Mathematical Analysis

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Remark. You are entitled to a reward of one point toward a homework assignment if you are the first person to report a bona-fide mathematical mistake (i.e., not including language typos and grammatical errors.)

Contents

CHAPTER 1

The real number system

1.1. Sets and functions

We speak the language of sets and functions in mathematics. In this section, we define the basic concepts of *sets* and *functions*.

Definition (Sets).

- Given a set A, $x \in A$ denotes that x is a member (or element, point) of A and $x \notin A$ denotes that x is not a member of A.
- Given two sets A and B, we say that A is a subset of B, denoted by $A \subset B$, if each member of A is a member of B, that is, $x \in A$ implies that $x \in B$.
- We say that two sets A and B are equal, denoted by $A = B$, if they have the same members.

Remark. Given two sets A and B, $A = B$ if and only if (denoted by "iff") $A \subset B$ and $B \subset A$. Therefore, to show that $A = B$, we need to prove that, on one hand $A \subset B$, that is, $x \in A$ implies that $x \in B$, and on the other hand $B \subset A$, that is, $x \in B$ implies that $x \in A$.

Example.

- We say that a set A is finite, if there are finite number of members of A. In this case, we use $|A|$ to denote the size (that is, number of members) of A.
- $\mathbb{N} = \{1, 2, 3, ...\}$ denotes the set of natural numbers.[i]
- $\mathbb{Z} = \{..., -3, -2, -1, 0, 1, 2, 3, ...\}$ denotes the set of integers.
- $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z} \text{ and } q \neq 0\}$ denotes the set of rational numbers.
- We say that a set is countably infinite if its members can be arranged into an infinite sequence. For example, $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ are countably infinite.
- We say that a set is discrete if it is finite or countably infinite.

Definition (The empty set). The set that has no members is called the empty set and is denoted by $\emptyset$. A set that is not equal to the empty set is said to be nonempty.

Definition (A singleton set). A set that has a single member is called a singleton set.

Definition (The power set). Given a set A, the set of all the subsets of A is called the power set of A and is denoted by $\mathcal{P}(A)$.

Example. Let $A = \{1, 2, 3\}$. Then

$$\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}.$$

Remark. If A has n members, then $\mathcal{P}(A)$ has 2^n members. This is because each subset of A can be constructed via a selection process through all of its members (that is, selected or not). In the example above, we find all the subsets of $\{1, 2, 3\}$ via a selection process through members 1 (selected or not), 2 (selected or not), and 3 (selected or not).

[i] Notice that in the literature, $\mathbb{N}$ may refer to the set $\{0, 1, 2, 3, ...\}$.

Definition (Union, intersection, and complement). Let A and B be two sets.

- The union of A and B is $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
- The intersection of A and B is $A \cap B = \{x : x \in A \text{ and } x \in B\}$. We say that A and B are disjoint if $A \cap B = \emptyset$.
- The complement of A in B is $B \setminus A = \{x : x \in B \text{ and } x \notin A\}$, and is also denoted by $B \setminus A$. In particular, if all the set operations are within a universal set X, then for a set $A \subset X$, we simply call $X \setminus A$ the complement of A, and denote by A^c.

Example. Let $A = \{2,3,4\}$ and $B = \{3,4,5\}$. Then $A \cup B = \{2,3,4,5\}$, $A \cap B = \{3,4\}$, $A \setminus B = \{2\}$, and $B \setminus A = \{5\}$.

Remark. Let J be an indexing set. Then

$$\bigcup_{j \in J} A_j = \{x : x \in A_j \text{ for some } j \in J\},$$

and

$$\bigcap_{j \in J} A_j = \{x : x \in A_j \text{ for every } j \in J\}.$$

Example.

- If $J = \{1,2,...,n\}$, then we write

$$\bigcup_{j \in J} A_j = \bigcup_{j=1}^{n} A_j \quad \text{and} \quad \bigcap_{j \in J} A_j = \bigcap_{j=1}^{n} A_j.$$

- If $J = \mathbb{N}$, then we write

$$\bigcup_{j \in J} A_j = \bigcup_{j=1}^{\infty} A_j \quad \text{and} \quad \bigcap_{j \in J} A_j = \bigcap_{j=1}^{\infty} A_j.$$

THEOREM (De Morgan). *Let A_j, $j \in J$, be sets. Then*

$$\left(\bigcup_{j \in J} A_j\right)^c = \bigcap_{j \in J} A_j^c \quad \text{and} \quad \left(\bigcap_{j \in J} A_j\right)^c = \bigcup_{j \in J} A_j^c.$$

PROOF. First we show that $(\cup_{j\in J} A_j)^c \subset \cap_{j\in J} A_j^c$. To this end, let $x \in (\cup_{j\in J} A_j)^c$. Then $x \notin \cup_{j\in J} A_j$, which means that $x \notin A_j$ for every $j \in J$. (If not, then $x \in A_j$ for some $j \in J$, which means that $x \in \cup_{j\in J} A_j$.) Thus, $x \in A_j^c$ for every $j \in J$. So $x \in \cap_{j\in J} A_j^c$.

Then we show that $\cap_{j\in J} A_j^c \subset (\cup_{j\in J} A_j)^c$. To this end, let $x \in \cap_{j\in J} A_j^c$. Then $x \in A_j^c$ for every $j \in J$, which means that $x \notin A_j$ for every $j \in J$. Hence, $x \notin \cup_{j\in J} A_j$. (If not, then $x \in \cup_{j\in J} A_j$, which means that $x \in A_j$ for some $j \in J$.) Thus, $x \in (\cup_{j\in J} A_j)^c$.

Since $(\cup_{j\in J} A_j)^c \subset \cap_{j\in J} A_j^c$ and $\cap_{j\in J} A_j^c \subset (\cup_{j\in J} A_j)^c$, we have that $(\cup_{j\in J} A_j)^c = \cap_{j\in J} A_j^c$.

See Question 1.1 for the proof of the other identity. □

Definition (Functions). Let X and Y be sets. A function $f : X \to Y$ is a correspondence that assigns to each member $x \in X$ a member in Y, denoted by $f(x)$.

Definition. Let $f : X \to Y$.

- We say that f is injective if $x_1, x_2 \in X$ and $x_1 \neq x_2$ imply that $f(x_1) \neq f(x_2)$.
- We say that f is surjective if for every $y \in Y$, there is $x \in X$ such that $f(x) = y$.
- We say that f is bijective if it is injective and surjective.

Example.

- Let $f : X \to X$ be defined by $f(x) = x$. Then f is bijective.
- Let $f : \{a, b\} \to \{1\}$ be defined by $f(a) = f(b) = 1$. Then f is not injective.
- The function $f : \mathbb{Z} \to \mathbb{N}$ defined by $f(x) = x^2$ is not injective.
- The function $f : \mathbb{N} \to \mathbb{N}$ defined by $f(x) = x^2$ is not surjective.
- The function $f : \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x + 1$ is bijective.

Definition (Images). Let $f : X \to Y$. Given a subset $E \subset X$, we define $f(E) = \{f(x) \in Y : x \in E\}$ the image of E.

Proposition 1.1. *Let $f : X \to Y$ and $A_1, A_2 \subset X$. Then $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$.*

Proof. To show $f(A_1 \cap A_2) \subset f(A_1) \cap f(A_2)$, let $y \in f(A_1 \cap A_2)$. Then there is an $x \in A_1 \cap A_2$ with $f(x) = y$. Now $x \in A_1$ and $x \in A_2$. Thus, $y = f(x) \in f(A_1)$ and $y = f(x) \in f(A_2)$. So $y \in f(A_1) \cap f(A_2)$. □

Remark. Let $f : X \to Y$ and $A_1, A_2 \subset X$. It is not necessarily true that $f(A_1 \cap A_2) = f(A_1) \cap f(A_2)$. Indeed, let $f : \{a, b\} \to \{1\}$ be defined by $f(a) = f(b) = 1$. Let $A_1 = \{a\}$ and $A_2 = \{b\}$. Then $f(A_1) = \{1\}$ and $f(A_2) = \{1\}$ so $f(A_1) \cap f(A_2) = \{1\}$. Since $A_1 \cap A_2 = \emptyset$, $f(A_1 \cap A_2) = \emptyset$. Hence, $f(A_1 \cap A_2) \neq f(A_1) \cap f(A_2)$.

Definition (Inverse images). Let $f : X \to Y$. Given a subset $E \subset Y$, we define $f^{-1}(E) = \{x \in X : f(x) \in E\}$ the inverse image of E.

Example. Let $f : \mathbb{Z} \to \mathbb{N}$ be defined by $f(x) = x^2$. Then $f^{-1}(\{1\}) = \{-1, 1\}$, $f^{-1}(\{2\}) = \emptyset$, $f^{-1}(\{4, 9\}) = \{-2, -3, 2, 3\}$.

Proposition 1.2. *Let $f : X \to Y$ and $B_1, B_2 \subset Y$. Then $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$.*

Proof. First we show that $f^{-1}(B_1 \cup B_2) \subset f^{-1}(B_1) \cup f^{-1}(B_2)$. To this end, let $x \in f^{-1}(B_1 \cup B_2)$. Then $f(x) \in B_1 \cup B_2$. Now $f(x) \in B_1$ or $f(x) \in B_2$. If $f(x) \in B_1$, then $x \in f^{-1}(B_1)$. If $f(x) \in B_2$, then $x \in f^{-1}(B_2)$. Thus, $x \in f^{-1}(B_1)$ or $x \in f^{-1}(B_2)$. So $x \in f^{-1}(B_1) \cup f^{-1}(B_2)$.

Then we show that $f^{-1}(B_1) \cup f^{-1}(B_2) \subset f^{-1}(B_1 \cup B_2)$. To this end, let $x \in f^{-1}(B_1) \cup f^{-1}(B_2)$. Then $x \in f^{-1}(B_1)$ or $x \in f^{-1}(B_2)$. If $x \in f^{-1}(B_1)$, then $f(x) \in B_1$. If $x \in f^{-1}(B_2)$, then $f(x) \in B_2$. Thus, $f(x) \in B_1$ or $f(x) \in f(B_2)$. So $f(x) \in B_1 \cup B_2$, that is, $x \in f^{-1}(B_1 \cup B_2)$.

Since $f^{-1}(B_1 \cup B_2) \subset f^{-1}(B_1) \cup f^{-1}(B_2)$ and $f^{-1}(B_1) \cup f^{-1}(B_2) \subset f^{-1}(B_1 \cup B_2)$, we have that $f^{-1}(B_1 \cup B_2) = f^{-1}(B_1) \cup f^{-1}(B_2)$. □

Homework Assignment.

Question 1.1. Show that

$$\left(\bigcap_{j \in J} A_j\right)^c = \bigcup_{j \in J} A_j^c.$$

Question 1.2. Let $f : X \to Y$ and $A_1, A_2 \subset X$. Show that $f(A_1 \cup A_2) = f(A_1) \cup f(A_2)$.

Question 1.3. Let $f : X \to Y$ and $B_1, B_2 \subset Y$. Show that $f^{-1}(B_1 \cap B_2) = f^{-1}(B_1) \cap f^{-1}(B_2)$.

Question 1.4. Let $f : X \to Y$ and $B \subset Y$. Show that $f^{-1}(Y \setminus B) = X \setminus f^{-1}(B)$.

1.2. Ordered fields

The set of real numbers is an example of ordered fields. In this section, we define the basic concept of *fields*, in which we can perform the arithmetic operations of addition and multiplication. We then define *order*, through the basic concept of positive numbers.

Definition (Fields). A field $\mathbb{F}$ is a non-empty set with two operations $+$ and $\cdot$, called addition and multiplication, respectively, which satisfy the following axioms.

(A). Axioms of addition:

(A1). For all $a, b \in \mathbb{F}$, $a + b \in \mathbb{F}$.

(A2). Addition is commutative: For all $a, b \in \mathbb{F}$, $a + b = b + a$.

(A3). Addition is associative: For all $a, b, c \in \mathbb{F}$, $(a + b) + c = a + (b + c)$.

(A4). There is an element $0 \in \mathbb{F}$, called the additive identity, such that $0 + a = a$ for all $a \in \mathbb{F}$.

(A5). For each $a \in \mathbb{F}$, there is an element $-a \in \mathbb{F}$, called the additive inverse of a, such that $(-a) + a = 0$.

(M). Axioms of multiplication:

(M1). For all $a, b \in \mathbb{F}$, $a \cdot b \in \mathbb{F}$.

(M2). Multiplication is commutative: For all $a, b \in \mathbb{F}$, $a \cdot b = b \cdot a$.

(M3). Multiplication is associative: For all $a, b, c \in \mathbb{F}$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

(M4). There is an element $1 \in \mathbb{F}$, called the multiplicative identity, such that $1 \cdot a = a$ for all $a \in \mathbb{F}$.

(M5). For each $a \in \mathbb{F}$ such that $a \neq 0$, there is an element $\frac{1}{a} \in \mathbb{F}$, called the multiplicative inverse of a, such that $\frac{1}{a} \cdot a = 1$.

(D). Multiplication is distributive with respect to the addition: For all $a, b, c \in \mathbb{F}$, $a \cdot (b + c) = a \cdot b + a \cdot c$.

Example. We see that $\mathbb{Q}$ is a field, whereas $\mathbb{N}$ are $\mathbb{Z}$ are not. Indeed, 2 does not have an additive inverse in $\mathbb{N}$ and does not have a multiplicative inverse in $\mathbb{Z}$.

Example. There are a variety of operations which fail the commutative property, or associative property, or both. For example, the subtraction fails both (that is, $a - b \neq b - a$ and $(a - b) - c \neq a - (b - c)$), and similarly, the division. The following are more examples.

- The matrix multiplication is not commutative, for instance,
$$\begin{pmatrix} 0 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$
Moreover, let A be a $m \times n$ matrix and B be a $n \times m$ matrix. Then $AB \neq BA$ if $m \neq n$, since the former is a $m \times m$ matrix but the latter is a $n \times n$ matrix.
- The cross product of vectors is not commutative. In fact, it is an example of anti-communtative operation: $\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$.
- The cross product of vectors is not associative, for instance, let $\vec{i}$, $\vec{j}$, and $\vec{k}$ be the unit vectors on the x, y, and z axis, respectively. Then
$$0 = \left(\vec{i} \times \vec{i}\right) \times \vec{j} \neq \vec{i} \times \left(\vec{i} \times \vec{j}\right) = \vec{i} \times \vec{k} = -\vec{j}.$$

We next prove some properties of the arithmetic operations of addition and multiplication in a field. Though common knowledge (but only for the case when $\mathbb{F} = \mathbb{Q}$), we need to verify them as consequences of the axioms listed above. Therefore, in the proofs below, we specify each axiom when needed.

Proposition 1.3. *Let $\mathbb{F}$ be a field and $a, b \in \mathbb{F}$.*

(i). If $a + b = a + c$, then $b = c$.

(ii). If $a + b = a$, then $b = 0$.

(iii). If $a + b = 0$, then $b = -a$.

(iv). $-(-a) = a$.

PROOF.

(i). Compute that

$$\begin{aligned} b &= 0+b \quad \text{by the identity axiom (A4) of addition} \\ &= (-a+a)+b \quad \text{by the inverse axiom (A5) of addition} \\ &= -a+(a+b) \quad \text{by the associative axiom (A3) of addition} \\ &= -a+(a+c) \quad \text{by the assumption that } a+b=a+c \\ &= (-a+a)+c \quad \text{by the associative axiom (A3) of addition} \\ &= 0+c \quad \text{by the inverse axiom (A5) of addition} \\ &= c \quad \text{by the identity axiom (A4) of addition.} \end{aligned}$$

(ii). Since $a+b=a=0+a=a+0$ by the identity axiom (A4) and the commutative axiom (A2) of addition, $b=0$ by (i) with c taken as 0.

(iii). Since $a+b=0=(-a)+a=a+(-a)$ by the inverse axioms (A5) and the commutative axiom (A2) of addition, $b=-a$ by (i) with c taken as $-a$.

(iv). Since $(-a)+a=0$ by the inverse axioms (A5) and the commutative axiom (A2) of addition, $a=-(-a)$ by (iii) with a taken as $-a$ and b taken as a.

□

PROPOSITION 1.4. *Let $\mathbb{F}$ be a field.*

(i). The multiplicative identity is unique.

(ii). Suppose that $a \in \mathbb{F}$ and $a \neq 0$. Then the multiplicative inverse of a is unique.

PROOF.

(i). Suppose that $\overline{1}$ is another multiplicative identity. Then by the identity axiom (M4) of multiplication,

$$\overline{1} \cdot 1 = 1.$$

Since 1 is multiplicative identity, by the identity axiom (M4) of multiplication again,

$$1 \cdot \overline{1} = \overline{1}.$$

But multiplication is communicative by Axiom (M2),

$$\overline{1} \cdot 1 = 1 \cdot \overline{1},$$

which implies that $\overline{1}=1$, that is, the multiplicative identity is unique.

(ii). Write $b=\frac{1}{a}$, that is, b is the multiplicative inverse of a. Hence, $a \cdot b = b \cdot a = 1$ by the inverse axiom (M5) and the commutative axiom (M2) of multiplication.

Suppose that $\overline{b}$ is another multiplicative inverse of a. Then $a \cdot \overline{b} = \overline{b} \cdot a = 1$ by the inverse axiom (M5) and the commutative axiom (M2) of multiplication.

Hence, by the identity axiom (M4) and the associative axiom (M3) of multiplication,

$$\overline{b} = \overline{b} \cdot 1 = \overline{b} \cdot (a \cdot b) = \left(\overline{b} \cdot a\right) \cdot b = 1 \cdot b = b.$$

□

PROPOSITION 1.5. *Let $\mathbb{F}$ be a field and $a \in \mathbb{F}$. Then $a \cdot 0 = 0$.*

PROOF. By the identity axiom (A4) and the commutative axiom (A2) of addition, $a = 0+a=a+0$. Then by the distributive axiom (D),

$$a \cdot a = a \cdot (a+0) = a \cdot a + a \cdot 0.$$

Adding $-a \cdot a$ on both sides, we have that

$$-a \cdot a + a \cdot a = -a \cdot a + a \cdot a + a \cdot 0.$$

But $-a \cdot a + a \cdot a = 0$ by the inverse axiom (A5) of addition. Hence,

$$0 = 0 + a \cdot 0 = a \cdot 0,$$

in which the last equation follows from the identity axiom (A4) of addition. □

Remark. Let $\mathbb{F}$ be a field and $a, b, c \in \mathbb{F}$.

- $a - b$ means $a + (-b)$.
- $a + b + c$ means $(a + b) + c$.
- na for $n \in \mathbb{N}$ means $a + a + \cdots + a$, the addition of a by n times.
- ab means $a \cdot b$.
- $\frac{a}{b}$ (when $b \neq 0$) means $a(\frac{1}{b})$.
- abc means $(ab)c$.
- a^n for $n \in \mathbb{N}$ means $aa \cdots a$, the multiplication of a by n times.

We have verified several basic properties of the arithmetic operations of addition and multiplication in a field $\mathbb{F}$ (see also questions in this section): They behave exactly as in the case when $\mathbb{F} = \mathbb{Q}$. In the following, we perform the arithmetic operations as usual, without specifying which axiom is used.

We now discuss the concept of order in a field.

