L^1(\mathbb{R})}$.

Proof.

(i). We verify the translation invariance similar to the stages in the construction of Lebesgue integration.

Stage 1: If $f = \chi_E$ for some measurable set E, then $\|f\|_{L^1(\mathbb{R})} = m(E) = m(E_h) = \|f_h\|_{L(\mathbb{R})}$, in which $E_h = E + h$, since Lebesgue measure is invariant under translation. Then $\|f\|_{L^1(\mathbb{R})} = \|f_h\|_{L(\mathbb{R})}$ for all simple functions by linearity of Lebesgue integration.

Stage 3: If f is measurable and nonnegative, then there is a sequence of simple functions $\{\varphi_n\} \nearrow f$ pointwise. Then $\int \varphi_n \to \int f$ as $n \to \infty$ by monotone convergence theorem in Theorem 4.12. Now $\{(\varphi_n)_h\} \nearrow f_h$ and $\int (\varphi_n)_h = \int \varphi_n$ by Stage 1, we have that $\int (\varphi_n)_h \to \int f_h$ as $n \to \infty$ so $\int f_h = \int f$.

Stage 4. If f is integrable, then $f = f^+ - f^-$ and $\|f\|_{L^1(\mathbb{R})} = \int f^+ + \int f^-$. Now $f_h = (f^+)_h - (f^-)_h$ and $\int (f^+)_h = \int f^+$ and $\int (f^-)_h = \int f^-$. So

$$\|f_h\|_{L^1(\mathbb{R})} = \int (f^+)_h + \int (f^-)_h = \int f^+ + \int f^- = \|f\|_{L^1(\mathbb{R})}.$$

(ii). Similar to (i), we only need to verify that $f = \chi_E$ for some measurable set E. In this case,

$$\|f\|_{L^1(\mathbb{R})} = \int f = m(E),$$

and

$$\|f(-x)\|_{L^1(\mathbb{R})} = m(-E), \quad \text{in which } -E = \{x \in \mathbb{R} : -x \in E\}.$$

So it suffices to show that $m(E) = m(-E)$, and we only need to show this for intervals $E = (a, b)$. The proof is complete by noticing that

$$m(E) = b - a, \quad \text{and} \quad m(-E) = m\left((-b, -a)\right) = b - a.$$

(iii). Similar to (i), we only need to verify that $f = \chi_E$ for some measurable set E. In this case,

$$\|f\|_{L^1(\mathbb{R})} = \int f = m(E),$$

and

$$\|f(\delta x)\|_{L^1(\mathbb{R})} = \int f(\delta x) = m(E_\delta), \quad \text{in which } E_\delta = \{x \in \mathbb{R} : \delta x \in E\}.$$

So it suffices to show that $m(E) = \delta m(E_\delta)$, and we only need to show this for intervals $E = (a, b)$. The proof is complete by noticing that

$$m(E) = b - a, \quad \text{and} \quad m(E_\delta) = m\left(\left(\frac{a}{\delta}, \frac{b}{\delta}\right)\right) = \frac{b-a}{\delta}.$$

$\square$

THEOREM 4.19 (Continuity under translation). *Let $f \in L^1(\mathbb{R})$. Then $f_h \to f$ in $L^1(\mathbb{R})$ as $h \to 0$, that is,*

$$\|f_h - f\|_{L^1(\mathbb{R})} \to 0 \quad \text{as } h \to 0.$$

PROOF. Given any $\varepsilon > 0$, by Theorem 4.17, there is a continuous function of compact support g such that $\|g - f\|_{L^1(\mathbb{R})} < \varepsilon$. Then by Theorem 4.18,

$$\|g_h - f_h\|_{L^1(\mathbb{R})} = \|(g - f)_h\|_{L^1(\mathbb{R})} = \|g - f\|_{L^1(\mathbb{R})} < \varepsilon.$$

Since g is continuous supported on a compact set F, g is uniformly continuous on F and $m(F) < \infty$. (F is closed and bounded.) Thus, there is $\delta > 0$ such that $|g(x-h) - g(x)| < \varepsilon/m(F)$ for all $x \in F$ if $0 < h < \delta$. This means that

$$|g_h(x) - g(x)| = |g(x-h) - g(x)| < \frac{\varepsilon}{m(F)}.$$

Therefore,

$$\|g_h - g\|_{L^1(\mathbb{R})} \le \int_F \frac{\varepsilon}{m(F)} = \varepsilon.$$

Then,

$$\|f_h - f\|_{L^1(\mathbb{R})} \le \|f_h - g_h\|_{L^1(\mathbb{R})} + \|g_h - g\|_{L^1(\mathbb{R})} + \|g - f\|_{L^1(\mathbb{R})} < 3\varepsilon.$$

$\square$

$\boxed{\textbf{Homework Assignment}}$.