Definition (Ordered fields). Let $\mathbb{F}$ be a field. We say that $\mathbb{F}$ is an ordered field if there is a nonempty subset P of $\mathbb{F}$ (called the positive subset) for which the following conditions hold.

(i). P is closed under addition: If $a, b \in P$, then $a + b \in P$.
(ii). P is closed under multiplication: If $a, b \in P$, then $ab \in P$.
(iii). Law of trichotomy: For any $a \in \mathbb{F}$, exactly one of the following holds: $a \in P$, $-a \in P$, or $a = 0$.

For $a, b \in \mathbb{F}$, we denote by $a < b$ (or equivalently, $b > a$) if $b - a \in P$, moreover, we denote by $a \leq b$ (or equivalently, $b \geq a$) if $a < b$ or $a = b$.

Remark (Positive and negative numbers). Let $\mathbb{F}$ be an ordered field and $P \subset \mathbb{F}$ be the positive subset. We say that $a \in \mathbb{F}$ is a positive number if $a \in P$ and is a negative number if $-a \in P$. Therefore, $a \in \mathbb{F}$ is either positive, negative, or zero.

Moreover, $a \in \mathbb{F}$ is positive if and only if $a = a - 0 \in P$, that is, $a > 0$; $a \in \mathbb{F}$ is negative if and only if $-a = 0 - a \in P$, that is, $a < 0$.

Example. We see that $\mathbb{Q}$ is an ordered field, whereas one cannot define an order in the field of complex numbers with rational coordinates $\{a + bi : a, b \in \mathbb{Q}\}$. (Indeed, the field $\mathbb{C} = \{a + bi : a, b \in \mathbb{R}\}$ of complex numbers does not support an order either. But we have not introduced the real numbers yet. The other example of fields that does not support an order is the finite field $\mathbb{F}_p = \{0, 1, ..., p - 1\}$, in which p is a prime number.)

Remark. Let $\mathbb{F}$ be an ordered field, in which P is the subset of positive subset. Suppose that $a, b \in \mathbb{F}$. Then $b - a \in \mathbb{F}$. By the law of trichotomy, exactly one of the following holds:

- $b - a \in P$, that is, $a < b$,
- $-(b - a) = -b + a = a - b \in P$, that is, $a > b$,
- $b - a = 0$, that is, $a = b$.

That is, in an ordered field, we can compare any given pair of members. In fact, we can arrange any finite collection of members according to order, via its transitive property below.

Similarly as before in the field, we prove some properties of order in an ordered field $\mathbb{F}$, specifying each condition in the definition when needed.

THEOREM 1.6 (Transitivity of orders). *Let $\mathbb{F}$ be an ordered field and $a, b, c \in \mathbb{F}$ such that $a < b$ and $b < c$. Then $a < c$.*

PROOF. Since $a < b$, $b - a \in P$. Since $b < c$, $c - b \in P$. Because P is closed under addition,

$$(b-a)+(c-b) = b-a+c-b = -a+b-b+c = -a+c = c-a \in P,$$

which implies that $a < c$. □

PROPOSITION 1.7. *Let $\mathbb{F}$ be an ordered field and $a, b, c \in \mathbb{F}$.*

(i). Suppose that $a < b$. Then $a + c < b + c$.
(ii). Suppose that $a < b$ and $c > 0$. Then $ac < bc$.

PROOF.

(i). Since $a < b$, $b - a \in P$. Hence,

$$(b+c)-(a+c) = b+c-a-c = b-a+c-c = b-a \in P.$$

Therefore, $a + c < b + c$.

(ii). Since $a < b$, $b - a \in P$. Since $c > 0$, $c \in P$. Because P is closed under multiplication,

$$c(b-a) = cb - ca = bc - ac \in P.$$

Therefore, $ac < bc$.

□

Remark (Mathematical induction). Mathematical induction is a powerful approach to prove statements that depend on $\mathbb{N}$. That is, to prove that statements $S(n)$ are true for all $n \in \mathbb{N}$, it suffices to establish the following.

(i). $S(1)$ is true.
(ii). Suppose that $S(n)$ is true. Then $S(n + 1)$ is true.

Example. Let $\mathbb{F}$ be an ordered field and $a \in \mathbb{F}$. We prove that if $a > 0$, then $a^n > 0$ for all $n \in \mathbb{N}$ by mathematical induction.

(i). If $n = 1$, then $a^1 = a > 0$.
(ii). Suppose that $a^n > 0$. Then $a^n \in P$, in which P is the positive set. Since P is closed under multiplication, $a^{n+1} = a^n \cdot a \in P$. Therefore, $a^{n+1} > 0$.

PROPOSITION 1.8. *Let $\mathbb{F}$ be an ordered field and $0 < a < b$. Then $a^n < b^n$ for each $n \in \mathbb{N}$.*

PROOF. We prove by mathematical induction on n.

If $n = 1$, then $a^1 = a$ and $b^1 = b$. Hence, $a^1 < b^1$ follows from the assumption.

Suppose that $a^n < b^n$. Because $b > 0$, $a^n \cdot b < b^n \cdot b = b^{n+1}$ by (ii) of the proposition above. Moreover, since $a^n > 0$ (by the example above) and $a < b$, $a^{n+1} = a \cdot a^n < b \cdot a^n = a^n \cdot b$ by the inductive assumption and (ii) of the proposition above again. Hence , $a^{n+1} < b^{n+1}$ by the transitivity property of orders in Theorem 1.6. □

Definition (Absolute values). Let $\mathbb{F}$ be an ordered field. The absolute value of $a \in \mathbb{F}$ is defined as

$$|a| = \begin{cases} a & \text{if } a \geq 0, \\ -a & \text{if } a < 0. \end{cases}$$

Remark (The triangle inequality). Let $\mathbb{F}$ be an ordered field and $a, b \in \mathbb{F}$. Then the triangle inequality yields that

$$|a + b| \leq |a| + |b|.$$

$\boxed{\textbf{Homework Assignment}}$.

Question 1.5. Let $\mathbb{F}$ be a field and $a, b, c \in \mathbb{F}$ such that $a \neq 0$.

(i). Suppose that $ab = ac$. Show that $b = c$.
(ii). Suppose that $ab = a$. Show that $b = 1$.
(iii). Suppose that $ab = 1$. Show that $b = \frac{1}{a}$.
(iv). Show that $1/(\frac{1}{a}) = a$.

Question 1.6. Let $\mathbb{F}$ be a field.

(i). Show that the additive identity is unique.
(ii). Suppose that $a \in \mathbb{F}$. Show that the additive inverse of a is unique.

Question 1.7. Let $\mathbb{F}$ be a field and $a, b \in \mathbb{F}$. Show that $-(a + b) = -a - b$.

Question 1.8. Let $\mathbb{F}$ be an ordered field and $a, b, c \in \mathbb{F}$.

(i). Suppose that $a < b$ and $c < 0$. Show that $ac > bc$.
(ii). Suppose that $a \neq 0$. Show that $a^2 > 0$.
(iii). Suppose that $0 < a < b$. Show that $0 < \frac{1}{b} < \frac{1}{a}$.

Question 1.9. Let $\mathbb{F}$ be an ordered field and $a, b, c, d \in F$ such that $a < b$ and $c < d$.

(i). Show that $a + c < b + d$.
(ii). Is $ac < bd$ true? Prove your assertion.

1.3. The completeness axiom of the real numbers

In the previous section, we defined the ordered fields, for which the set of rational numbers

$$\mathbb{Q} = \left\{ \frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0 \right\}$$

is an important example. The structure of an ordered field allows us to perform arithmetic operations (addition, subtraction, multiplication, and division) and arrangement (according to the order). However, such a structure is not sufficient for our discussion of mathematical analysis for several reasons. For example, while we can define square $a^2 = a \cdot a$ in $\mathbb{Q}$ for each $a \in \mathbb{Q}$, the radical $\sqrt{a}$ (such that $(\sqrt{a})^2 = a$) is not always defined as a rational number. Indeed,

THEOREM 1.9. *There is no solution in $\mathbb{Q}$ to the equation $x^2 = 2$.*

The following is a typical example of proof by contradiction.

PROOF. Suppose that $x = \frac{p}{q} \in \mathbb{Q}$ in the simplified form solves the equation. In particular, $p, q \in \mathbb{Z}$ do not share any common factor. Then

$$\left(\frac{p}{q}\right)^2 = \frac{p^2}{q^2} = 2.$$

That is,

$$p^2 = 2q^2,$$

which implies that p^2 is even. It then follows that p is also even. (If not, that is, p is odd, then p^2 is also odd.) Write $p = 2m$ for $m \in \mathbb{Z}$. Then

$$p^2 = (2m)^2 = 4m^2 = 2q^2,$$

that is, $q^2 = 2m^2$. Hence, q^2 is even so q is also even. Now both p and q are even so they share a common factor of 2. This contradicts with our assumption that p and q do not share any common factor. Therefore, $x \notin \mathbb{Q}$. $\square$

The theorem above asserts that we need to enlarge the set $\mathbb{Q}$ to define radicals. In addition, a satisfactory discussion of the main concepts of mathematical analysis, such as limits, continuity, differentiation, and integration, must be based on an accurately defined concept of the real number system. This is done through the completeness axiom.

We define the set $\mathbb{R}$ of real numbers as a complete ordered field which contains $\mathbb{Q}$. There are several approaches to this definition (which are all equivalent). In fact, $\mathbb{R}$ is the *smallest* complete ordered field which contains $\mathbb{Q}$, that is, $\mathbb{R}$ is the *completion* of $\mathbb{Q}$ (with respect to the metric topology).

In this section, we define the completeness axiom through the "*least-upper-bound*" property.

Definition (Upper bounds and lower bounds). Let $\mathbb{F}$ be an ordered field and $E \subset \mathbb{F}$.

- We say that E is bounded above if there is $a \in \mathbb{F}$ such that $x \leq a$ for all $x \in E$. In this case, a is said to be an upper bound of E.
- We say that E is bounded below if there is $a \in \mathbb{F}$ such that $x \geq a$ for all $x \in E$. In this case, a is said to be a lower bound of E.
- We say that E is bounded if it is bounded above and bounded below.

Remark. Let $\mathbb{F}$ be an ordered field and $E \subset \mathbb{F}$.

- We say that E is unbounded above if it is not bounded above.
- We say that E is unbounded below if it is not bounded below.
- We say that E is unbounded if it is unbounded above or unbounded below.

Example.

- Let $\mathbb{F}$ be an ordered field and $E \subset \mathbb{F}$ be finite. Then E is bounded. Indeed, the largest member in E is an upper bound of E, while the smallest member of E is a lower bound. For example, $\{1, 2, 3\}$ is bounded above by 3 and is bounded below by 1.
- $\mathbb{N}$ is bounded below in $\mathbb{Q}$ because $1 \in \mathbb{Q}$ is a lower bound of $\mathbb{N}$. In fact, if $a \in \mathbb{Q}$ and $a \leq 1$, then a is a lower bound of $\mathbb{N}$.
- Let $E = \{x \in \mathbb{Q} : x^2 < 2\}$. Then E is bounded in $\mathbb{Q}$. Indeed, 2 is an upper bound and -2 is a lower bound.

Proposition 1.10. *Let $\mathbb{F}$ be an ordered field and $E \subset \mathbb{F}$.*

(i). Suppose that E is bounded above with a as an upper bound. Then any $b \in \mathbb{F}$ such that $b \geq a$ is also an upper bound of E.

(ii). Suppose that E is bounded below with a as a lower bound. Then any $b \in \mathbb{F}$ such that $b \leq a$ is also a lower bound of E.

Proof. We prove (i) and leave (ii) in Question 1.10.

Since a is an upper bound, $x \leq a$ for all $x \in E$. Since $a \leq b$, $x \leq b$ because of the transitivity property of orders in Theorem 1.6. Therefore, b is also an upper bound of E. $\square$

Proposition 1.11. $\mathbb{N}$ *is unbounded above in* $\mathbb{Q}$.

Proof. Suppose that $\mathbb{N}$ is bounded above with $a \in \mathbb{Q}$ as an upper bound. Let $a = \frac{p}{q}$ in the simplified form, that is, $p \in \mathbb{Z}$ and $q \geq 1$ do not share any common factors.

Firstly, we have that $p > 0$. If not, that is, $p \leq 0$, then $\frac{p}{q} \leq 0$. Hence, $\frac{p}{q} < 1 \in \mathbb{N}$, which is not possible by the assumption that a is an upper bound of $\mathbb{N}$.

Now $p \in \mathbb{N}$ so $2p \in \mathbb{N}$. Then $2p > \frac{p}{q}$ since $q \geq 1$. This contradicts again with the assumption that a is an upper bound of $\mathbb{N}$. Therefore, $\mathbb{N}$ is unbounded above in $\mathbb{Q}$. $\square$

Definition (Supremum and infimum). Let $\mathbb{F}$ be an ordered field and $E \subset \mathbb{F}$.

- Let E be bounded above. We say that a is the supremum (or the least upper bound) of E, denoted by $a = \sup E$, if a is an upper bound of E and any $b < a$ is not an upper bound of E, that is, any upper bound of E must be greater than or equal to a.
- Let E be bounded below. We say that a is the infimum (or the greatest lower bound) of E, denoted by $a = \inf E$, if a is a lower bound of E and any $b > a$ is not a lower bound of E, that is, any lower bound of E must be smaller than or equal to a.
- We denoted by $\sup E = \infty$ if E is unbounded above, and by $\inf E = -\infty$ if E is unbounded below.

Remark. Let $\mathbb{F}$ be an ordered field and $E \subset \mathbb{F}$. Define $-E = \{-x : x \in E\}$. Then E is bounded above iff $-E$ is bounded below, and E is bounded below iff $-E$ is bounded above. Indeed, any upper bound a of E supplies a lower bound $-a$ of $-E$, and any lower bound a of E supplies an upper bound $-a$ of $-E$, see Question 1.12.

Moreover, if $a = \sup E$, then $-a = \inf(-E)$, and if $a = \inf E$, then $-a = \sup(-E)$, see Question 1.15.

As a consequence, all statements involving the upper bound and the supremum can be directly transformed to ones of the lower bounds and of the infimum.

Example. Let $\mathbb{F}$ be an ordered field and $E \subset \mathbb{F}$ be finite. Then $\sup E$ is the largest member in E and $\inf E$ is the smallest member of E. For example, let $E = \{1, 2, 3\} \subset \mathbb{Q}$. Then $\sup E = 3$ and $\inf E = 1$.

However, we shall point out that the supremum and infimum of a set, if exist, are not necessarily a member of E. For example, let

$$E = \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\} \subset \mathbb{Q}.$$

Then $\sup E = 1 \notin E$.

Proposition 1.12. *Let $\mathbb{F}$ be an ordered field and $E \subset \mathbb{F}$ be bounded above. Suppose that $\sup E$ exists. Then it is unique.*

Proof. Write $a = \sup E$. Then a is an upper bound of E. Suppose that $\overline{a}$ is also a supremum of E. Then $\overline{a}$ is an upper bound of E.

Notice that any upper bound of E must be greater than or equal to $\sup E$. Hence, on one hand, $\overline{a} \geq a$, since $a = \sup E$ and $\overline{a}$ is an upper bound of E; on the other hand, $a \geq \overline{a}$, since $\overline{a} = \sup E$ and a is an upper bound of E.

Therefore, $\overline{a} = a$. $\square$

Definition. Let $\mathbb{F}$ be an ordered field. We say that $\mathbb{F}$ is complete is every subset $E \subset \mathbb{F}$ which is bounded above has its supremum $\sup E \in \mathbb{F}$.

Theorem 1.13. *$\mathbb{Q}$ is not complete.*

Remark. An ordered field $\mathbb{F}$ is complete requires that *each* subset E which is bounded above has its supremum $\sup E \in \mathbb{F}$. Hence, to prove that $\mathbb{Q}$ is not complete, it suffices to find *some* subset E which is bounded above but does not have its supremum defined in $\mathbb{Q}$. However, notice that there are plenty of examples of subsets of $\mathbb{Q}$ which are bounded above and have their supremum in $\mathbb{Q}$. These examples include all the finite subsets and infinite subsets such as

$$E = \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\},$$

for which $\sup E = 1 \in \mathbb{Q}$.

Proof. It suffices to find a set in $\mathbb{Q}$ which is bounded above but does not have its supremum in $\mathbb{Q}$. To this end, set

$$E = \{x \in \mathbb{Q} : x^2 < 2\}.$$

We showed that E is bounded in $\mathbb{Q}$. Suppose that $a = \sup E \in \mathbb{Q}$. Write

$$b = a - \frac{a^2 - 2}{a+2} = \frac{2a+2}{a+2}.$$

Then

$$b^2 - 2 = \frac{2\,(a^2 - 2)}{(a+2)^2}.$$

- If $a \in E$, that is, $a^2 < 2$, then $b > a$ by the first equation above. But the second equation implies that $b^2 < 2$ so $b \in E$. Since a is an upper bound of E, $b \leq a$, which leads to a contradiction.
- If $a \notin E$, that is, $a^2 \geq 2$, then $a^2 > 2$, since $a^2 \neq 2$ for all $a \in \mathbb{Q}$. By the first equation above, $b < a$. But the second equation implies that $b^2 > 2 > x^2$ for all $x \in E$. Hence, $b > x$ for all $x \in E$. Thus, b is also an upper bound of E, which leads to a contradiction since $a = \sup E$.

Therefore, $\sup E$ is not a rational number. □

Remark. The above theorem shows that $\mathbb{Q}$ has certain "gaps" (that is, $\sqrt{2} \notin \mathbb{Q}$), in spite of the fact that between any two rationals there is another. Indeed, if $r < s$, then $r < \frac{r+s}{2} < s$. The real number system fills these gaps. For example, $\sup E$ for which $E = \{x \in \mathbb{Q} : x^2 < 2\}$ defines $\sqrt{2} \in \mathbb{R}$.

Definition. We define the set $\mathbb{R}$ of real numbers as the (smallest) complete ordered field which contains $\mathbb{Q}$.

Remark. The real numbers in $\mathbb{R} \setminus \mathbb{Q}$ are said to be irrational.

Theorem 1.14. *Let $E \subset \mathbb{R}$ be bounded above. Then $a = \sup E$ iff*

(i). a is an upper bound of E,

(ii). for each $\varepsilon > 0$, there is $x = x(\varepsilon) \in E$ such that $x > a - \varepsilon$.

Proof. We first show necessity and only need to prove (ii). For each $\varepsilon > 0$, $a - \varepsilon < a$. Since $a = \sup E$, $a - \varepsilon$ is not an upper bound. Hence, there is $x = x(\varepsilon) \in E$ such that $x > a - \varepsilon$.

We next show sufficiency and only need to prove that any $b \in \mathbb{R}$ such that $b < a$ cannot be an upper bound of E. Set $\varepsilon = a - b > 0$. Then by (ii). there is $x = x(\varepsilon) \in E$ such that $x > a - \varepsilon = a - (a - b) = b$. Hence, b is not an upper bound of E. □

Remark. The theorem above is incredibly useful to prove statements involving supremum. For example, let

$$E = \left\{ \frac{n}{n+1} : n \in \mathbb{N} \right\} \subset \mathbb{R}.$$

We prove that $\sup E = 1$.

(i). Since

$$\frac{n}{n+1} < 1 \quad \text{for all } n \in \mathbb{N},$$

1 is an upper bound of E.

(ii). Set $\varepsilon > 0$. There is $N \in \mathbb{N}$ such that

$$N > \frac{1-\varepsilon}{\varepsilon},$$

because $\mathbb{N}$ is unbounded above (so $\frac{1-\varepsilon}{\varepsilon}$ is not an upper bound of $\mathbb{N}$). It then follows that

$$\frac{N}{N+1} > 1 - \varepsilon.$$

Therefore, $\sup E = 1$.

THEOREM 1.15 (The Archimedean Principle). *Let $a, b \in \mathbb{R}$ and $a > 0$. Then there is $n \in \mathbb{N}$ such that $na > b$.*

PROOF. We prove the theorem by contradiction. Suppose that the conclusion is false, that is, $na \leq b$ for all $n \in \mathbb{N}$. Write

$$E = \{na : n \in \mathbb{N}\}.$$

Then b is an upper bound of E and E is bounded above. Put $c = \sup E$. Since $a > 0$, $c - a < c$ so $c - a$ is not an upper bound of E. Hence, there is $n \in \mathbb{N}$ such that $na > c - a$, which implies that $(n+1)a > c$. But $(n+1)a \in E$, which contradicts with the assumption that c is an upper bound of E. Therefore, there is $n \in \mathbb{N}$ such that $na > b$. □

ALTERNATIVE PROOF. Since $\mathbb{N}$ is unbounded above, $\frac{b}{a}$ is not an upper bound. Hence, there is $n \in \mathbb{N}$ such that $n > \frac{b}{a}$, that is, $na > b$. □

Definition (Intervals). Let $a, b \in \mathbb{R}$ such that $a < b$.

- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$.
- $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$.
- $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$.
- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$.
- $(a, \infty) = \{x \in \mathbb{R} : x > a\}$.
- $[a, \infty) = \{x \in \mathbb{R} : x \geq a\}$.
- $(-\infty, a) = \{x \in \mathbb{R} : x < a\}$.
- $(-\infty, a] = \{x \in \mathbb{R} : x \leq a\}$.

We write $\mathbb{R} = (-\infty, \infty)$.

THEOREM 1.16. *There is a rational number in (a, b).*

PROOF. Since $a < b$, $\frac{1}{b-a} > 0$. By the Archimedean principle, there is $n \in \mathbb{N}$ such that

$$n > \frac{1}{b-a}.$$

So $an + 1 < bn$. Select k to be the largest integer such that $k \leq an$. Then $k + 1 > an$. (If not, then $k + 1 \leq an$, which contradicts with the fact that k is the largest integer such that $k \leq an$.) Now $k + 1 \leq an + 1 < bn$. Thus, $an < k + 1 < bn$. So

$$a < \frac{k+1}{n} < b,$$

in which $\frac{k+1}{n}$ is a rational number. □

COROLLARY 1.17. *There are infinitely many rational numbers in* (a, b).

PROOF. Suppose that there are finitely many rational numbers, $x_1, ..., x_n$ for some $n \in \mathbb{N}$, in (a, b). Select the largest one, called x, that is, $x = \max\{x_1, ...x_n\}$. Then there is a rational number $\bar{x}$ in (x, b), that is, $a < x < \bar{x} < b$. This contradicts with the assumption that x is the largest rational number in (a, b). Therefore, there are infinitely many rational numbers in (a, b). $\square$

Each of the intervals listed above contains an interval of the form (a, b), for instance, $(a, b] \supset (a, c)$ with $c = \frac{a+b}{2}$. As a consequence, each interval contains infinitely many rational numbers. In Question 1.17, we show that the same conclusions hold for irrational numbers.

Homework Assignment.

Question 1.10. Let $\mathbb{F}$ be an ordered field and $E \subset \mathbb{F}$. Suppose that E is bounded below with a as a lower bound. Show that any $b \in \mathbb{F}$ such that $b \leq a$ is also a lower bound of E.

Question 1.11. Let $\mathbb{F}$ be an ordered field and $E \subset \mathbb{F}$ be bounded below. Suppose that $\inf E$ exists. Show that it is unique.

Question 1.12. Let $\mathbb{F}$ be an ordered field and $E \subset \mathbb{F}$ be bounded above. Show that $-E$ is bounded below.

Question 1.13. Let $E \subset \mathbb{R}$ be bounded below. Show that $a = \inf E$ iff

(i). a is a lower bound of E,

(ii). for each $\varepsilon > 0$, there is $x = x(\varepsilon) \in E$ such that $x < a + \varepsilon$.

Question 1.14. Let

$$E = \left\{ \frac{n+1}{n} : n \in \mathbb{N} \right\} \subset \mathbb{R}.$$

Show that $\inf E = 1$.

Question 1.15. Let $E \subset \mathbb{R}$ be bounded above and $a = \sup E$. Show that $-a = \inf(-E)$.

Question 1.16. Show that there is an irrational number in (a, b). (Hint: If r is rational, then $r - \sqrt{2}$ is irrational.)

Question 1.17. Show that there are infinitely many irrational numbers in (a, b).

Question 1.18. Let A and B be bounded sets of real numbers.

(i). Define a set $A + B$ by

$$A + B = \{a + b : a \in A \text{ and } b \in B\}.$$

Show that $\sup(A + B) = \sup A + \sup B$.

(ii). Define a set AB by

$$AB = \{ab : a \in A \text{ and } b \in B\}.$$

Is $\sup(AB) = (\sup A)(\sup B)$ true? Prove your assertion.

CHAPTER 2

Sequences of real numbers

In this chapter, we discuss sequences of real numbers, particularly, their limits. The fundamental "ε-N" language of mathematical analysis is presented in this discussion. (In fact, we already had a taste of it in Section 1.3, in which we discussed the supremum and infimum of sets in $\mathbb{R}$.)

2.1. Convergence and divergence

Definition (Sequences). A sequence of real numbers is a function from $\mathbb{N}$ to $\mathbb{R}$.

Remark. Let $f : \mathbb{N} \to \mathbb{R}$. Then $f(1), f(2), f(3), ...$ is a sequence of real numbers. On the other hand, a sequence of real numbers $a_1, a_2, a_3, ...$ define a function $f : \mathbb{N} \to \mathbb{R}$ by setting $f(n) = a_n$ for all $n \in \mathbb{N}$.

We use $\{a_n\}_{n=1}^{\infty}$ (or simply $\{a_n\}$) to denote a sequence (of real numbers), in which a_n is called the n-th term. A basic difference between a sequence and a set is that the terms in a sequence can have repeated values while all members in a set are distinct.

Example.

- $1, 2, 3, ...$, or $\{n\}_{n=1}^{\infty}$, is the sequence of natural numbers.
- $2, 4, 6, ...$, or $\{2n\}_{n=1}^{\infty}$, is the sequence of positive even numbers.
- $1, 3, 5, ...$, or $\{2n-1\}_{n=1}^{\infty}$, is the sequence of positive odd numbers.
- $1, 4, 9, ...$, or $\{n^2\}_{n=1}^{\infty}$, is the sequence of squares of natural numbers.
- Let $c \in \mathbb{R}$. Then $\{c\}_{n=1}^{\infty}$ is a constant sequence.
- $1, \frac{1}{2}, \frac{1}{3}, ...$, or $\{\frac{1}{n}\}_{n=1}^{\infty}$.
- $-1, 1, -1, 1, ...$, or $\{(-1)^n\}_{n=1}^{\infty}$.
- $2, 3, 5, 7, 11, ...$, in which the n-th term is the n-th prime.

Definition (Convergence and divergence). We say that a sequence $\{a_n\}_{n=1}^{\infty}$ converges, if there is $L \in \mathbb{R}$ with the following property: For each $\varepsilon > 0$, there is $N = N(\varepsilon) \in \mathbb{N}$ such that

$$|a_n - L| < \varepsilon \quad \text{for all } n \geq N.$$

In this case, we say that L is the limit of $\{a_n\}_{n=1}^{\infty}$, or a_n converges to L, and write

$$\lim_{n\to\infty} a_n = L \quad \text{or} \quad a_n \to L \text{ as } n \to \infty.$$

A sequence which is not convergent is said to be divergent.

Remark (The ε-N argument). A sequence $a_n \to L$ means that a_n gets *arbitrarily close* to L if n is *sufficiently large*. In the ε-N argument, ε plays the goal of *closeness* while N indicates the condition of *largeness*, that is, how deep we need to go in the sequence. That is, to achieve a given goal that the terms are in $(L-\varepsilon, L+\varepsilon)$ (that is, $|a_n - L| < \varepsilon$), we can find a sufficiently large N (which depends on ε) such that all terms starting from the N-th term in the sequence are in $(L-\varepsilon, L+\varepsilon)$.

Example. A constant sequence converges (to that constant). Indeed, let $a_n = c$ for all $n \in \mathbb{N}$. For each $\varepsilon > 0$, take $N = 1 \in \mathbb{N}$. Then $|a_n - c| = 0 < \varepsilon$ for all $n \geq N$. Hence, the sequence converges to c.

Example. We prove that the sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ converges. Indeed, we show that

$$\lim_{n\to\infty} \frac{1}{n} = 0.$$

To this end, let $\varepsilon > 0$. Then there is $N = N(\varepsilon) \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon}$ by the Archimedean principle. Now if $n \geq N$, then

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon.$$

Hence, the sequence converges to 0.

Remark. To prove convergence $a_n \to L$ by the ε-N argument, we find $N = N(\varepsilon) \in \mathbb{N}$ such that the terms a_n when $n \geq N$ are in $(L - \varepsilon, L + \varepsilon)$. Usually, we need to choose larger N for smaller ε (that is, to meet the demand that a_n are closer to L). For instance, in the example above,

- If $\varepsilon = 1$, then we can take $N = N(1) = 2$. So $|\frac{1}{n} - 0| < 1$ for all $n \geq 2$.
- If $\varepsilon = 0.1$, then we can take $N = N(0.1) = 11$. So $|\frac{1}{n} - 0| < 0.1$ for all $n \geq 11$.
- If $\varepsilon = 0.01$, then we can take $N = N(0.01) = 101$. So $|\frac{1}{n} - 0| < 0.01$ for all $n \geq 101$.

Notice that the choice of $N(\varepsilon)$ for a given ε is not unique. In particular, with $\varepsilon = 0.01$, the choice of $N(0.01) = 200$ (or any integer greater than 100) would also work.

Example. We prove that the sequence $\{\frac{1}{\sqrt{n}}\}_{n=1}^{\infty}$ converges. Indeed, we show that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0.$$

To this end, let $\varepsilon > 0$. Then there is $N = N(\varepsilon) \in \mathbb{N}$ such that $N > \frac{1}{\varepsilon^2}$ by the Archimedean principle. Now if $n \geq N$, then

$$\left|\frac{1}{\sqrt{n}} - 0\right| = \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \varepsilon.$$

Therefore, the sequence converges to 0.

Remark. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence which is defined by $a_n = f(n)$ for all $n \in \mathbb{N}$. To prove that $a_n \to L$ in the ε-N argument, it amounts to solving the inequality $|f(n) - L| < \varepsilon$. For instance, the examples above correspond to

$$\left|\frac{1}{n} - 0\right| < \varepsilon, \quad \text{which solves to } n > \frac{1}{\varepsilon},$$

and

$$\left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon, \quad \text{which solves to } n > \frac{1}{\varepsilon^2}.$$

Example. We prove that the sequence $\{\frac{n^2}{n^2+1}\}_{n=1}^{\infty}$ converges. Indeed, we show that

$$\lim_{n\to\infty} \frac{n^2}{n^2 + 1} = 1.$$

To this end, let $\varepsilon > 0$. If $\varepsilon > 1$, then for all $n \in \mathbb{N}$,

$$\left|\frac{n^2}{n^2+1} - 1\right| = \frac{1}{n^2+1} < 1 < \varepsilon.$$

If $0 < \varepsilon \le 1$, then there is $N = N(\varepsilon) \in \mathbb{N}$ such that $N > \sqrt{\frac{1}{\varepsilon} - 1}$ by the Archimedean principle. Now if $n \ge N$, then

$$\left|\frac{n^2}{n^2+1} - 1\right| = \frac{1}{n^2+1} \le \frac{1}{N^2+1} < \varepsilon.$$

Therefore, the sequence converges to 0.

Remark. In the example above, the inequality

$$\left|\frac{n^2}{n^2+1} - 1\right| < \varepsilon.$$

solves to

$$N > \sqrt{\frac{1}{\varepsilon} - 1},$$

which requires that $\frac{1}{\varepsilon} > 1$, that is, $\varepsilon \le 1$. We therefore argued in two cases of $\varepsilon > 1$ and $0 < \varepsilon \le 1$.

We shall again remark the flexibility when choosing $N = N(\varepsilon)$ in the ε-N argument. Indeed, the choice of $N = N(\varepsilon)$ such that $N > \sqrt{\frac{1}{\varepsilon}}$ also works in the example above, since

$$\left|\frac{n^2}{n^2+1} - 1\right| = \frac{1}{n^2+1} \le \frac{1}{N^2+1} < \frac{1}{N^2} < \varepsilon.$$

This choice also spares us the trouble to divide the argument into two cases of $\varepsilon > 1$ and $0 < \varepsilon \le 1$.

Example. We prove that the sequence $\{e^{-n}\}_{n=1}^{\infty}$ converges. Indeed, we show that

$$\lim_{n\to\infty} e^{-n} = 0.$$

To this end, let $\varepsilon > 0$. Then there is $N = N(\varepsilon) \in \mathbb{N}$ such that $N > -\ln\varepsilon$ by the Archimedean principle. Now if $n \ge N$, then

$$\left|e^{-n} - 0\right| = e^{-n} \le e^{-N} < \varepsilon.$$

Therefore, the sequence converges to 0.

Remark. To prove convergence of a sequence a_n by the ε-N argument, the particular interest lies in the case when ε is small. For instance, in the example above, $N > -\ln\varepsilon > 0$ and gets larger when $\varepsilon < 1$ and gets smaller.

Example. We prove that the sequence $\{2n\}_{n=1}^{\infty}$ diverges by contradiction. Suppose that

$$\lim_{n\to\infty} 2n = L \quad \text{for some } L \in \mathbb{R}.$$

That is, for each $\varepsilon > 0$, there is $N = N(\varepsilon) \in \mathbb{N}$ such that

$$|2n - L| < \varepsilon \quad \text{for all } n \ge N.$$

In particular, set $\varepsilon = 1$. Then there is $N = N(1) \in \mathbb{N}$ such that

$$|2n - L| < 1 \quad \text{for all } n \ge N,$$

which implies that

$$2n < |L| + 1 \quad \text{for all } n \geq N.$$

But this is impossible because of the Archimedean principle. Therefore, the sequence diverges.

Remark (Triangle inequality). The triangle inequality and its variances are frequently used in arguing convergence and divergence: Let $a, b \in \mathbb{R}$. Then

$$|a| - |b| \leq |a + b| \leq |a| + |b|.$$

Example. We prove that the sequence $\{(-1)^n\}_{n=1}^{\infty}$ diverges by contradiction. Suppose that

$$\lim_{n\to\infty} (-1)^n = L \quad \text{for some } L \in \mathbb{R}.$$

That is, for each $\varepsilon > 0$, there is $N = N(\varepsilon) \in \mathbb{N}$ such that

$$|(-1)^n - L| < \varepsilon \quad \text{for all } n \geq N.$$

In particular, set $\varepsilon = \frac{1}{2}$. Then there is $N = N(\frac{1}{2}) \in \mathbb{N}$ such that

$$|(-1)^n - L| < \frac{1}{2} \quad \text{for all } n \geq N.$$

Notice that $(-1)^n$ and $(-1)^{n+1}$ always have opposite signs. Hence,

$$\left|(-1)^n - (-1)^{n+1}\right| = 2.$$

By the triangle inequality,

$$\begin{aligned} 2 &= \left|(-1)^n - (-1)^{n+1}\right| \\ &= \left|(-1)^n - L + L - (-1)^{n+1}\right| \\ &\leq |(-1)^n - L| + \left|L - (-1)^{n+1}\right| \\ &< \frac{1}{2} + \frac{1}{2} \\ &= 1, \end{aligned}$$

which is impossible. Therefore, the sequence diverges.

Theorem 2.1. *The limit of a convergent sequence is unique.*

Proof. We prove by contradiction. Suppose that $a_n \to L$ and $a_n \to M$ with $L \neq M$. Then $|L - M| > 0$. Set

$$\varepsilon = \frac{|L - M|}{3} > 0.$$

Then $(L - \varepsilon, L + \varepsilon) \cap (M - \varepsilon, M + \varepsilon) = \emptyset$.

Since $a_n \to L$, there is $N_1 = N_1(\varepsilon) \in \mathbb{N}$ such that

$$|a_n - L| < \varepsilon \quad \text{for all } n \geq N_1.$$

That is, $a_n \in (L - \varepsilon, L + \varepsilon)$ if $n \geq N_1$.

Since $a_n \to M$, there is $N_2 = N_2(\varepsilon) \in \mathbb{N}$ such that

$$|a_n - M| < \varepsilon \quad \text{for all } n \geq N_2.$$

That is, $a_n \in (M - \varepsilon, M + \varepsilon)$ if $n \geq N_2$.

Now if $n \geq \max\{N_1, N_2\}$, then $a_n \in (L - \varepsilon, L + \varepsilon) \cap (M - \varepsilon, M + \varepsilon)$, which contradicts with the fact that $(L - \varepsilon, L + \varepsilon) \cap (M - \varepsilon, M + \varepsilon) = \emptyset$. Therefore, $L = M$. $\square$

ALTERNATIVE PROOF. Suppose that $a_n \to L$ and $a_n \to M$. Let $\varepsilon > 0$. Since $a_n \to L$, there is $N_1 = N_1(\frac{\varepsilon}{2}) \in \mathbb{N}$ such that

$$|a_n - L| < \frac{\varepsilon}{2} \quad \text{for all } n \geq N_1.$$

Since $a_n \to M$, there is $N_2 = N_2(\frac{\varepsilon}{2}) \in \mathbb{N}$ such that

$$|a_n - M| < \frac{\varepsilon}{2} \quad \text{for all } n \geq N_2.$$

Now if $n \geq \max\{N_1, N_2\}$, then by the triangle inequality,

$$\begin{aligned} |L - M| &= |L - a_n + a_n - M| \\ &\leq |L - a_n| + |a_n - M| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon. \end{aligned}$$

That is, $|L - M| < \varepsilon$ for all $\varepsilon > 0$. It then follows that $L = M$. (If not, that is, $L \neq M$, then $|L - M| > \varepsilon$ by setting $\varepsilon = \frac{|L-M|}{2} > 0$.) □

Remark. Let $a, b \in \mathbb{R}$.

- If $|a - b| < \varepsilon$ for all $\varepsilon > 0$, then $a = b$.
- If $a \neq b$, then there is $\varepsilon > 0$ (e.g., $\varepsilon = \frac{|a-b|}{2}$) such that $|a - b| > \varepsilon$.

THEOREM 2.2. *A sequence $a_n \to L$ iff for each $\varepsilon > 0$, all but a finite number of terms of $\{a_n\}_{n=1}^{\infty}$ are in $(L - \varepsilon, L + \varepsilon)$.*

PROOF. We first show necessity. Suppose that $a_n \to L$. Then for each $\varepsilon > 0$, there is $N = N(\varepsilon) \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for all $n \geq N$, that is, $a_n \in (L - \varepsilon, L + \varepsilon)$. Hence, all but possibly a finite number of terms from $a_1, ..., a_{N-1}$ are in $(L - \varepsilon, L + \varepsilon)$.