4-14. Let $f \in L^1(\mathbb{R})$ and $f_n \in L^1(\mathbb{R})$ for all $n \in \mathbb{N}$. Suppose that $\|f_n - f\|_{L^1(\mathbb{R})} \to 0$ as $n \to \infty$. Does $f_n \to f$ as $n \to \infty$ pointwise a.e. on $\mathbb{R}$? Prove your assertion.

4-15. Suppose that $f \in L^1(\mathbb{R})$. Does $\lim_{x\to\infty} f(x) = 0$? Prove your assertion.

4-16. Suppose that $f \in L^1(\mathbb{R})$ and f is uniformly continuous on $\mathbb{R}$. Prove that $\lim_{|x|\to\infty} f(x) = 0$.

4-17. Prove that if $f \in L^1(\mathbb{R})$, then $F(x) = \int_{(-\infty,x]} f(t)\,dt$ is uniformly continuous on $\mathbb{R}$.

4.4. The L^p spaces

Definition (L^p spaces). Let E be a measurable set with positive measure and $p \in [1, \infty)$. We define $L^p(E)$ as the space of all measurable functions on E that satisfy

$$\int_E |f(x)|^p\,dx < \infty.$$

If $f \in L^p(E)$, we define the $L^p(E)$ norm of f by

$$\|f\|_{L^p(E)} := \left(\int_E |f(x)|^p\,dx\right)^{\frac{1}{p}}.$$

Definition (L^∞ space). Let E be a measurable set with positive measure. We define $L^\infty(E)$ as the space of all measurable functions on E that are essentially bounded, that is, there is $M \in \mathbb{R}$ such that

$$|f(x)| \le M \quad \text{a.e. on } E.$$

If $f \in L^\infty(E)$, we define the $L^\infty(E)$ norm of f by

$$\|f\|_{L^\infty(E)} := \inf_M \{|f(x)| \le M \text{ a.e. on } E\}.$$

Here, $\|f\|_{L^\infty(E)}$ is called the essential supremum of f on E.

It is evident that $|f| \le \|f\|_{L^\infty(E)}$ a.e. on E. Indeed,

$$\left\{x \in E : |f(x)| > \|f\|_{L^\infty(E)}\right\} = \bigcup_{k=1}^{\infty} \left\{x \in E : |f(x)| > \|f\|_{L^\infty(E)} + \frac{1}{k}\right\}$$

is of measure zero since each set in the union is of measure zero.

Remark. For $1 \le p \le \infty$, we as before identify $L^p(E)$ the space of equivalence classes that f and g are equivalent if $f = g$ a.e. on E. It is evident that under such equivalence relation, $L^p(E)$ is a vector space.

Definition (Conjugate exponents). If the two exponents p and q satisfy $1 \le p, q \le \infty$ and the relation

$$\frac{1}{p} + \frac{1}{q} = 1$$

holds, we say that p and q are conjugate or dual exponents. Here, we use the convention that $1/\infty = 0$ so 1 and ∞ are conjugate exponents. We denote p' the conjugate exponent of p.

THEOREM 4.20 (Hölder's inequality). *Let $1 \le p, q \le \infty$ be conjugate exponents. If $f \in L^p(E)$ and $g \in L^q(E)$, then $fg \in L^1(E)$ and*

$$\|fg\|_{L^1(E)} \le \|f\|_{L^p(E)} \|g\|_{L^q(E)}.$$

PROOF. The cases when $p = 1$ or $p = \infty$ are trivial.