We next show sufficiency. Let $\varepsilon > 0$. Then all but a finite number of terms of $\{a_n\}_{n=1}^{\infty}$ are in $(L - \varepsilon, L + \varepsilon)$. Suppose that the terms which are not in $(L - \varepsilon, L + \varepsilon)$ are $a_{n_1}, ..., a_{n_k}$. Set $N = \max\{n_1, ..., n_k\} + 1$. Now if $n \geq N$, then n is greater than $n_1, ..., n_k$. Hence, $a_n \in (L - \varepsilon, L + \varepsilon)$, that is, $|a_n - L| < \varepsilon$. Therefore, $a_n \to L$. □

Definition. We say that a sequence $\{a_n\}_{n=1}^{\infty}$ is bounded (above, below, or both) if the set of the values of the terms of $\{a_n\}$ is bounded (above, below, or both). In particular,

- $\{a_n\}_{n=1}^{\infty}$ is bounded above if there is $M \in \mathbb{R}$ such that $a_n \leq M$ for all $n \in \mathbb{N}$;
- $\{a_n\}_{n=1}^{\infty}$ is bounded below if there is $M \in \mathbb{R}$ such that $a_n \geq M$ for all $n \in \mathbb{N}$;
- $\{a_n\}_{n=1}^{\infty}$ is bounded if it is bounded above and bounded below, that is, there are $M_1, M_2 \in \mathbb{R}$ such that $M_1 \leq a_n \leq M_2$ for all $n \in \mathbb{N}$.

Remark. If $\{a_n\}_{n=1}^{\infty}$ is bounded above, then there is always $K > 0$ such that $a_n \leq K$ for all $n \in \mathbb{N}$. Indeed, there is $M \in \mathbb{R}$ such that $a_n \leq M$ for all $n \in \mathbb{N}$. Set $K = |M| + 1$. Then $K > 0$ and $a_n \leq M \leq |M| + 1 = K$ for all $n \in \mathbb{N}$. Similarly, if $\{a_n\}_{n=1}^{\infty}$ is bounded below, then there is always $K > 0$ such that $a_n \geq -K$ for all $n \in \mathbb{N}$.

Moreover, if $\{a_n\}_{n=1}^{\infty}$ is bounded, then there is always $K > 0$ such that $-K \leq a_n \leq K$ (that is, $|a| \leq K$) for all $n \in \mathbb{N}$. Indeed, there are $M_1, M_2 \in \mathbb{R}$ such that $M_1 \leq a_n \leq M_2$ for all $n \in \mathbb{N}$. Set $K = \max\{|M_1| + 1, |M_2| + 1\}$. Then $K > 0$, and on one hand, $a_n \leq M_2 \leq |M_2| + 1 \leq K$ for all $n \in \mathbb{N}$, on the other hand, $a_n \geq M_2 \geq -|M_2| - 1 \geq -K$.

Example.

- $\{n\}_{n=1}^{\infty}$ is bounded below and is unbounded above.

- $\{c\}_{n=1}^{\infty}$ is bounded.
- $\{\frac{1}{n}\}_{n=1}^{\infty}$ is bounded.
- $\{(-1)^n\}_{n=1}^{\infty}$ is bounded.

COROLLARY 2.3. *A convergent sequence is bounded in* $\mathbb{R}$.

PROOF. Suppose that $a_n \to L$. Then there is $N = N(1) \in \mathbb{N}$ such that a_n are in $(L-1, L+1)$ for all $n \geq N$. Set

$$M_1 = \min\{-|a_1|, ..., -|a_N|, L-1\} \quad \text{and} \quad M_2 = \max\{|a_1|, ..., |a_N|, L+1\}.$$

Then $M_1 \leq a_n \leq M_2$ for all $n \in \mathbb{N}$. Therefore, $\{a_n\}$ is bounded. □

Remark. A bounded sequence is not necessarily convergent. For example, $\{(-1)^n\}_{n=1}^{\infty}$ is bounded but diverges.

Definition (Monotone sequences).

- We say that a sequence $\{a_n\}_{n=1}^{\infty}$ is increasing if $a_{n+1} \geq a_n$ for all $n \in \mathbb{N}$.
- We say that a sequence $\{a_n\}_{n=1}^{\infty}$ is decreasing if $a_{n+1} \leq a_n$ for all $n \in \mathbb{N}$.
- We say that a sequence is monotone if it is either increasing or decreasing.

We say that a sequence is strictly increasing, decreasing, or monotone, if the above inequalities are strict.

Remark. By the definition, a constant sequence is both increasing and decreasing.

THEOREM 2.4. *A decreasing sequence which is bounded below converges to the infimum of the set of the values of its terms.*

PROOF. Let $\{a_n\}_{n=1}^{\infty}$ be decreasing and bounded below. Then the set E of the values of its terms is bounded below.

Write $L = \inf E$. For each $\varepsilon > 0$, there is $x \in E$ such that $x < L + \varepsilon$. This means that there is $N \in \mathbb{N}$ such that $a_N = x < L + \varepsilon$.

Now if $n \geq N$, then $a_n \leq a_N < L + \varepsilon$ since the sequence is decreasing. On the other hand, L is a lower bound of the values of a_n for all $n \in \mathbb{N}$. It then follows that $a_n \geq L > L - \varepsilon$. Hence,

$$L - \varepsilon < a_n < L + \varepsilon \quad \text{for all } n \geq N,$$

that is, $|a_n - L| < \varepsilon$. Therefore, $a_n \to L$. □

Example. The sequence $\{\frac{1}{n}\}_{n=1}^{\infty}$ is decreasing and bounded below. Therefore, it converges and converges to 0, the infimum of the set $\{1, \frac{1}{2}, \frac{1}{3}, ...\}$.

THEOREM 2.5. *Let $a_n \to a$ and $b_n \to b$.*

(i). $a_n + b_n \to a + b$.
(ii). Suppose that $c \in \mathbb{R}$. Then $ca_n \to ca$.
(iii). $a_nb_n \to ab$.
(iv). Suppose that $b \neq 0$ and $b_n \neq 0$ for all $n \in \mathbb{N}$. Then $\frac{a_n}{b_n} \to \frac{a}{b}$.

PROOF.

(i). Let $\varepsilon > 0$. Since $a_n \to a$, there is $N_1 = N_1(\frac{\varepsilon}{2}) \in \mathbb{N}$ such that

$$|a_n - a| < \frac{\varepsilon}{2} \quad \text{for all } n \geq N_1.$$

Since $b_n \to b$, there is $N_2 = N_2(\frac{\varepsilon}{2}) \in \mathbb{N}$ such that

$$|b_n - b| < \frac{\varepsilon}{2} \quad \text{for all } n \geq N_2.$$

Set $N = \max\{N_1, N_2\}$. Now if $n \geq N$, then by the triangle inequality,

$$|(a_n + b_n) - (a + b)| = |(a_n - a) + (b_n - b)| \leq |a_n - a| + |b_n - b| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $a_n + b_n \to a + b$.

(ii). If $c = 0$, then $ca_n = 0$ for all $n \in \mathbb{N}$ and $ca = 0$. Therefore, $ca_n \to ca$ in this case.

Suppose that $c \neq 0$. Then $|c| > 0$. Let $\varepsilon > 0$. Since $a_n \to a$, there is $N = N(\frac{\varepsilon}{|c|}) \in \mathbb{N}$ such that

$$|a_n - a| < \frac{\varepsilon}{|c|} \quad \text{for all } n \geq N.$$

Now if $n \geq N$, then

$$|ca_n - ca| = |c|\,|a_n - a| < |c| \cdot \frac{\varepsilon}{|c|} = \varepsilon.$$

Therefore, $ca_n \to ca$.

(iii). See Question 2.8.

(iv). Since $b \neq 0$, $|b| > 0$. Since $b_n \to b$, there is $N_1 = N_1(\frac{|b|}{2}) \in \mathbb{N}$ such that

$$|b_n - b| < \frac{|b|}{2} \quad \text{for all } n \geq N_1.$$

By the triangle inequality,

$$|b_n| = |b_n + b - b| \geq |b| - |b_n - b| > |b| - \frac{|b|}{2} = \frac{|b|}{2}.$$

Since $a_n \to a$, there is $N_2 = N_2(\frac{|b|\varepsilon}{4}) \in \mathbb{N}$ such that

$$|a_n - a| < \frac{|b|\varepsilon}{4} \quad \text{for all } n \geq N_2.$$

Since $b_n \to b$, there is $N_3 = N_3(\frac{|b|^2\varepsilon}{4(|a|+1)}) \in \mathbb{N}$ such that

$$|b_n - b| < \frac{|b|^2\varepsilon}{4(|a|+1)} \quad \text{for all } n \geq N_3.$$

Set $N = \max\{N_1, N_2, N_3\}$. Now if $n \geq N$, then by the triangle inequality,

$$\begin{aligned}
\left|\frac{a_n}{b_n} - \frac{a}{b}\right| &= \frac{|a_n b - b_n a|}{|b_n b|} \\
&= \frac{|a_n b - ab + ab - b_n a|}{|b_n b|} \\
&\leq \frac{|a_n b - ab| + |ab - b_n a|}{|b_n b|} \\
&= \frac{|b|\,|a_n - a| + |a|\,|b_n - b|}{|b|\,|b_n|} \\
&< \frac{|b| \cdot \frac{|b|\varepsilon}{4} + |a| \cdot \frac{|b|^2\varepsilon}{4(|a|+1)}}{|b| \cdot \frac{|b|}{2}} \\
&< \frac{\frac{|b|^2\varepsilon}{4} + \frac{|b|^2\varepsilon}{4}}{\frac{|b|^2}{2}} \\
&= \varepsilon.
\end{aligned}$$

Therefore, $\frac{a_n}{b_n} \to \frac{a}{b}$.

$\square$

Remark. More generally, one can interchange taking limits and evaluating according to continuous functions. For example, if $a_n \to L$, then $a_n^p \to L^p$ for all $p > 0$ and $e^{a_n} \to e^L$.

$\boxed{\textbf{Homework Assignment}}$.

Question 2.1. Show that the following sequences are convergent.

(i). Let $p > 0$.

$$\left\{\frac{1}{n^p}\right\}_{n=1}^{\infty}.$$

(ii).

$$\left\{\frac{2n+3}{3n+4}\right\}_{n=1}^{\infty}.$$

Question 2.2. Show that $\{\sin(\frac{n\pi}{2})\}_{n=1}^{\infty}$ diverges.

Question 2.3.

(i). Let $a_n \geq 0$ for all $n \in \mathbb{N}$ and $a_n \to a$. Show that $a \geq 0$. (Hint: Prove by contradiction. Assume that $a < 0$. Then set ε depending on a, e.g., $\varepsilon = -\frac{a}{2} > 0$, in the ε-N argument.)

(ii). Let $a_n > 0$ for all $n \in \mathbb{N}$ and $a_n \to a$. Is $a > 0$ true? Prove your assertion.

Question 2.4 (The squeezing theorem). Let $a_n \leq b_n \leq c_n$ for all $n \in \mathbb{N}$. Suppose that $a_n \to L$ and $c_n \to L$. Show that $b_n \to L$.

Question 2.5.

(i). Let $a_n \to a$. Show that $|a_n| \to |a|$. (Hint: Use the triangle inequality to find an upper bound of $||a_n| - |a||$.)

(ii). Let $|a_n| \to 0$. Show that $a_n \to 0$.

(iii). Let $|a_n| \to a$. Is $\{a_n\}$ convergent? Prove your assertion.

Question 2.6. Let $\{a_n\}_{n=1}^{\infty}$ be increasing and bounded above. Show that $\{a_n\}_{n=1}^{\infty}$ converges to the supremum of the set of the values of its terms.

Question 2.7. Show that the following sequences are convergent.

(i).

$$a_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots (2n)}.$$

(ii). $a_1 = 0.1$, $a_2 = 0.12$,...,$a_{10} = 0,12345678910$, $a_{11} = 0.1234567891011$,...

Question 2.8. Let $a_n \to a$ and $b_n \to b$. Show that $a_n b_n \to ab$.

Question 2.9. Let $a_n \to 0$ and $\{b_n\}_{n=1}^{\infty}$ be bounded. Show that $a_n b_n \to 0$.

Question 2.10.

(i). Let $a_n \leq K$ for all $n \in \mathbb{N}$ and $a_n \to a$. Show that $a \leq K$.

(ii). Let $a_n \leq b_n$ for all $n \in \mathbb{N}$, $a_n \to a$, and $b_n \to b$. Show that $a \leq b$.

2.2. Divergence to infinity

In this section, we investigate a special class of divergent sequences. Such a sequence diverges to infinity (or negative infinity). We then introduce the concept of subsequences, which, in the case of divergence to infinity, present a rich structure within sequences.

Definition.

- We say that a sequence $\{a_n\}_{n=1}^{\infty}$ diverges to infinity, if for each $M \in \mathbb{R}$, there is $N = N(M) \in \mathbb{N}$ such that
$$a_n > M \quad \text{for all } n \geq N.$$
In this case, we write
$$\lim_{n\to\infty} a_n = \infty \quad \text{or} \quad a_n \to \infty \text{ as } n \to \infty.$$
- We say that a sequence $\{a_n\}_{n=1}^{\infty}$ diverges to negative infinity, if for each $M \in \mathbb{R}$, there is $N = N(M) \in \mathbb{N}$ such that
$$a_n < M \quad \text{for all } n \geq N.$$
In this case, we write
$$\lim_{n\to\infty} a_n = -\infty \quad \text{or} \quad a_n \to -\infty \text{ as } n \to \infty.$$

Remark. With a bit abuse of notations, we write $a_n \to \infty$ and $a_n \to -\infty$, despite the fact that, in these cases the sequences are divergent.

Example. We prove that the sequence $\{2n\}_{n=1}^{\infty}$ diverges to infinity. Let $M \in \mathbb{R}$. There is $N = N(M) \in \mathbb{N}$ such that $N > \frac{|M|}{2}$. Now if $n \geq N$, then
$$2n \geq 2N > 2 \cdot \frac{|M|}{2} = |M| \geq M.$$
Hence, the sequence diverges to infinity.

Remark. We see again the flexibility in choosing $N = N(M)$. For instance, the choice of $N = N(M) > |M|$ also works in the example above. Moreover, to prove divergence to infinity by the M-N argument, it suffices to consider only positive M.

Example. We prove that the sequence $\{\ln \frac{1}{n}\}_{n=1}^{\infty}$ diverges to negative infinity. Let $M \in \mathbb{R}$. If $M > 0$, then $\ln \frac{1}{n} \leq 0 < M$ for all $n \in \mathbb{N}$.

If $M \leq 0$, then there is $N = N(M) \in \mathbb{N}$ such that $N > e^{-M}$. Now if $n \geq N$, then
$$\ln \frac{1}{n} \leq \ln \frac{1}{N} < \ln \frac{1}{e^{-M}} = M.$$
Hence, the sequence diverges to negative infinity.

Remark. To prove divergence to negative infinity by the M-N argument, it suffices to consider only negative M (or equivalently, $-M$ with $M > 0$).

Example. It is immediate to see that if $a_n \to \infty$, then $\{a_n\}_{n=1}^{\infty}$ is unbounded above, and if $a_n \to -\infty$, then $\{a_n\}_{n=1}^{\infty}$ is unbounded below. However, the reverse statements are false. For instance, the sequence $\{0, 1, 0, 2, 0, 3, ...\}$ is unbounded above but does not diverge to infinity, while the sequence $\{-1, 0, -2, 0, -3, 0, ...\}$ is unbounded below but does not diverge to negative infinity.

On the other hand, if we select only the even-numbered terms of $\{0, 1, 0, 2, 0, 3, ...\}$, the resulting *subsequence* $\{1, 2, 3, ...\}$ diverges to infinity, while if we select only the odd-numbered terms of $\{-1, 0, -2, 0, -3, 0, ...\}$, the resulting *subsequence* $\{-1, -2, -3, ...\}$ diverges to negative infinity. These examples indicate that if a sequence is unbounded above (or below), then one can select terms to form a subsequence which diverges to infinity (or negative infinity).

Definition (Subsequences). Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. Suppose that $n_1 < n_2 < n_3 < \cdots$ is a strictly increasing sequence in $\mathbb{N}$. Then $\{a_{n_1}, a_{n_2}, a_{n_3}, ...\} = \{a_{n_k}\}_{k=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$.

Remark. The indices $n_1 < n_2 < n_3 < \cdots$ provide a rule of selecting terms among $\{a_n\}_{n=1}^{\infty}$ to form a subsequence. For instance, if the indices are given by $2 < 4 < 6 < \cdots$, then we select even-numbered terms only.

THEOREM 2.6. *Let $\{a_n\}_{n=1}^{\infty}$ be a sequence.*

(i). If $\{a_n\}_{n=1}^{\infty}$ is unbounded above, then there is a subsequence which is strictly increasing and diverges to infinity.

(ii). If $\{a_n\}_{n=1}^{\infty}$ is unbounded below, then there is a subsequence which is strictly decreasing and diverges to negative infinity.

PROOF.

(i). See Question 2.15.

(ii). We inductively select terms from $\{a_n\}_{n=1}^{\infty}$ to form a subsequence which is strictly decreasing and diverges to negative infinity.

- Step 1. We select a_{n_1}. Since $\{a_n\}_{n=1}^{\infty}$ is unbounded below, -1 is not a lower bound. Hence, there is $n_1 \in \mathbb{N}$ such that $a_{n_1} < -1$.
- Step 2. We select a_{n_2}. Since $\{a_n\}_{n=1}^{\infty}$ is unbounded below, $\min\{a_1, a_2, ..., a_{n_1}, -2\}$ is not a lower bound. Hence, there is $n_2 \in \mathbb{N}$ such that $a_{n_2} < \min\{a_1, a_2, ..., a_{n_1}, -2\}$. This means that $n_2 > n_1$, $a_{n_2} < a_{n_1}$, and $a_{n_2} < -2$.
- Inductive step. Suppose that $a_{n_1}, a_{n_2}, ..., a_{n_k}$ are selected, in which $n_1 < n_2 < \cdots < n_k$, $a_{n_j} < a_{n_{j-1}}$ and $a_{n_j} < -j$ for all $j = 2, ..., k$. Now we select $a_{n_{k+1}}$.
  Since $\{a_n\}_{n=1}^{\infty}$ is unbounded below, $\min\{a_1, a_2, ..., a_{n_k}, -(k+1)\}$ is not a lower bound. Hence, there is $n_{k+1} \in \mathbb{N}$ such that $a_{n_{k+1}} < \min\{a_1, a_2, ..., a_{n_k}, -(k+1)\}$. This means that $n_{k+1} > n_k$, $a_{n_{k+1}} < a_{n_k}$, and $a_{n_{k+1}} < -(k+1)$.

Therefore, we have selected a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$, in which $a_{n_{k+1}} < a_{n_k}$ and $a_{n_k} < -k$ for all $k \in \mathbb{N}$. Hence, this subsequence is strictly decreasing and diverges to negative infinity. Indeed, let $M > 0$. Then there is $K \in \mathbb{N}$ such that $K > M$ (that is, $-K < -M$). Now if $k \geq K$, then $a_{n_k} < -k \leq -K < -M$.

$\square$

Homework Assignment.

Question 2.11.