The proof of the theorem relies on a generalized form of the arithmetic-geometric mean inequality: If $A, B \geq 0$ and $0 \leq \theta \leq 1$, then

$$A^\theta B^{1-\theta} \leq \theta A + (1-\theta)B. \tag{4.1}$$

Note that when $\theta = 1/2$, the above inequality states the fact that the geometric mean of two numbers is majorized by their arithmetic mean.

To establish (4.1), we observe first that we may assume $B \neq 0$ and replace A by AB. So it suffices to prove that

$$A^\theta \leq \theta A + (1-\theta).$$

To this end, let

$$f(x) = x^\theta - \theta x - (1-\theta).$$

Then

$$f'(x) = \theta(x^{\theta-1} - 1) \begin{cases} \geq 0 & \text{if } 0 \leq x \leq 1; \\ \leq 0 & \text{if } x \geq 1. \end{cases}$$

Hence, the continuous function f on $[0, \infty)$ attains its maximum value at $x = 1$. But $f(1) = 0$ so $f(x) \leq 0$ for all $x \in [0, \infty)$, as desired.

To prove Hölder's inequality, we may assume $\|f\|_{L^p(E)} > 0$ and $\|g\|_{L^q(E)} > 0$. (Otherwise, $f = 0$ a.e. on E or $g = 0$ a.e. on E so $fg = 0$ a.e. on E and Hölder's inequality becomes trivial.) Dividing both sides of Hölder's inequality by $\|f\|_{L^p(E)}\|g\|_{L^q(E)}$, we may assume that $\|f\|_{L^p(E)} = \|g\|_{L^q(E)} = 1$ and we only need to verify that $\|fg\|_{L^1(E)} \leq 1$. To this end, set $A = |f(x)|^p$, $B = |g(x)|^q$, and $\theta = 1/p$ so that $1 - \theta = 1/q$, then (4.1) gives

$$|f(x)g(x)| \leq \frac{1}{p}|f(x)|^p + \frac{1}{q}|g(x)|^q.$$

Integrating both side of the above inequality over E yields

$$\|fg\|_{L^1(E)} = \int_E |fg| \leq \frac{1}{p}\int_E |f|^p + \frac{1}{q}\int_E |g|^q = \frac{1}{p}\|f\|^p_{L^p(E)} + \frac{1}{q}\|g\|^q_{L^q(E)} = 1.$$

□

Remark (Cauchy-Schwarz inequality). A special case of Hölder inequality when $p = q = 2$ is called Cauchy-Schwarz inequality: If $f, g \in L^2(E)$, then $fg \in L^1(E)$, and

$$\|fg\|_{L^1(E)} \leq \|f\|_{L^2(E)}\|g\|_{L^2(E)}.$$

Question. Is there any inclusion relation among $L^p(E)$ for different exponents of p in general?

COROLLARY 4.21. *Let $m(E) < \infty$ and $1 \leq p \leq r \leq \infty$. Then $L^r(E) \subset L^p(E)$ if $r \geq p$, and*

$$\|f\|_{L^p(E)} \leq m(E)^{\frac{1}{p}-\frac{1}{r}}\|f\|_{L^r(E)}.$$

PROOF. By Hölder's inequality with two conjugate exponents r/p and $r/(r-p)$, we have that

$$\int_E |f|^p \leq \left[\int_E (|f|^p)^{\frac{r}{p}}\right]^{\frac{p}{r}} \left[\int_E 1^{\frac{r}{r-p}}\right]^{\frac{r-p}{r}} = \left[\int_E |f|^r\right]^{\frac{p}{r}} [m(E)]^{1-\frac{p}{r}}.$$

Raising to power $1/p$ to both sides, we have that

$$\left(\int_E |f|^p\right)^{\frac{1}{p}} \leq \left[\int_E (|f|^p)^{\frac{r}{p}}\right]^{\frac{1}{r}} \left[\int_E 1^{\frac{r}{r-p}}\right]^{\frac{r-p}{rp}} = \left[\int_E |f|^r\right]^{\frac{1}{r}} [m(E)]^{\frac{1}{p}-\frac{1}{r}},$$

which is

$$\|f\|_{L^p(E)} \leq m(E)^{\frac{1}{p}-\frac{1}{r}} \|f\|_{L^r(E)}.$$

□

COROLLARY 4.22. *Let $m(E) < \infty$ and $f \in L^\infty(E)$. Then $f \in L^p(E)$ for all $1 \leq p \leq \infty$, and*

$$\lim_{p\to\infty} \|f\|_{L^p(E)} = \|f\|_{L^\infty(E)}.$$

PROOF. On one hand, by Hölder's inequality,

$$\|f\|_{L^p(E)} \leq m(E)^{\frac{1}{p}} \|f\|_{L^\infty(E)}.$$

Taking $p \to \infty$, we have that

$$\limsup_{p\to\infty} \|f\|_{L^p(E)} \leq \|f\|_{L^\infty(E)},$$

since $(m(E))^{1/p} \to 1$ as $p \to \infty$.

On the other hand, for any $\varepsilon > 0$, $\|f\|_{L^\infty(E)} - \varepsilon$ not the essential supremum of $|f|$ so

$$m(E_1) > \delta \quad \text{for some } \delta > 0,$$

in which

$$E_1 = \left\{x \in E : |f(x)| \geq \|f\|_{L^\infty(E)} - \varepsilon\right\}.$$

Hence,

$$\|f\|^p_{L^p(E)} = \int_E |f|^p \geq \int_{E_1} |f|^p \geq m(E_1)\left(\|f\|_{L^\infty(E)} - \varepsilon\right)^p \geq \delta\left(\|f\|_{L^\infty(E)} - \varepsilon\right)^p.$$

Then

$$\|f\|_{L^p(E)} \geq \delta^{\frac{1}{p}}\left(\|f\|_{L^\infty(E)} - \varepsilon\right),$$

which implies that

$$\liminf_{p\to\infty} \|f\|_{L^p(E)} \geq \|f\|_{L^\infty(E)} - \varepsilon,$$

since $\delta^{1/p} \to 1$ as $p \to \infty$. But the above inequality is true for all $\varepsilon > 0$ so

$$\liminf_{p\to\infty} \|f\|_{L^p(E)} \geq \|f\|_{L^\infty(E)},$$

and the proof is finished. □

THEOREM 4.23 (Minkowski's inequality). *If $1 \leq p \leq \infty$ and $f, g \in L^p(E)$, then $f+g \in L^p(E)$ and*

$$\|f+g\|_{L^p(E)} \leq \|f\|_{L^p(E)} + \|g\|_{L^p(E)}.$$

That is, the triangle inequality holds for the $L^p(E)$ norm.

PROOF. The case when $p = 1$ is proved in Section 4.3, while the case when $p = \infty$ is trivial. When $1 < p < \infty$, first note by checking the cases when $|f(x)| \leq |g(x)|$ and when $|f(x) > |g(x)|$ that

$$|f(x) + g(x)|^p \leq 2^p\left(|f(x)|^p + |g(x)|^p\right).$$

Integrating over E:

$$\int_E |f+g|^p \leq 2^p\left(\int_E |f|^p + \int_E |g|^p\right) < \infty,$$

since $f, g \in L^p(E)$. This shows that $f + g \in L^p(E)$.

By triangle inequality, we have that

$$|f(x) + g(x)|^p \leq |f(x)||f(x) + g(x)|^{p-1} + |g(x)||f(x) + g(x)|^{p-1}.$$

Then by Hölder's inequality with exponents p and $q = p' = p/(p-1)$, we have that

$$\begin{aligned}
\|f+g\|_{L^p(E)}^p &= \int_E |f+g|^p \\
&\le \int_E |f||f+g|^{p-1} + \int_E |g||f+g|^{p-1} \\
&\le \left(\int_E |f|^p\right)^{\frac{1}{p}} \left(\int_E |f+g|^{(p-1)q}\right)^{\frac{1}{q}} + \left(\int_E |g|^p\right)^{\frac{1}{p}} \left(\int_E |f+g|^{(p-1)q}\right)^{\frac{1}{q}} \\
&\le \|f\|_{L^p(E)} \left[\left(\int_E |f+g|^p\right)^{\frac{1}{p}}\right]^{\frac{p}{q}} + \|g\|_{L^p(E)} \left[\left(\int_E |f+g|^p\right)^{\frac{1}{p}}\right]^{\frac{p}{q}} \\
&= \|f\|_{L^p(E)} \|f+g\|_{L^p(E)}^{\frac{p}{q}} + \|g\|_{L^p(E)} \|f+g\|_{L^p(E)}^{\frac{p}{q}} \\
&= \|f+g\|_{L^p(E)}^{\frac{p}{q}} \left(\|f\|_{L^p(E)} + \|g\|_{L^p(E)}\right).
\end{aligned}$$