(i). Let $a_n \to \infty$. Suppose that $a_n \neq 0$ for all $n \in \mathbb{N}$. Show that $\frac{1}{a_n} \to 0$.

(ii). Let $b_n \to 0$. Suppose that $b_n \neq 0$ for all $n \in \mathbb{N}$. Is $\frac{1}{b_n} \to \infty$? Prove your assertion.

Question 2.12. Let $a_n \leq b_n$ for all $n \in \mathbb{N}$.

(i). Suppose that $a_n \to \infty$. Show that $b_n \to \infty$.

(ii). Suppose that $b_n \to -\infty$. Show that $a_n \to -\infty$.

Question 2.13. Let $a_n \to \infty$ and $b_n \to \infty$.

(i). Is $a_n + b_n \to \infty$ true? Prove your assertion.

(ii). Is $a_n - b_n \to \infty$ true? Prove your assertion.

(iii). Is $a_n b_n \to \infty$ true? Prove your assertion.
(iv). Suppose that $b_n \neq 0$ for all $n \in \mathbb{N}$. Is $\frac{a_n}{b_n} \to \infty$ true? Prove your assertion.

Question 2.14.

(i). Construct a sequence which contains a subsequence that diverges to negative infinity and a subsequence that diverges to infinity.
(ii). Construct a sequence which contains a subsequence that converges to -1, a subsequence that converges to 1, a subsequence that diverges to negative infinity, and a subsequence that diverges to infinity.

Question 2.15. Let $\{a_n\}_{n=1}^{\infty}$ be unbounded above. Show that there is a subsequence which is strictly increasing and diverges to infinity.

2.3. Subseqences

In the previous section, we saw that a sequence which is unbounded above (or below) contains a subsequence which diverges to infinity (or negative infinity). In this section, we study the subsequences in the case of bounded sequences. To this end, we begin by additional remarks concerning subsequences.

Remark. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. One can think of a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ as a result of an inductive selecting process of choosing terms from $\{a_n\}_{n=1}^{\infty}$.

- Step 1. In the first selection, we select the n_1-th term. Notice that $n_1 \geq 1$.
- Step 2. In the second selection, we select the n_2-th term beyond the n_1-th term, that is, $n_2 > n_1$. This implies that $n_2 \geq 2$.
- Inductive step. Suppose that $a_{n_1}, a_{n_2}, ..., a_{n_k}$ are chosen such that $n_1 < n_2 < \cdots < n_k$ and $n_1 \geq 1$, $n_2 \geq 2$,...,$n_k \geq k$. In the $(k+1)$-th selection, we select n_{k+1}-th term beyond the n_k-th term, that is, $n_{k+1} > n_k$. This implies that $n_{k+1} \geq k+1$.

Therefore, the index k in $\{a_{n_k}\}_{k=1}^{\infty}$ indicates that a_{n_k} is the k-th term in this subsequence, which is in turn the n_k-th term from the original sequence $\{a_n\}_{n=1}^{\infty}$.

For instance, if we set $n_k = 2k$, then we select the even-numbered terms from $\{a_n\}_{n=1}^{\infty}$ to form a subsequence. Whereas if we set $n_k = k$, then we select the every term from $\{a_n\}_{n=1}^{\infty}$ to form a subsequence, which coincides with the original sequence. That is, a sequence is considered as a subsequence of itself.

Theorem 2.7. *A sequence converges iff every subsequence converges.*

Proof. We first show sufficiency. Suppose that every subsequence converges. Then the sequence converges because it is a subsequence of itself.

We next show necessity. Suppose that $a_n \to L$. Let $\varepsilon > 0$. Then there is $N \in \mathbb{N}$ such that $|a_n - L| < \varepsilon$ for all $n \geq N$. Consider a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$. Now if $k \geq N$, then $n_k \geq k \geq N$. Hence, $|a_{n_k} - L| < \varepsilon$. Therefore, $a_{n_k} \to L$. □

Definition (Subsequential limits). We say that $L \in \mathbb{R}$ is a subsequential limit of a sequence, if there is a subsequence which converges to L.

Remark. It is obvious that a sequence converges iff there is only one subsequential limit, that is, each subsequence converges to the same limit. For instance, $\{\frac{(-1)^n}{n}\}_{n=1}^{\infty}$ converges to 0. So the subsequence of even-numbered terms converges to 0 and so is the subsequence of odd-numbered terms. In fact, regardless of how one selects terms to form a subsequence, it always converges to 0.

Example.

- The subsequential limit of $\{\frac{1}{n}\}_{n=1}^{\infty}$ is 0.
- The subsequential limits of $\{(-1)^n\}_{n=1}^{\infty}$ are $-1, 1$.
- The subsequential limits of $\{\sin(\frac{n\pi}{2})\}_{n=1}^{\infty}$ are $-1, 0, 1$.
- The subsequential limits of $\{1, 1, 2, 1, 2, 3, 1, 2, 3, 4, ...\}$ are $1, 2, 3, 4, ...$
- The subsequential limits of $\{1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$ are $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...$

THEOREM 2.8. *$L \in \mathbb{R}$ is a subsequential limit of $\{a_n\}_{n=1}^{\infty}$ iff for each $\varepsilon > 0$, there are infinitely many terms in $(L-\varepsilon, L+\varepsilon)$.*

Remark. We know that $\{a_n\}_{n=1}^{\infty}$ converges to L iff for each $\varepsilon > 0$, all but finitely many terms are in $(L-\varepsilon, L+\varepsilon)$. This, in particular, means that there are infinitely many terms in $(L-\varepsilon, L+\varepsilon)$, and moreover, the terms which are not in $(L-\varepsilon, L+\varepsilon)$ can only be finitely many.

In comparison, L is a subsequential limit of $\{a_n\}_{n=1}^{\infty}$ iff for each $\varepsilon > 0$, there are infinitely many terms in $(L-\varepsilon, L+\varepsilon)$. However, it is possible that there are also infinitely many terms which are not in $(L-\varepsilon, L+\varepsilon)$. For instance, 1 is a subsequential limit of $\{(-1)^n\}_{n=1}^{\infty}$. For each $0 < \varepsilon < 1$, there are infinitely many terms (that is, the even-numbered terms) in $(1-\varepsilon, 1+\varepsilon)$. However, there are also infinitely many terms (that is, the odd-numbered terms) which are not in $(1-\varepsilon, 1+\varepsilon)$.

PROOF. We first show necessity. Suppose that $L \in \mathbb{R}$ is a subsequential limit of $\{a_n\}_{n=1}^{\infty}$. Then there is a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$ which converges to L. Hence, for each ε, there is $K = K(\varepsilon) \in \mathbb{N}$ such that $|a_{n_k} - L| < \varepsilon$, that is, $a_{n_k} \in (L-\varepsilon, L+\varepsilon)$, for all $k \geq K$. This means that there are infinitely many terms of $\{a_{n_k}\}_{k=1}^{\infty}$, which are also terms of $\{a_n\}_{n=1}^{\infty}$, that are in $(L-\varepsilon, L+\varepsilon)$.

We next show sufficiency. Suppose that for each $\varepsilon > 0$, there are infinite many terms in $(L-\varepsilon, L+\varepsilon)$. We inductively select terms from $\{a_n\}_{n=1}^{\infty}$ to form a subsequence which converges to L.

- Step 1. Since there are infinitely many terms of $\{a_n\}_{n=1}^{\infty}$ in $(L-1, L+1)$, we select such a term, denoted as the n_1-th term, such that $a_{n_1} \in (L-1, L+1)$.
- Step 2. Since there are infinitely many terms of $\{a_n\}_{n=1}^{\infty}$ in $(L-\frac{1}{2}, L+\frac{1}{2})$, there are infinitely many terms beyond the n_1-th term which are in $(L-\frac{1}{2}, L+\frac{1}{2})$. (Otherwise, there would be at most n_1 terms in $(L-\frac{1}{2}, L+\frac{1}{2})$.) We select such a term, denoted as the n_2-th term, such that $n_2 > n_1$ and $a_{n_2} \in (L-\frac{1}{2}, L+\frac{1}{2})$.
- Inductive step. Suppose that $a_{n_1}, a_{n_2}, ..., a_{n_k}$ are chosen such that $n_1 < n_2 < \cdots < n_k$ and $a_{n_j} \in (L-\frac{1}{j}, L+\frac{1}{j})$ for all $j = 1, ..., k$. Now we select $a_{n_{k+1}}$.

  Since there are infinitely many terms of $\{a_n\}_{n=1}^{\infty}$ in $(L-\frac{1}{k+1}, L+\frac{1}{k+1})$, there are infinitely many terms beyond the n_k-th term which are in $(L-\frac{1}{k+1}, L+\frac{1}{k+1})$. (Otherwise, there would be at most n_k terms in $(L-\frac{1}{k+1}, L+\frac{1}{k+1})$.) We select such a term, denoted as the n_{k+1}-th term, such that $n_{k+1} > n_k$ and $a_{n_{k+1}} \in (L-\frac{1}{k+1}, L+\frac{1}{k+1})$.

Therefore, we have selected a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$, in which $a_{n_k} \in (L-\frac{1}{k}, L+\frac{1}{k})$ for all $k \in \mathbb{N}$. Hence, this subsequence converges to L. Indeed, for each $\varepsilon > 0$, there is $K = K(\varepsilon) \in \mathbb{N}$ such that $K > \frac{1}{\varepsilon}$. Now if $k \geq K$, then

$$|a_{n_k} - L| \leq \frac{1}{k} \leq \frac{1}{K} < \varepsilon.$$

□

The theorem above can be useful to derive the limits of some sequences.

Example. Let $0 < r < 1$. We prove that $r^n \to 0$ and first show that it converges. Indeed, the sequence is bounded below (by 0). Moreover, it is decreasing: $r^{n+1} = r \cdot r^n < r^n$ for all $n \in \mathbb{N}$. Hence, the sequence converges. We denote the limit by L.

Consider the subsequence $\{r^{n+1}\}_{n=1}^{\infty}$, that is, the one which contains all the terms of $\{r^n\}_{n=1}^{\infty}$ but the first. This subsequence converges to the same limit L. But

$$L = \lim_{n\to\infty} r^{n+1} = r \cdot \lim_{n\to\infty} r^n = rL,$$

which implies that $L = 0$.

Example. Let $0 < r < 1$. We prove that $\sqrt[n]{r} \to 1$ and first show that it converges. Indeed, the sequence is bounded above (by 1). Moreover, it is increasing: $r^{\frac{1}{n+1}} > r^{\frac{1}{n}}$ for all $n \in \mathbb{N}$. Hence, the sequence converges. We denote the limit by L.

Consider the subsequence $\{\sqrt[2n]{r}\}_{n=1}^{\infty}$, that is, the one which contains all the even-numbered terms of $\{\sqrt[n]{r}\}_{n=1}^{\infty}$. This subsequence converges to the same limit L. But

$$L = \lim_{n\to\infty} \sqrt[2n]{r} = \sqrt{\lim_{n\to\infty} \sqrt[n]{r}} = \sqrt{L},$$

which implies that $L = 1$.

Definition. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence and E be the set of subsequential limits. Define

$$\limsup_{n\to\infty} a_n = \sup E \quad \text{and} \quad \liminf_{n\to\infty} a_n = \inf E.$$

Example.

- $$\limsup_{n\to\infty} \frac{1}{n} = \liminf_{n\to\infty} \frac{1}{n} = 0.$$
- $$\limsup_{n\to\infty} (-1)^n = 1 \quad \text{and} \quad \liminf_{n\to\infty} (-1)^n = -1.$$
- $$\limsup_{n\to\infty} \sin\left(\frac{n\pi}{2}\right) = 1 \quad \text{and} \quad \liminf_{n\to\infty} \sin\left(\frac{n\pi}{2}\right) = -1.$$
- Consider the sequence $\{1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$. The set of subsequential limits are $\{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$. Therefore, $\limsup = 1$ and $\liminf = 0$.

Remark. The $\limsup$ and $\liminf$ of a sequence are *not* necessarily the supremum or the infimum of the set of the values of the terms. For instance, $\{\frac{(-1)^n}{n}\}_{n=1}^{\infty}$ converges to 0 so $\limsup$ and $\liminf$ are both 0. However, the supremum and the infimum of the set of the values of the terms are $\frac{1}{2}$ and -1, respectively.

THEOREM 2.9. *Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence and E be the set of subsequential limits.*

(i). $\limsup a_n \in E$*, that is,* $\limsup a_n$ *is also a subsequential limit of* $\{a_n\}_{n=1}^{\infty}$.

(ii). $\liminf a_n \in E$*, that is,* $\liminf a_n$ *is also a subsequential limit of* $\{a_n\}_{n=1}^{\infty}$.

Remark. In general, the supremum and infimum of a set are not necessarily elements of the set. For instance, we have seen that $\sup\{\frac{n}{n+1} : n \in \mathbb{N}\} = 1$, which is not an element of the set, and that $\inf\{\frac{n+1}{n} : n \in \mathbb{N}\} = 1$, which is not an element of the set.

However, the theorem above asserts that the set of subsequential limits of a bounded sequence always has its supremum and infimum as its elements. In particular, this implies that such a set is closed, in the topology of the real numbers, see Chapter 3. For instance, consider the sequence $\{1, 1, \frac{1}{2}, 1, \frac{1}{2}, \frac{1}{3}, 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$. The set of the subsequential limits $E = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, ...\}$.

Then $\sup E = 1$ and $\inf E = 0$ are both subsequential limits. Indeed, there is a subsequence 1, which converges to 1 and a subsequence $\frac{1}{n}$, which converges to 0.

Proof.

(i). Write $L = \limsup a_n$. We inductively select terms from $\{a_n\}_{n=1}^{\infty}$ to form a subsequence which converges to L.

- Step 1. Since $L = \sup E$, there is $L_1 \in E$ such that $L_1 > L - \frac{1}{2}$. Since L_1 is a subsequential limit of $\{a_n\}_{n=1}^{\infty}$, there are infinitely many terms in $(L_1 - \frac{1}{2}, L_1 + \frac{1}{2})$. In particular, there is $n_1 \in \mathbb{N}$ such that $a_{n_1} \in (L_1 - \frac{1}{2}, L_1 + \frac{1}{2})$. Hence,
$$a_{n_1} > L_1 - \frac{1}{2} > \left(L - \frac{1}{2}\right) - \frac{1}{2} = L - 1,$$
and
$$a_{n_1} < L_1 + \frac{1}{2} \leq L + \frac{1}{2} < L + 1.$$
Here, we used the fact that $L_1 \leq L$ because L is an upper bound of E.
- Step 2. Since $L = \sup E$, there is $L_2 \in E$ such that $L_2 > L - \frac{1}{4}$. Since L_2 is a subsequential limit of $\{a_n\}_{n=1}^{\infty}$, there are infinitely many terms in $(L_1 - \frac{1}{4}, L_1 + \frac{1}{4})$. In particular, there is $n_2 > n_1$ such that $a_{n_2} \in (L_2 - \frac{1}{4}, L_2 + \frac{1}{4})$. Hence,
$$a_{n_2} > L_2 - \frac{1}{4} > \left(L - \frac{1}{4}\right) - \frac{1}{4} = L - \frac{1}{2},$$
and
$$a_{n_2} < L_2 + \frac{1}{4} \leq L + \frac{1}{4} < L + \frac{1}{2}.$$
- Inductive step. Suppose that $a_{n_1}, a_{n_2}, ..., a_{n_k}$ are chosen such that $n_1 < n_2 < \cdots < n_k$ and $a_{n_j} \in (L - \frac{1}{j}, L + \frac{1}{j})$ for $j = 1, ..., k$. Now we select $a_{n_{k+1}}$.
Since $L = \sup E$, there is $L_{k+1} \in E$ such that $L_{k+1} > L - \frac{1}{2(k+1)}$. Since L_{k+1} is a subsequential limit of $\{a_n\}_{n=1}^{\infty}$, there are infinitely many terms in $(L_{k+1} - \frac{1}{2(k+1)}, L_{k+1} + \frac{1}{2(k+1)})$. In particular, there is $n_{k+1} > n_k$ such that $a_{n_{k+1}} \in (L_{k+1} - \frac{1}{2(k+1)}, L_{k+1} + \frac{1}{2(k+1)})$. Hence,
$$a_{n_{k+1}} > L_{k+1} - \frac{1}{2(k+1)} > \left(L - \frac{1}{2(k+1)}\right) - \frac{1}{2(k+1)} = L - \frac{1}{k+1},$$
and
$$a_{n_{k+1}} < L_{k+1} + \frac{1}{2(k+1)} \leq L + \frac{1}{2(k+1)} < L + \frac{1}{k+1}.$$

Therefore, we have selected a subsequence $\{a_{n_k}\}_{k=1}^{\infty}$, in which $a_{n_k} \in (L - \frac{1}{k}, L + \frac{1}{k})$ for all $k \in \mathbb{N}$. Hence, this subsequence converges to L. Indeed, for each $\varepsilon > 0$, there is $K = K(\varepsilon) \in \mathbb{N}$ such that $K > \frac{1}{\varepsilon}$. Now if $k \geq K$, then
$$|a_{n_k} - L| < \frac{1}{k} \leq \frac{1}{K} < \varepsilon.$$

(ii). See Question 2.19.

□

$\boxed{\textbf{Homework Assignment}}$.

Question 2.16. Let $r > 1$. Show that $r^n \to \infty$.

Question 2.17. Let $r > 1$. Show that $\sqrt[n]{r} \to 1$.

Question 2.18. Let $a_1 = \sqrt{2}$ and $a_{n+1} = \sqrt{a_n + 2}$ for all $n \geq 1$. Show that $a_n \to 2$. (Hint: First use mathematical induction to show that $\{a_n\}_{n=1}^{\infty}$ is increasing and bounded above by 2.)

Question 2.19. Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence and E be the set of subsequential limits. Show that $\liminf a_n \in E$, that is, $\liminf a_n$ is also a subsequential limit of $\{a_n\}_{n=1}^{\infty}$.

Question 2.20. Let $a_n \to L$. Suppose that $a_n \leq L$ for infinitely many values of n. Show that there is a subsequence which is increasing and converges to L.

Question 2.21. Let $a_n \to L$. Show that either there is an increasing subsequence which converges to L or there is a decreasing subsequence which converges to L.

CHAPTER 3

Topology of the real number system

In this chapter, we discuss the *topology* of the real number system $\mathbb{R}$, which is the study of the open sets and continuous functions. We present these fundamental concepts, the closely related ones of closed sets and compact sets, as well as the Heine-Borel theorem, which asserts that a set in $\mathbb{R}$ is compact iff it is closed and bounded. Moreover, we prove the Bolzano-Weierstrass theorem, which allows us to revisit the concept of subsequences in the previous chapter. From where, we also prove that a sequence converges iff it is Cauchy.

3.1. Open sets and closed sets

Definition (Open sets and closed sets). We say that a set $E \subset \mathbb{R}$ is open if for each $x \in E$, there is $\delta = \delta(x) > 0$ such that $(x-\delta, x+\delta) \subset E$.

We say that a set $F \subset \mathbb{R}$ is closed if $F^c = \mathbb{R} \setminus F$ is open.

Example. By the definition, $\mathbb{R}$ is open. Hence, $\emptyset$ is closed since $\emptyset^c = \mathbb{R}$. On the other hand, $\emptyset$ is also open by the definition (because the condition trivially holds). Hence, $\mathbb{R}$ is closed since $\mathbb{R}^c = \emptyset$. Therefore, $\mathbb{R}$ and $\emptyset$ are both open and closed.