If $\|f+g\|_{L^p(E)} = 0$, then the theorem is complete; if $\|f+g\|_{L^p(E)} > 0$, then dividing both sides in the above inequality by $\|f+g\|_{L^p(E)}^{\frac{p}{q}}$ gives that

$$\|f+g\|_{L^p(E)}^{p-\frac{p}{q}} = \|f+g\|_{L^p(E)} \le \|f\|_{L^p(E)} + \|g\|_{L^p(E)},$$

since $p - p/q = p(1 - 1/q) = 1$. □

Remark. With Minkowski's inequality, $L^p(E)$ is a normed vector space. When $p = 1$, $L^1(E)$ is also complete, therefore is a Banach space in Theorem 4.15. Similarly, $L^p(E)$ is also a Banach space.

THEOREM 4.24. *Let $1 \le p \le \infty$. Then $L^p(E)$ is a Banach space.*

Homework Assignment.

4-18. Prove that the linear vector space of all polynomials on $[0, 1]$ is not complete under the L^p norm, $1 \le p \le \infty$. (Hint: Construct a sequence of polynomials that converges to e^x in L^p norm.)

4-19. (Interpolation Theorem) Let $1 \le p < q < r \le \infty$. Suppose that $f \in L^p(E) \cap L^r(E)$. Prove that

$$\|f\|_{L^q(E)} \le \|f\|_{L^p(E)}^{1-t} \|f\|_{L^r(E)}^t, \quad \text{where } t \text{ satisfies that } \frac{1}{q} = \frac{1-t}{p} + \frac{t}{r}.$$

As a corollary, this shows that $f \in L^q(E)$, and

$$L^p(E) \cap L^r(E) \subset L^q(E).$$

4-20. Let $C([a, b])$ be the space of continuous functions on a bounded interval $[a, b]$ with the norm

$$\|f\|_{C([a,b])} = \max_{x \in [a,b]} |f(x)|.$$

Prove that $C([a, b])$ is a Banach space.

4.5. Fubini's Theorem

Fubini's theorem concerns interchanging order of integration in the multi-variable setting. It is of vital importance.

To set up Fubini's Theorem, let $\mathbb{R}^d = \mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, where $d = d_1 + d_2$ and $d_1, d_2 \geq 1$. A point in $\mathbb{R}^d$ then takes the form (x, y), where $x \in \mathbb{R}^{d_1}$ and $y \in \mathbb{R}^{d_2}$. If f is a function in $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, define a "slice" of f corresponding to $y \in \mathbb{R}^{d_2}$ as

$$f^y(x) = f(x, y),$$

similarly, the slice of f for a fixed $x \in \mathbb{R}^{d_1}$ is $f_x(y) = f(x, y)$.

THEOREM 4.25 (Fubini's Theorem). *Suppose that $f(x, y)$ is integrable on $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$. Then for a.e. $y \in \mathbb{R}^{d_2}$*

(i). The slice f^y is integrable on $\mathbb{R}^{d_1}$.

(ii). The function defined by

$$\int_{\mathbb{R}^{d_1}} f^y(x)\, dx$$

is integrable on $\mathbb{R}^{d_2}$.

Moreover,

(iii).

$$\int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x, y)\, dx \right) dy = \int_{\mathbb{R}^d} f.$$

Clearly, the theorem is symmetric in x and y so that we also may conclude that the slice f_x is integrable on $\mathbb{R}^{d_2}$ for a.e. x. Moreover, $\int_{\mathbb{R}^{d_2}} f_x(y)\, dy$ is integrable, and

$$\int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x, y)\, dy \right) dx = \int_{\mathbb{R}^d} f.$$

In particular, Fubini's Theorem states that the integral of f on $\mathbb{R}^d$ can be computed by iterating lower-dimensional integrals, and that the iteration can be taken in any order.

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