Theorem 3.1. *The intervals (a,b), (a,∞), and $(-\infty,a)$ are open.*

Proof.

(i). Let $x \in (a,b)$. Then $a < x < b$. Set $\delta = \min\{x-a, b-x\}$. Then $\delta > 0$, and $(x-\delta, x+\delta) \subset (a,b)$. Therefore, (a,b) is open.

(ii). Let $x \in (a,\infty)$. Then $x > a$. Set $\delta = x-a$. Then $\delta > 0$, $(x-\delta, x+\delta) \subset (a,\infty)$. Therefore, (a,∞) is open.

(iii). See Question 3.1.

□

Remark. Since $(-\infty,a]^c = (a,\infty)$, which is open, $(-\infty,a]$ is closed. Similarly, $[a,\infty)$ is closed. To prove that $[a,b]$ is closed, we need to show that $[a,b]^c = (-\infty,a) \cup (b,\infty)$ is open. This is a consequence of the following theorem.

Theorem 3.2.

(i). Let E_j be open for all $j \in J$. Then $\cup_{j\in J} E_j$ is open.

(ii). Let $E_1, ..., E_n$ be open. Then $\cap_{j=1}^n E_j$ is open.

Proof.

(i). Let $x \in \cup_{j\in J} E_j$. Then $x \in E_j$ for some $j \in J$. Since E_j is open, there is $\delta > 0$ such that $(x-\delta, x+\delta) \subset E_j$. Hence, $(x-\delta, x+\delta) \subset E_j \subset \cup_{j\in J} E_j$. Therefore, $\cup_{j\in J} E_j$ is open.

(ii). Let $x \in \cap_{j=1}^n E_j$. Then $x \in E_j$ for all $j = 1, ..., n$. Since E_j is open, there is $\delta_j > 0$ such that $(x-\delta_j, x+\delta_j) \subset E_j$. Set

$$\delta = \min\{\delta_1, \delta_2, ..., \delta_n\}.$$

Then $(x-\delta, x+\delta) \subset (x-\delta_j, x+\delta_j) \subset E_j$ for all $j = 1, ..., n$. Hence, $(x-\delta, x+\delta) \subset \cap_{j=1}^n E_j$. Therefore, $\cap_{j=1}^n E_j$ is open.

□

Remark (Finite sets are closed). Let $E = \{a\}$. Then $E^c = (-\infty, a) \cup (a, \infty)$ is open. Therefore, each singleton set is closed. More generally, let $E = \{a_1, ..., a_n\}$, in which $a_1 < a_2 < \cdots < a_{n-1} < a_n$. Then $E^c = (-\infty, a_1) \cup (a_1, a_2) \cup \cdots \cup (a_{n-1}, a_n) \cup (a_n, \infty)$ is open. Therefore, each finite set is closed.

By De Morgan's Law in Theorem 1.1, we have that

THEOREM 3.3.
(i). Let E_j be closed for all $j \in J$. Then $\cap_{j\in J} E_j$ is closed.
(ii). Let $E_1, ..., E_n$ be closed. Then $\cup_{j=1}^n E_j$ is closed.

PROOF.
(i). See Question 3.2
(ii). By De Morgan's Law,

$$\left(\bigcup_{j=1}^n E_j\right)^c = \bigcap_{j=1}^n E_j^c.$$

Since E_j is closed, E_j^c is open. By the theorem above, $\cap_{j=1}^n E_j^c$ is open. Therefore, $\cup_{j=1}^n E_j$ is closed.

□

Remark. The previous two theorems state that finite union and intersection of open sets is open, and finite union and intersection of closed sets is closed. However, these assertions are false for infinite union or intersection.

For open sets, infinite union stays open but infinite intersection is not necessarily open. For instance, let $E_j = (-\frac{1}{j}, \frac{1}{j})$. Then $\cap_{j=1}^\infty E_j = \{0\}$ is closed. (Indeed, if $x \in E_j$ for all $j \in \mathbb{N}$, then $-\frac{1}{j} < x < \frac{1}{j}$. This implies that $x = 0$.)

For closed sets, infinite intersection stays closed but infinite union is not necessarily closed. See Question 3.2.

Example. The interval $(a, b]$ is neither open nor closed. Indeed, it is not open, because for $b \in (a, b]$, $(b - \delta, b + \delta) \not\subset (a, b]$ for $\delta = 1$. (In fact, this is true for each $\delta > 0$.) On the other hand, $(a, b]^c = (-\infty, a] \cup (b, \infty)$ is not open, because for $a \in (-\infty, a] \cup (b, \infty)$, $(a - \delta, a + \delta) \not\subset (-\infty, a] \cup (b, \infty)$ for $\delta = 1$. (In fact, this is true for each $\delta > 0$.) See Question 3.3 for a similar statement.

Definition (Limit points). We say that p is a limit point (or a cluster point, an accumulation point) of a set $E \subset \mathbb{R}$ if for each $\delta > 0$, $(p - \delta, p + \delta)$ contains a point $q \in E$ such that $q \neq p$. We denote by E' the set of limit points of E.

Remark. A point p is a limit point of a set $E \subset \mathbb{R}$ iff for each $\delta > 0$, $(p - \delta, p + \delta) \setminus \{p\} = (p - \delta, p) \cup (p, p + \delta)$ contains a point $q \in E$.

PROPOSITION 3.4. *Let E be a finite set. Then $E' = \emptyset$, that is, E has no limit points.*

PROOF. Let $E = \{x_1, ..., x_n\}$ for some $n \in \mathbb{N}$.
First consider x_j for $j = 1, ..., n$. Set

$$\delta = \min\{|x_j - x_k| : k = 1, ..., n \text{ and } k \neq j\}.$$

Then $\delta > 0$, and $(x_j - \delta, x_j + \delta) \cap E = \{x_j\}$, that is, $(x_j - \delta, x_j + \delta)$ does not contain any point of E which is distinct from x_j. Hence, x_j is not a limit point of E.

Then consider $p \in \mathbb{R}$ such that $p \notin E$, that is, $p \neq x_j$ for all $j = 1, ..., n$. Set

$$\delta = \min \{|p - x_j| : j = 1, ..., n\}.$$

Then $\delta > 0$, and $(p - \delta, p + \delta) \cap E = \emptyset$, that is, $(p - \delta, p + \delta)$ does not contain any point of E. Hence, p is not a limit point of E. □

Example.

- Let $E = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then $E' = \{0\}$, that is, 0 is the only limit point of E.
- Notice that $\mathbb{N}' = \emptyset$, that is, $\mathbb{N}$ has no limit points.

THEOREM 3.5. *A point p is a limit point of E iff for each $\delta > 0$, $(p - \delta, p + \delta)$ contains infinitely many points of E.*

PROOF. We first show sufficiency. Suppose that for each $\delta > 0$, $(p - \delta, p + \delta)$ contains infinitely many points of E. Then $(p - \delta, p + \delta)$ contains a point of E which is distinct from p. (If not, then $(p - \delta, p + \delta)$ contains at most one point, that is, p, of E, which is impossible.)

We then show necessity. Suppose that p is a limit point of E. Let $\delta > 0$. Suppose that there are finitely many points, $x_1, ..., x_n$ for some $n \in \mathbb{N}$, in $(p - \delta, p + \delta)$. Set

$$\delta_1 = \min_{j=1,...,n} \{|x_j - p| : x_j \neq p\}.$$

Then $\delta_1 > 0$, and $(p - \delta_1, p + \delta_1)$ contains no points of E which are distinct with p. This contradicts with the assumption that p is a limit point of E. Therefore, $(p - \delta, p + \delta)$ contains infinitely many points of E. □

Example.

- Let $\mathbb{Q}$ be the set of rational numbers. Then $\mathbb{Q}' = \mathbb{R}$, that is, every real number is a limit point of $\mathbb{Q}$. Indeed, let $p \in \mathbb{R}$ and $\delta > 0$. Then $(p - \delta, p + \delta)$ contains infinitely many points of $\mathbb{Q}$ by Corollary 1.17, which states that every interval contains infinitely many rational numbers.
- Let $E = \mathbb{R} \setminus \mathbb{Q}$, that is, the set of irrational numbers. Then $E' = \mathbb{R}$, that is, every real number is a limit point of E. Indeed, let $p \in \mathbb{R}$ and $\delta > 0$. Then $(p - \delta, p + \delta)$ contains infinitely many points of E by Question 1.17, which states that every interval contains infinitely many irrational numbers.

PROPOSITION 3.6. $[a, b]' = [a, b]$.

PROOF. We first show that if $p \in [a, b]$, then p is a limit point of $[a, b]$, that is, $[a, b] \subset [a, b]'$.

If $p = a$, then for each $\delta > 0$, set $\delta_1 = \min\{\delta, b - a\}$. Then $\delta_1 > 0$, and $(a - \delta, a + \delta) \cap [a, b] = [a, a + \delta_1)$ contains infinitely many points. Hence, a is a limit point of $[a, b]$.

If $p = b$, then for each $\delta > 0$, set $\delta_1 = \min\{\delta, b - a\}$. Then $\delta_1 > 0$, and $(b - \delta, b + \delta) \cap [a, b] = (b - \delta_1, b]$ contains infinitely many points. Hence, b is a limit point of $[a, b]$.

If $a < p < b$, then for each $\delta > 0$, set $\delta_1 = \min\{\delta, b - p, p - a\}$. Then $\delta_1 > 0$, and $(p - \delta, p + \delta) \cap [a, b] = (p - \delta_1, p + \delta_1)$ contains infinitely many points. Hence, p is a limit point of $[a, b]$.

We then show that if $p \notin [a, b]$, then p is not a limit point of $[a, b]$, that is, $[a, b]' \subset [a, b]$. Set $\delta = \min\{|p - a|, |p - b|\}$. Then $\delta > 0$, and $(p - \delta, p + \delta) \cap [a, b] = \emptyset$. Hence, p is not a limit point of $[a, b]$. □

Definition. Let $E \subset \mathbb{R}$. We define the closure of E, denoted as $\overline{E}$, as $\overline{E} = E \cup E'$.

Example.

- Let $E = \{\frac{1}{n} : n \in \mathbb{N}\}$. Then $\overline{E} = E \cup \{0\}$.

- $\overline{\mathbb{N}} = \mathbb{N}$.
- $\overline{\mathbb{Q}} = \mathbb{R}$.
- $\overline{\mathbb{R} \setminus \mathbb{Q}} = \mathbb{R}$.
- $\overline{[a, b]} = [a, b]$.
- $\overline{\emptyset} = \emptyset$ and $\overline{\mathbb{R}} = \mathbb{R}$.

THEOREM 3.7. *Let $E \subset \mathbb{R}$. Then $\overline{E}$ is closed.*

Remark (Closure is always closed). This theorem asserts that the closure of a set, regardless of closed, open, both closed and open, or neither, is always closed.

PROOF. Let $E \subset \mathbb{R}$. We show that $\overline{E}^c = (E \cup E')^c$ is open. To this end, let $p \in (E \cup E')^c$. Then $p \notin E$ and p is not a limit point of E. Hence, there is $\delta > 0$ such that $(p - \delta, p + \delta)$ contains no points of E which are distinct from p. Together with the condition that $p \notin E$, $(p - \delta, p + \delta) \cap E = \emptyset$, which implies that $(p - \delta, p + \delta) \subset E^c$.

Now set $\delta_1 = \frac{\delta}{3}$. Then $(p - \delta_1, p + \delta_1) \subset E^c$. Moreover, let $q \in (p - \delta_1, p + \delta_1)$. Then $(q - \delta_1, q + \delta_1) \subset (p - \delta, p + \delta) \subset E^c$, which implies that q is not a limit point of E. This shows that $(p - \delta_1, p + \delta_1) \cap E' = \emptyset$, that is, $(p - \delta_1, p + \delta_1) \subset (E')^c$. By De Morgan's law in Theorem 1.1,

$$(p - \delta_1, p + \delta_1) \subset E^c \cap (E')^c = (E \cup E')^c = \overline{E}^c.$$

Therefore, $\overline{E}^c$ is open. □

THEOREM 3.8. *A set $E \subset \mathbb{R}$ is closed iff $\overline{E} = E$.*

Remark. Notice that $E \subset \overline{E} = E \cup E'$ is true for each $E \subset \mathbb{R}$. The condition $\overline{E} = E$ is equivalent to $E' \subset E$, that is, each limit point of E is a point of E.

PROOF. We first show sufficiency. Suppose that $\overline{E} = E$. Then E is closed, since $\overline{E}$ is always closed.

We then show necessity. Suppose that E is closed. Then E^c is open. We prove that $\overline{E} = E$, that is, each limit point of E is a point of E. To this end, let $p \in E^c$. There is $\delta > 0$ such that $(p - \delta, p + \delta) \subset E^c$. Hence, $(p - \delta, p + \delta)$ contains no points of E, which implies that p is not a limit point of E. Therefore, $E' \subset E$. □

Homework Assignment.

Question 3.1. Show that $(-\infty, a)$ is open.

Question 3.2.

(i). Let E_j be closed for all $j \in J$. Show that $\cap_{j \in J} E_j$ is closed.

(ii). Let E_j be closed for all $j \in J$. Is $\cup_{j \in J} E_j$ closed? Prove your assertion.

Question 3.3. Show that $[a, b)$ is neither open nor closed.

Question 3.4. Show that $(a, b)' = [a, b]$.

Question 3.5. Let $E \subset \mathbb{R}$ be bounded below. Is $\inf E$ a limit point of E? Prove your assertion.

Question 3.6. Let $E \subset \mathbb{R}$ be bounded above. Show that $\sup E \in \overline{E}$.

Question 3.7. Let $A \subset B$. Show that $\overline{A} \subset \overline{B}$.

3.2. The Bolzano-Weierstrass theorem

In this section, we prove the Bolzano-Weierstrass theorem, which asserts that every infinite set which is bounded has at least one limit point. This theorem is closely related to convergence and divergence, as well as the subsequential limits of sequences. We establish such a connection. To this end, we first prepare the following lemma.

LEMMA 3.9. *Let $[a_n, b_n]$ for $n \in \mathbb{N}$ be a sequence of intervals such that $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for all $n \in \mathbb{N}$. Suppose that $(b_n - a_n) \to 0$. Then*

$$\bigcap_{n=1}^{\infty} [a_n, b_n] = \{p\} \quad \textit{for some } p \in \mathbb{R}.$$

Remark. Firstly, we must have $a_n < b_n$ in each interval $[a_n, b_n]$. Moreover, the condition $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ in the lemma (which is usually referred as the "nested" condition) implies that $a_{n+1} \geq a_n$ and $b_{n+1} \leq b_n$ for all $n \in \mathbb{N}$.

See Question 3.8 for the proof.

Example. We demonstrate by examples how each condition is necessary in the lemma above.

- $$\bigcap_{n=1}^{\infty} \left[-\frac{1}{n}, \frac{1}{n}\right] = \{0\},$$
  in which 0 is the limit of the sequences $\{\frac{1}{n}\}_{n=1}^{\infty}$ and $\{-\frac{1}{n}\}_{n=1}^{\infty}$.
- The lemma is not necessarily true for intervals (a_n, b_n) with the same conditions. Indeed,
  $$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}, \quad \text{but} \quad \bigcap_{n=1}^{\infty} \left(0, \frac{1}{n}\right) = \emptyset,$$
  both of which is nested and have the lengths convergent to 0.
- The lemma is not necessarily true without the condition that $(b_n - a_n) \to 0$. Indeed,
  $$\bigcap_{n=1}^{\infty} \left[0, 1 + \frac{1}{n}\right] = [0, 1],$$
  which is not a singleton set.

THEOREM 3.10 (Bolzano-Weierstrass). *Let $E \subset \mathbb{R}$ be infinite and bounded. Then $E' \neq \emptyset$, that is, there is at least one limit point of E.*

Remark. The conditions of infinity and boundedness are necessary for the conclusion in the theorem. In particular, a finite set has no limit points, while $\mathbb{N}$, which is infinite but is not bounded, has no limit points.

PROOF. Since E is bounded, there is $M > 0$ such that $-M \leq x \leq M$ for all $x \in E$. Denote $[-M, M]$ by $[a_0, b_0]$. Then $E \subset [a_0, b_0]$, which has length $2M$.

Divide $[a_0, b_0]$ into two closed intervals of equal length. Then at least one of the two intervals contains infinitely many points of E. (If not, then both intervals contain finitely many points of E, which implies that E has finitely many points. This is impossible.) Denote such an interval by $[a_1, b_1]$, which has length $\frac{2M}{2} = M$.

Divide $[a_1, b_1]$ into two closed intervals of equal length. Then at least one of the two intervals contains infinitely many points of E. (If not, then both intervals contain finitely many points of E, which implies that $[a_1, b_1]$ contains finitely many points of E. This is impossible.) Denote such an interval by $[a_2, b_2]$, which has length $\frac{M}{2}$.

Suppose that $[a_1, b_1], [a_2, b_2], ..., [a_k, b_k]$ are chosen such that for all $j = 1, ..., k$, $[a_j, b_j]$ contains infinitely many points of E, and has length $(b_j - a_j) = \frac{M}{2^{j-1}}$, and for all $j = 1, ..., k-1$, $[a_{j+1}, b_{j+1}] \subset [a_j, b_j]$.

Divide $[a_k, b_k]$ into two closed intervals of equal length. Then at least one of the two intervals contains infinitely many points of E. (If not, then both intervals contain finitely many points of E, which implies that $[a_k, b_k]$ contains finitely many points of E. This is impossible.) Denote such an interval by $[a_{k+1}, b_{k+1}]$, which has length $\frac{M}{2^{k-1}}/2 = \frac{M}{2^k}$.

Therefore, we have selected a sequence of intervals $[a_n, b_n]$ for $n \in \mathbb{N}$ such that $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for all $n \in \mathbb{N}$ and $(b_n - a_n) = \frac{M}{2^{n-1}} \to 0$. By Lemma 3.9, there is $p \in \mathbb{R}$ such that

$$\bigcap_{n=1}^{\infty} [a_n, b_n] = \{p\}.$$

Hence, $p \in E'$, that is, p is a limit point of E. Indeed, for each $\delta > 0$, there is $n \in \mathbb{N}$ such that

$$\frac{M}{2^{n-1}} < \delta.$$

Since $p \in [a_n, b_n]$, $[a_n, b_n] \subset (p-\delta, p+\delta)$. This means that $(p-\delta, p+\delta)$ contains infinitely many points of E. Therefore, $p \in E'$. □

The concept of the limit point of a set is closely related to the one of the limit of a sequence, when the set is given by the values of the terms of the sequence. However, these two are also different: The values of the terms can repeat in a sequence but the elements of a set are distinct. In particular, let $\{a_n\}_{n=1}^{\infty}$ be a sequence and E be the set of the values of the terms. If E is finite, then at least one value must be repeated by infinitely many terms of $\{a_n\}_{n=1}^{\infty}$.

THEOREM 3.11. *Each bounded sequence has at least one subsequential limit.*

Remark. The theorem is false without the boundedness condition. Indeed, the sequence $\{n\}_{n=1}^{\infty}$, which is not bounded, has no subsequential limits.

PROOF. Let $\{a_n\}_{n=1}^{\infty}$ be a bounded sequence. Let E be the set of the values of the terms of $\{a_n\}_{n=1}^{\infty}$. We prove the theorem in two cases.

We first assume that E is finite. Then at least one value must be repeated by infinitely many terms. (If not, that is, each value is repeated by finitely many terms, then there are finitely many terms in $\{a_n\}_{n=1}^{\infty}$, which is impossible.) Select these terms which share the same value. Then the corresponding subsequence is a constant sequence and converges to that constant, which is a subsequential limit of $\{a_n\}_{n=1}^{\infty}$.

We then assume that E is infinite. Since E is bounded, there is at least one limit point p of E. We show that p is a subsequential limit of $\{a_n\}_{n=1}^{\infty}$. To this end, for each $\varepsilon > 0$, $(p-\varepsilon, p+\varepsilon)$ contains infinitely many points of E. Each point of E is taken by at least one term of $\{a_n\}_{n=1}^{\infty}$. Hence, $(p-\varepsilon, p+\varepsilon)$ contains infinitely many terms of $\{a_n\}_{n=1}^{\infty}$. Therefore, p is a subsequential limit of $\{a_n\}_{n=1}^{\infty}$. □

Definition (Cauchy sequences). We say that a sequence $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence, if for each $\varepsilon > 0$, there is $N = N(\varepsilon) \in \mathbb{N}$ such that

$$|a_n - a_m| < \varepsilon \quad \text{for all } n, m \geq N.$$

Example. Let $a_1 = 0.9$, $a_2 = 0.99$,... We prove that $\{a_n\}_{n=1}^{\infty}$ is Cauchy. To this end, let $\varepsilon > 0$. Then there is $N \in \mathbb{N}$ such that

$$N > \frac{\ln \frac{1}{\varepsilon}}{\ln 10}.$$

Now if $n, m \geq N$, assuming that $n \geq m$, then $a_n - a_m = 0.0 \cdots 09 \cdots 9$, whose k-th digit is 0 for $k = 1, ..., m$ and is 9 for $k = m+1, ..., n$. Hence,

$$|a_n - a_m| = 0.0 \cdots 09 \cdots 9 < 0.0 \cdots 1 = \frac{1}{10^m} \leq \frac{1}{10^N} < \varepsilon.$$

Therefore, $\{a_n\}_{n=1}^{\infty}$ is Cauchy.

In fact, we see that $\{a_n\}_{n=1}^{\infty}$ is increasing and bounded above (by 1). Hence, it converges. The following theorem asserts that a sequence converges iff it is Cauchy.

Theorem 3.12. *A sequence converges iff it is Cauchy.*

Proof. We first show necessity. Suppose that $a_n \to L$. Then for each $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that

$$|a_n - L| < \frac{\varepsilon}{2} \quad \text{for all } n \geq N.$$

Now if $n, m \geq N$, then by the triangle inequality,

$$|a_n - a_m| = |a_n - L + L - a_m| \leq |a_n - L| + |L - a_m| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore, $\{a_n\}_{n=1}^{\infty}$ is a Cauchy sequence.

See Question 3.9 for sufficiency. $\square$

Homework Assignment.

Question 3.8. Let $[a_n, b_n]$ for $n \in \mathbb{N}$ be a sequence of intervals such that $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for all $n \in \mathbb{N}$. Suppose that $(b_n - a_n) \to 0$. Show that

$$\bigcap_{n=1}^{\infty} [a_n, b_n] = \{p\} \quad \text{for some } p \in \mathbb{R},$$

by the following steps.

(i). Show that $\{a_n\}_{n=1}^{\infty}$ converges and the limit $a \in [a_n, b_n]$ for all $n \in \mathbb{N}$.
(ii). Show that $\{b_n\}_{n=1}^{\infty}$ converges and the limit $b \in [a_n, b_n]$ for all $n \in \mathbb{N}$.
(iii). Show that $a = b$, which is denoted by p.
(iv). Show that $\cap_{n=1}^{\infty}[a_n, b_n] = \{p\}$, that is, p is the only point in the intersection of $[a_n, b_n]$ for all $n \in \mathbb{N}$.

Question 3.9. Let $\{a_n\}_{n=1}^{\infty}$ be a Cauchy sequence. Show that it converges by the following steps.

(i). Show that $\{a_n\}_{n=1}^{\infty}$ is bounded. (Hence, there is at least one subsequential limit, denoted by a.)
(ii). Show that $\{a_n\}_{n=1}^{\infty}$ has exactly one subsequential limit a. (Hence, $a_n \to a$.)

Question 3.10. Let $a_1 = 0.9$, $a_2 = 0.99$,... Show that $a_n \to 1$.

Question 3.11. Let $a_1 = \frac{1}{2}$ and

$$a_n = a_{n-1} + \frac{1}{2^n} \quad \text{for } n \geq 2.$$

(i). Show that $\{a_n\}_{n=1}^{\infty}$ is Cauchy. (Hence, it converges.)
(ii). Find the limit of $\{a_n\}_{n=1}^{\infty}$.

3.3. Compact sets

In this section, we introduce the important concept of compact sets in topology via the open covers. The Heine-Borel theorem asserts that a set in $\mathbb{R}$ is compact iff it is bounded and closed.

Definition (Open covers). Let $E \subset \mathbb{R}$. We say that a collection of open sets A_j, $j \in J$, is an open cover of E if

$$E \subset \bigcup_{j \in J} A_j.$$

We say that an open cover A_j, $j \in J$, of E is finite if J is finite.

Remark (Trivial open covers). Let $E \subset \mathbb{R}$.

- $\{\mathbb{R}\}$ is always an open cover of E.
- For each $\delta > 0$,
$$\bigcup_{x \in E} (x - \delta, x + \delta)$$
is an open cover of E. In addition, if E is finite, then this is a finite open cover.

Example. The following are open covers of $(0, 1)$.

- $\{(0, 1)\}$.
- $\{(0, \frac{2}{3}), (\frac{1}{3}, 1)\}$.
- $\{(0, 1 - \frac{1}{n}) : n \in \mathbb{N}\}$.
- $\{(\frac{1}{n}, 1) : n \in \mathbb{N}\}$.

Example. The following are open covers of $[0, 1]$.

- $\{(-1, 2)\}$.
- $\{(-1, \frac{2}{3}), (\frac{1}{3}, 2)\}$.
- $\{(-1, 1 + \frac{1}{n}) : n \in \mathbb{N}\}$.
- $\{(-\frac{1}{n}, 2) : n \in \mathbb{N}\}$.

Example. The following are open covers of $(0, \infty)$.

- $\{(0, \infty)\}$.
- $\{(0, \frac{2}{3}), (\frac{1}{3}, \infty)\}$.
- $\{(0, n) : n \in \mathbb{N}\}$.
- $\{(-\frac{1}{n}, \infty) : n \in \mathbb{N}\}$.

Example. The following are open covers of $[0, \infty)$.

- $\{(-1, \infty)\}$.
- $\{(-1, \frac{2}{3}), (\frac{1}{3}, \infty)\}$.
- $\{(-1, n) : n \in \mathbb{N}\}$.
- $\{(-\frac{1}{n}, \infty) : n \in \mathbb{N}\}$.

Definition (Compact sets). A set $E \subset \mathbb{R}$ is said to be compact if each open cover of E has a finite subcover.

Example.

- $(0, 1)$ is not compact. Indeed, we prove that the open cover $\{(\frac{1}{n}, 1) : n \in \mathbb{N}\}$ does not have a finite subcover by contradiction. Suppose that there is a finite subcover $\{(\frac{1}{n_1}, 1), ..., (\frac{1}{n_k}, 1)\}$. Set $N = \max\{n_1, ..., n_k\}$. Then
$$\bigcup_{j=1}^{k} \left(\frac{1}{n_j}, 1 \right) = \left(\frac{1}{N}, 1 \right),$$

which does not cover $(0,1)$. Therefore, $(0,1)$ is not compact.

- $(0,\infty)$ is not compact. Indeed, we prove that the open cover $\{(0,n) : n \in \mathbb{N}\}$. does not have a finite subcover by contradiction. Suppose that there is a finite subcover $\{(0,n_1),...,(0,n_k)\}$. Set $N = \max\{n_1,...,n_k\}$. Then
$$\bigcup_{j=1}^{k}(0,n_j) = (0,N),$$
which does not cover $(0,\infty)$. Therefore, $(0,\infty)$ is not compact.

THEOREM 3.13 (Heine-Borel). *A set $E \subset \mathbb{R}$ is compact iff it is bounded and closed.*

Remark. The intervals $[a,b]$ is compact, and $(a,b), [a,b), (a,b], (-\infty,a), (-\infty,a], (a,\infty), [a,\infty)$ are not compact.

$\boxed{\textbf{Homework Assignment}}$.

Question 3.12. Show that $(0,1]$ is not compact by finding an open cover which does not have a finite subcover.

Question 3.13. Show that $[0,\infty)$ is not compact by finding an open cover which does not have a finite subcover.

Question 3.14.

(i). Let A be compact and B be closed. Show that $A \cap B$ is compact.

(ii). Let A be compact and B be bounded. Is $A \cap B$ compact? Prove your assertion.

CHAPTER 4

Continuous functions

In this chapter, we investigate the limits and continuity of functions.

4.1. Limits of functions

Definition. Let $f : D \to \mathbb{R}$. Suppose that x_0 is a limit point of D. We say that the limit of f as x approaches x_0 is L if for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{for all } x \in D \text{ such that } 0 < |x - x_0| < \delta.$$

In this case, we write

$$\lim_{x \to x_0} f(x) = L.$$

Remark. In the definition of the limit of a function $f : D \to \mathbb{R}$ as x approaches x_0, x_0 needs not be in D. For instance, the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

has a (natural) domain $D = \mathbb{R} \setminus \{1\}$. But we prove that f has a limit of 2 as x approaches 1. To this end, let $\varepsilon > 0$. Set $\delta = \varepsilon$. Now if $0 < |x - 1| < \delta$, then

$$\left| \frac{x^2 - 1}{x - 1} - 2 \right| = |x - 1| < \delta = \varepsilon.$$

Therefore, $\lim_{x \to 1} f(x) = 2$.

Example. We prove that the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ has a limit of 1 as x approaches 1. (It is obvious that 1 is a limit point of $\mathbb{R} \setminus \{0\}$.) To this end, let $\varepsilon > 0$. Set

$$\delta = \min \left\{ \frac{1}{2}, \frac{\varepsilon}{2} \right\}.$$

Now if $0 < |x - 1| < \delta$, then by the triangle inequality,

$$|x| \geq 1 - |1 - x| > 1 - \delta \geq 1 - \frac{1}{2} = \frac{1}{2}.$$

Hence,

$$\left| \frac{1}{x} - 1 \right| = \frac{|x - 1|}{|x|} < \frac{\frac{\varepsilon}{2}}{\frac{1}{2}} = \varepsilon.$$

Therefore, $\lim_{x \to 1} f(x) = 1$.

Example. We prove that the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ has a limit of $\frac{1}{x_0}$ as x approaches $x_0 \neq 0$. (It is obvious that x_0 is a limit point of $\mathbb{R} \setminus \{0\}$.) To this end, let $\varepsilon > 0$. Set

$$\delta = \min \left\{ \frac{|x_0|}{2}, \frac{|x_0|^2 \varepsilon}{2} \right\}.$$

Now if $0 < |x - x_0| < \delta$, then by the triangle inequality,

$$|x| \geq |x_0| - |x_0 - x| > |x_0| - \delta \geq |x_0| - \frac{|x_0|}{2} = \frac{|x_0|}{2}.$$

Hence,

$$\left|\frac{1}{x} - \frac{1}{x_0}\right| = \frac{|x_0 - x|}{|x|\,|x_0|} < \frac{\frac{|x_0|^2 \varepsilon}{2}}{\frac{|x_0|^2}{2}} = \varepsilon.$$

Therefore, $\lim_{x \to x_0} f(x) = \frac{1}{x_0}$.

Example. We prove that the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ does not have a limit as x approaches 0 by contradiction. (It is obvious that 0 is a limit point of $\mathbb{R} \setminus \{0\}$.) Suppose that

$$\lim_{x \to 0} \frac{1}{x} = L \quad \text{for some } L \in \mathbb{R}.$$

That is, for each $\varepsilon > 0$, there is $\delta > 0$ such that

$$\left|\frac{1}{x} - L\right| < \varepsilon \quad \text{for all } 0 < |x| < \delta.$$

In particular, set $\varepsilon = 1$. Then there is $\delta > 0$ such that

$$\left|\frac{1}{x} - L\right| < 1 \quad \text{for all } 0 < |x| < \delta.$$

which implies that

$$\frac{1}{x} < |L| + 1 \quad \text{for all } 0 < |x| < \delta.$$

But this is impossible by choosing

$$x = \min\left\{\frac{\delta}{2}, \frac{1}{|L| + 1}\right\}.$$

Therefore, f does not have a limit as x approaches 0.

Example. We prove that the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = \sin(\frac{1}{x})$ does not have a limit as x approaches 0 by contradiction. (It is obvious that 0 is a limit point of $\mathbb{R} \setminus \{0\}$.) Suppose that

$$\lim_{x \to 0} f(x) = L \quad \text{for some } L \in \mathbb{R}.$$

That is, for each $\varepsilon > 0$, there is $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{for all } 0 < |x| < \delta.$$

In particular, set $\varepsilon = \frac{1}{2}$. Then there is $\delta > 0$ such that

$$|f(x) - L| < \frac{1}{2} \quad \text{for all } 0 < |x| < \delta.$$

Choose $n \in \mathbb{N}$ large enough such that

$$\frac{1}{(2n + \frac{1}{2})\pi}, \frac{1}{(2n + \frac{3}{2})\pi} \in (0, \delta).$$

Notice that

$$\left|f\left(\frac{1}{(2n + \frac{1}{2})\pi}\right) - f\left(\frac{1}{(2n + \frac{3}{2})\pi}\right)\right| = \left|\sin\left(2n\pi + \frac{\pi}{2}\right) - \sin\left(2n\pi + \frac{3\pi}{2}\right)\right| = 2.$$

By the triangle inequality,

$$\begin{aligned}
2 &= \left|f\left(\frac{1}{(2n+\frac{1}{2})\pi}\right) - f\left(\frac{1}{(2n+\frac{3}{2})\pi}\right)\right| \\
&= \left|f\left(\frac{1}{(2n+\frac{1}{2})\pi}\right) - L + L - f\left(\frac{1}{(2n+\frac{3}{2})\pi}\right)\right| \\
&\leq \left|f\left(\frac{1}{(2n+\frac{1}{2})\pi}\right) - L\right| + \left|L - f\left(\frac{1}{(2n+\frac{3}{2})\pi}\right)\right| \\
&< \frac{1}{2} + \frac{1}{2} \\
&= 1,
\end{aligned}$$

which is impossible. Therefore, f does not have a limit as x approaches 0.

THEOREM 4.1. *Let $f : D \to \mathbb{R}$. Suppose that x_0 is a limit point of D. Then*

$$\lim_{x\to x_0} f(x) = L$$

iff for every sequence $\{x_n\}_{n=1}^{\infty} \subset D$ such that $x_n \neq x_0$ for all $n \in \mathbb{N}$ and $x_n \to x_0$, we have that $f(x_n) \to L$.

PROOF. We first show necessity. Suppose that $\lim_{x\to x_0} f(x) = L$. Let $\varepsilon > 0$. Then there is $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{for all } x \in D \text{ such that } 0 < |x - x_0| < \delta.$$

Consider $\{x_n\}_{n=1}^{\infty} \subset D$ such that $x_n \neq x_0$ for all $n \in \mathbb{N}$ and $x_n \to x_0$. Then there is $N \in \mathbb{N}$ such that

$$0 < |x_n - x_0| < \delta \quad \text{for all } n \geq N.$$

Hence, $|f(x_n) - L| < \varepsilon$ for all $n \geq N$. Therefore, $f(x_n) \to L$.

We then show sufficiency. Suppose that for every sequence $\{x_n\}_{n=1}^{\infty} \subset D$ such that $x_n \neq x_0$ for all $n \in \mathbb{N}$ and $x_n \to x_0$, $f(x_n) \to L$. We prove that $\lim_{x\to x_0} f(x) = L$ by contradiction. That is, there is $\varepsilon > 0$ such that for each $\delta > 0$, there is $x = x(\delta) \in D$ such that $0 < |x - x_0| < \delta$ and $|f(x) - L| \geq \varepsilon$. Hence, for each $n \in \mathbb{N}$, there is $x_n \in D$ such that $0 < |x_n - x_0| < \delta$ and $|f(x_n) - L| \geq \varepsilon$. Now $\{x_n\}_{n=1}^{\infty} \subset D$, for which $x_n \neq x_0$ for all $n \in \mathbb{N}$ and $x_n \to x_0$. However, $f(x_n) \not\to L$. ☐

The sequence characterization allows us to immediately derive the following properties of the limits of functions from the corresponding ones of the limits of sequences in Section 2.1.

THEOREM 4.2. *Let $f : D \to \mathbb{R}$. Suppose that x_0 is a limit point of D. The limit of f as x approaches x_0, if exists, is unique.*

PROOF. Suppose that $\lim_{x\to x_0} f(x) = L$. Let $\{x_n\}_{n=1}^{\infty} \subset D$ such that $x_n \neq x_0$ for all $n \in \mathbb{N}$ and $x_n \to x_0$. Then $f(x_n) \to L$.

Now suppose that $\lim_{x\to x_0} f(x) = M$. Then $f(x_n) \to M$. But the limit of $\{f(x_n)\}_{n=1}^{\infty}$ is unique. Therefore, $M = L$, that is, the limit of f as x approaches x_0 is unique. ☐

THEOREM 4.3. *Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$. Suppose that x_0 is a limit point of D. Assume that*

$$\lim_{x\to x_0} f(x) = a \quad \textit{and} \quad \lim_{x\to x_0} g(x) = b.$$

(i).

$$\lim_{x\to x_0}\left(f(x)+g(x)\right)=a+b.$$

(ii). Suppose that $c\in\mathbb{R}$. Then

$$\lim_{x\to x_0} cf(x)=ca.$$

(iii).

$$\lim_{x\to x_0}\left(f(x)g(x)\right)=ab.$$

(iv). Suppose that $b\neq 0$ and $g(x)\neq 0$ for all $x\in D$. Then

$$\lim_{x\to x_0}\frac{f(x)}{g(x)}=\frac{a}{b}.$$

Proof. We only show (i) and the others are proved similarly.

Let $\{x_n\}_{n=1}^{\infty}\subset D$ such that $x_n\neq x_0$ for all $n\in\mathbb{N}$ and $x_n\to x_0$. Since $\lim_{x\to x_0} f(x)=a$, $f(x_n)\to a$; since $\lim_{x\to x_0} g(x)=b$, $g(x_n)\to b$. By Theorem 2.5,

$$\lim_{n\to\infty}\left(f\left(x_n\right)+g\left(x_n\right)\right)=a+b.$$

Therefore,

$$\lim_{x\to x_0}\left(f(x)+g(x)\right)=a+b.$$

□

Remark. The sequence characterization of limits of functions also provides an effective method to prove that a function does not have a limit at a point. That is, to show that a function $f(x):D\to\mathbb{R}$ does not have a limit as x approaches x_0, it suffices to find a sequence of points $\{a_n\}_{n=1}^{\infty}$ in D such that $a_n\neq x_0$ for all $n\in\mathbb{N}$ and $a_n\to x_0$. But $\{f(a_n)\}_{n=1}^{\infty}$ diverges.

Moreover, it also suffices to find two sequences of points $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ in D such that $a_n\neq x_0$, $b_n\neq x_0$ for all $n\in\mathbb{N}$, $a_n\to x_0$, and $b_n\to x_0$. But $\{f(a_n)\}_{n=1}^{\infty}$ and $\{f(b_n)\}_{n=1}^{\infty}$ converge to different limits.

Example. We prove that the function $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ defined by $f(x)=\frac{1}{x}$ does not have a limit as x approaches 0. Consider the sequence of points $\{\frac{1}{n}\}_{n=1}^{\infty}$. Then $\frac{1}{n}\to 0$. But $f(\frac{1}{n})=n$ does not converge. (Indeed, $\{f(\frac{1}{n})\}_{n=1}^{\infty}$ diverges to infinity.) Therefore, f does not have a limit as x approaches 0.

Example. We prove that the function $f:\mathbb{R}\setminus\{0\}\to\mathbb{R}$ defined by $f(x)=\sin(\frac{1}{x})$ does not have a limit as x approaches 0. Consider two sequences of points

$$\left\{\frac{1}{(2n+\frac{1}{2})\pi}\right\}_{n=1}^{\infty}\quad\text{and}\quad\left\{\frac{1}{(2n+\frac{3}{2})\pi}\right\}_{n=1}^{\infty}.$$

Then

$$\frac{1}{(2n+\frac{1}{2})\pi},\frac{1}{(2n+\frac{3}{2})\pi}\to 0.$$

But for all $n\in\mathbb{N}$,

$$f\left(\frac{1}{(2n+\frac{1}{2})\pi}\right)=\sin\left(2n\pi+\frac{\pi}{2}\right)=1\quad\text{and}\quad f\left(\frac{1}{(2n+\frac{3}{2})\pi}\right)=\sin\left(2n\pi+\frac{3\pi}{2}\right)=-1.$$

Therefore, f does not have a limit as x approaches 0.

In fact, for each $y \in [-1,1]$, there is $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ such that $\sin\theta = y$ (by taking $\theta = \arcsin y$). Consider the sequences of points

$$\left\{\frac{1}{2n\pi+\theta}\right\}_{n=1}^{\infty},$$

which converges to 0. But

$$f\left(\frac{1}{2n\pi+\theta}\right) = \sin(2n\pi+\theta) = y.$$

Definition (One-sided limits). Let $f : D \to \mathbb{R}$. Suppose that x_0 is a limit point of D.

- We say that the limit of f as x approaches x_0 from the right is L if for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that
$$|f(x) - L| < \varepsilon \quad \text{for all } x \in D \text{ such that } 0 < x - x_0 < \delta.$$
In this case, we write
$$\lim_{x\to x_0+} f(x) = L.$$
- We say that the limit of f as x approaches x_0 from the left is L if for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that
$$|f(x) - L| < \varepsilon \quad \text{for all } x \in D \text{ such that } 0 < x_0 - x < \delta.$$
In this case, we write
$$\lim_{x\to x_0-} f(x) = L.$$

THEOREM 4.4. *Let $f : D \to \mathbb{R}$. Suppose that x_0 is a limit point of D. Then*

$$\lim_{x\to x_0} f(x) = L$$

iff

$$\lim_{x\to x_0+} f(x) = \lim_{x\to x_0-} f(x) = L.$$

Example (Heaviside function). The Heaviside function $H : \mathbb{R} \to \mathbb{R}$ is defined by

$$H(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Then

$$\lim_{x\to 0+} f(x) = 1 \quad \text{and} \quad \lim_{x\to 0-} f(x) = 0.$$

As a consequence, H does not have a limit as x approaches 0. In fact, these can also be seen via the seqeunce characterization in Theorem 4.1: Consider two sequences of points

$$\left\{\frac{1}{n}\right\}_{n=1}^{\infty} \quad \text{and} \quad \left\{-\frac{1}{n}\right\}_{n=1}^{\infty}.$$

Then

$$\frac{1}{n}, -\frac{1}{n} \to 0.$$

But for all $n \in \mathbb{N}$,

$$H\left(\frac{1}{n}\right) = 1 \quad \text{and} \quad H\left(-\frac{1}{n}\right) = 0.$$

Therefore, H does not have a limit as x approaches 0.

$\boxed{\textbf{Homework Assignment}}$.

Question 4.1. Let $f(x) = x^2$.

(i). Show that $\lim_{x\to 0} f(x) = 0$.
(ii). Show that $\lim_{x\to x_0} f(x) = x_0^2$ for each $x_0 \in \mathbb{R}$.

Question 4.2. Show that

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0.$$

Question 4.3.

(i). Let $f(x) \geq 0$ for all $x \in D$ and $\lim_{x\to x_0} f(x) = L$. Show that $L \geq 0$.
(ii). Let $f(x) > 0$ for all $x \in D$ and $\lim_{x\to x_0} f(x) = L$. Is $L > 0$ true? Prove your assertion.

Question 4.4 (The squeezing theorem). Let $f(x) \leq g(x) \leq h(x)$ for all $x \in D$. Suppose that

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = L.$$

Show that $\lim_{x\to x_0} g(x) = L$.

Question 4.5. Let $f : D \to \mathbb{R}$ and $g : D \to \mathbb{R}$. Suppose that x_0 is a limit point of D. Assume that

$$\lim_{x\to x_0} f(x) = a \quad \text{and} \quad \lim_{x\to x_0} g(x) = b.$$

Show that

$$\lim_{x\to x_0} (f(x)g(x)) = ab.$$

4.2. Limits involving infinity

Definition. Let $f : D \to \mathbb{R}$. Suppose that x_0 is a limit point of D.

- We say that f diverges to infinity as x approaches x_0 if for each $M > 0$, there is $\delta = \delta(M) > 0$ such that
$$f(x) > M \quad \text{for all } x \in D \text{ such that } 0 < |x - x_0| < \delta.$$
In this case, we write
$$\lim_{x\to x_0} f(x) = \infty.$$
- We say that f diverges to negative infinity as x approaches x_0 if for each $M > 0$, there is $\delta = \delta(M) > 0$ such that
$$f(x) < -M \quad \text{for all } x \in D \text{ such that } 0 < |x - x_0| < \delta.$$
In this case, we write
$$\lim_{x\to x_0} f(x) = -\infty.$$

Example. We prove that the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x^2}$ diverges to infinity as x approaches 0. To this end, let $M > 0$. Set $\delta = \frac{1}{\sqrt{M}}$. Now if $0 < |x| < \delta$, then

$$|f(x)| = \frac{1}{x^2} > \frac{1}{\delta^2} = M.$$

Therefore, $\lim_{x\to 0} f(x) = \infty$.

THEOREM 4.5. *Let $f : D \to \mathbb{R}$. Suppose that x_0 is a limit point of D.*

(i).

$$\lim_{x \to x_0} f(x) = \infty$$

iff for every sequence $\{x_n\}_{n=1}^{\infty} \subset D$ *such that* $x_n \neq x_0$ *for all* $n \in \mathbb{N}$ *and* $x_n \to x_0$*, we have that* $f(x_n) \to \infty$*.*

(ii).

$$\lim_{x \to x_0} f(x) = -\infty$$

iff for every sequence $\{x_n\}_{n=1}^{\infty} \subset D$ *such that* $x_n \neq x_0$ *for all* $n \in \mathbb{N}$ *and* $x_n \to x_0$*, we have that* $f(x_n) \to -\infty$*.*

Proof.

(i). See Question 4.6.

(ii). We first show necessity. Suppose that $\lim_{x\to x_0} f(x) = -\infty$. Let $M > 0$. Then there is $\delta > 0$ such that

$$f(x) < -M \quad \text{for all } x \in D \text{ such that } 0 < |x - x_0| < \delta.$$

Consider $\{x_n\}_{n=1}^{\infty} \subset D$ such that $x_n \neq x_0$ for all $n \in \mathbb{N}$ and $x_n \to x_0$. Then there is $N \in \mathbb{N}$ such that

$$0 < |x_n - x_0| < \delta \quad \text{for all } n \geq N.$$

Hence, $f(x_n) < -M$ for all $n \geq N$. Therefore, $f(x_n) \to -\infty$.

We then show sufficiency. Suppose that for every sequence $\{x_n\}_{n=1}^{\infty} \subset D$ such that $x_n \neq x_0$ for all $n \in \mathbb{N}$ and $x_n \to x_0$, $f(x_n) \to -\infty$. We prove that $\lim_{x\to x_0} f(x) = -\infty$ by contradiction. That is, there is $M \in \mathbb{R}$ such that for each $\delta > 0$, there is $x = x(\delta) \in D$ such that $0 < |x - x_0| < \delta$ and $f(x) \geq M$. Hence, for each $n \in \mathbb{N}$, there is $x_n \in D$ such that $0 < |x_n - x_0| < \delta$ and $f(x_n) \geq M$. Now $\{x_n\}_{n=1}^{\infty} \subset D$, for which $x_n \neq x_0$ for all $n \in \mathbb{N}$ and $x_n \to x_0$. However, $\{f(x_n)\}_{n=1}^{\infty}$ does not diverge to negative infinity.

□

We can similarly define the following concepts.

$$\lim_{x \to x_0+} f(x) = \infty \quad \text{and} \quad \lim_{x \to x_0+} f(x) = -\infty.$$

$$\lim_{x \to x_0-} f(x) = \infty \quad \text{and} \quad \lim_{x \to x_0-} f(x) = -\infty.$$

Definition. Let $f : D \to \mathbb{R}$. Suppose that x_0 is a limit point of D.

- We say that f diverges to infinity as x approaches x_0 from the right if for each $M > 0$, there is $\delta = \delta(M) > 0$ such that

  $$f(x) > M \quad \text{for all } x \in D \text{ such that } 0 < x - x_0 < \delta.$$

  In this case, we write

  $$\lim_{x \to x_0+} f(x) = \infty.$$

- We say that f diverges to negative infinity as x approaches x_0 from the right if for each $M > 0$, there is $\delta = \delta(M) > 0$ such that

  $$f(x) < -M \quad \text{for all } x \in D \text{ such that } 0 < x - x_0 < \delta.$$

  In this case, we write

  $$\lim_{x \to x_0+} f(x) = -\infty.$$

Example. We prove that the function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ diverges to infinity as x approaches 0 from the right. To this end, let $M > 0$. Set $\delta = \frac{1}{M}$. Now if $0 < x < \delta$, that is, $x \in (0, \delta)$, then

$$f(x) = \frac{1}{x} > \frac{1}{\delta} = M.$$

Therefore, $\lim_{x\to 0+} f(x) = \infty$. See Question 4.8 for $\lim_{x\to 0-} f(x) = -\infty$.

Moreover, we have shown in the previous section that f does not have a limit as x approaches 0. Therefore, we say that the function f as x approaches 0 does not have a limit and does not diverge to infinity or to negative infinity.

Homework Assignment.

Question 4.6. Let $f : D \to \mathbb{R}$. Suppose that x_0 is a limit point of D. Show that

$$\lim_{x\to x_0} f(x) = \infty$$

iff for every sequence $\{x_n\}_{n=1}^{\infty} \subset D$ such that $x_n \neq x_0$ for all $n \in \mathbb{N}$ and $x_n \to x_0$, we have that $f(x_n) \to \infty$.

Question 4.7. Write the definitions of the following equations.

$$\lim_{x\to x_0-} f(x) = \infty \quad \text{and} \quad \lim_{x\to x_0-} f(x) = -\infty.$$

Question 4.8. Show that

$$\lim_{x\to 0-} \frac{1}{x} = -\infty.$$

4.3. Continuity of functions

Definition. Let $f : D \to \mathbb{R}$. Suppose that $x_0 \in D$. We say that f is continuous at x_0 if for each $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that

$$|f(x) - f(x_0)| < \varepsilon \quad \text{for all } x \in D \text{ such that } |x - x_0| < \delta.$$

- We say that f is discontinuous at x_0 if it is not continuous at x_0.
- We say that f is continuous if it is continuous at each point $x_0 \in D$.

Remark. In the definition of continuity of a function $f : D \to \mathbb{R}$ at x_0, x_0 needs to be in D, but needs not be in D'. In particular, if $D' = \emptyset$, such as in the case when D is finite, then f is continuous at each point $x_0 \in D$.

Moreover, continuity is a "local" property, that is, whether f is continuous at x_0 depends only on its behavior in the neighborhood of x_0.

Example. Let $f : \mathbb{N} \to \mathbb{R}$. Then f is continuous at each $x_0 \in \mathbb{N}$.

Remark. Let $f : D \to \mathbb{R}$. If, in addition to $x_0 \in D$, x_0 is also a limit point of D, then f is continuous at x_0 iff

$$\lim_{x\to x_0} f(x) = f(x_0).$$

This equation contains three conditions:

(i). $f(x_0)$ exists, that is, $x_0 \in D$.
(ii). $\lim_{x\to x_0} f(x)$ exists.
(iii). $\lim_{x\to x_0} f(x)$ coincides with $f(x_0)$.

Example.

- Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by
$$f(x) = \begin{cases} \frac{x^2-1}{x-1} & \text{if } x \neq 1, \\ 0 & \text{if } x = 1. \end{cases}$$
Notice that $\lim_{x\to 1} f(x) = 2$. But $f(1) = 0$. Therefore, f is discontinuous at 1. This is an example of "removable discontinuity", in which case the limit of f exists at x_0 but differs with the function value $f(x_0)$ at x_0. Such discontinuous point can be removed by changing the value $f(x_0)$ to the limit of the function. In particular, if we change $f(1)$ to 2, then the function is continuous. (In fact, the function is $x + 1$ on $\mathbb{R}$.)
- Consider the Heaviside function $f : \mathbb{R} \to \mathbb{R}$ defined by
$$H(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$
Notice that $f(0) = 1$. But $\lim_{x\to 0+} = 1$ and $\lim_{x\to 0-} = 0$. As a consequence, f does not have a limit as x approaches 0. Therefore, f is discontinuous at 0. This is an example of "jump discontinuity", in which case the limit from right differs from the one from left. From one side to the other of such a discontinuous point, there is a jump of function values.
- Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by
$$f(x) = \begin{cases} \sin\left(\frac{1}{x}\right) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$
Notice that $\lim_{x\to 0} f(x)$ does not exist. In fact, the one-sided limits $\lim_{x\to 0+} f(x)$ and $\lim_{x\to 0-} f(x)$ do not exist. This is an example of "essential discontinuity", in which case the limit does not exist. In particular, such a discontinuous point x_0 cannot be removed by changing the value of the function at x_0.

Example. We have shown in the previous section that at each $x_0 \neq 0$,
$$\lim_{x\to x_0} \frac{1}{x} = \frac{1}{x_0}.$$
Therefore, the function $\frac{1}{x}$ is continuous on $\mathbb{R} \setminus \{0\}$.

Example (Dirichlet function). Define the Dirichlet function $f : \mathbb{R} \to \mathbb{R}$ by
$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$
Then f is discontinuous at each $x_0 \in \mathbb{R}$. Indeed,

- if $x_0 \in \mathbb{Q}$, then for each $\delta > 0$, there is an irrational point $x \in (x_0 - \delta, x_0 + \delta)$ such that
$$|f(x) - f(x_0)| = |0 - 1| > \frac{1}{2}.$$
Therefore, f is discontinuous at x_0.
- if $x_0 \in \mathbb{R} \setminus \mathbb{Q}$, then for each $\delta > 0$, there is a rational point $x \in (x_0 - \delta, x_0 + \delta)$ such that
$$|f(x) - f(x_0)| = |1 - 0| > \frac{1}{2}.$$
Therefore, f is discontinuous at x_0.

Example. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ with } p \in \mathbb{Z} \setminus \{0\} \text{ and } q \in \mathbb{N} \text{ in the simplified form,} \\ 1 & \text{if } x = 0, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Then f is discontinuous at each point $x_0 \in \mathbb{Q}$ and is continuous at each point $x_0 \in \mathbb{R}\setminus\mathbb{Q}$. Indeed,

- if $x_0 = 0$, then for each $\delta > 0$, there is an irrational point $x \in (x_0 - \delta, x_0 + \delta)$ such that
$$|f(x) - f(x_0)| = |1 - 0| > \frac{1}{2}.$$
Therefore, f is discontinuous at 0.
- See Question 4.11.
- For each $\varepsilon > 0$, there is $q \in \mathbb{N}$ such that $q > \frac{1}{\varepsilon}$. If $x_0 \in \mathbb{R} \setminus \mathbb{Q}$, then there is $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta)$ contains no rational points $x = \frac{p}{q}$ in the simplified form. (Firstly, $(x_0 - \frac{1}{2q}, x_0 + \frac{1}{2q})$ contains at most one such rational point x_1. We can then set $\delta = \min\{\frac{1}{2q}, |x_1 - x_0|\}$.)
  Now if $|x - x_0| < \delta$, then
$$|f(x) - f(x_0)| \leq \left|0 - \frac{1}{q}\right| < \varepsilon.$$
Therefore, f is continuous at x_0.

Theorem 4.6. *Let $f : D \to \mathbb{R}$ and $x_0 \in D$. Then f is continuous at x_0 iff for every sequence $\{x_n\}_{n=1}^{\infty} \subset D$ such that $x_n \to x_0$, we have that $f(x_n) \to f(x_0)$.*

Remark. The above theorem asserts that if f is continuous at $x_0 \in D$ and $\{x_n\}_{n=1}^{\infty} \subset D$ converges to x_0, then
$$\lim_{n\to\infty} f(x_n) = f(x_0) = f\left(\lim_{n\to\infty} x_n\right),$$
that is, we can interchange the operations of taking limits and evaluating.

The proof of this theorem is a reproduction of the one of Theorem 4.1 with necessary modification.

Proof. See Question 4.12 for necessity.

We then show sufficiency. Suppose that for every sequence $\{x_n\}_{n=1}^{\infty} \subset D$ such that $x_n \to x_0$, $f(x_n) \to f(x_0)$. We prove that $\lim_{x\to x_0} f(x) = f(x_0)$ by contradiction. That is, there is $\varepsilon > 0$ such that for each $\delta > 0$, there is $x = x(\delta) \in D$ such that $|x - x_0| < \delta$ and $|f(x) - f(x_0)| \geq \varepsilon$. Hence, for each $n \in \mathbb{N}$, there is $x_n \in D$ such that $|x_n - x_0| < \delta$ and $|f(x_n) - f(x_0)| \geq \varepsilon$. Now $\{x_n\}_{n=1}^{\infty} \subset D$, for which $x_n \to x_0$. However, $\{f(x_n)\}_{n=1}^{\infty}$ does not converge to $f(x_0)$. ☐

$\boxed{\textbf{Homework Assignment}}$.

Question 4.9. Let $f(x) = x^2$. Show that f is continuous on $\mathbb{R}$.

Question 4.10.

(i). Let f be continuous. Show that $|f|$ is continuous.

(ii). Let $|f|$ be continuous. Is f continuous? Prove your assertion.

Question 4.11. Define $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} \frac{1}{q} & \text{if } x = \frac{p}{q} \text{ with } p \in \mathbb{Z} \setminus \{0\} \text{ and } q \in \mathbb{N} \text{ in the simplified form,} \\ 1 & \text{if } x = 0, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

Show that f is discontinuous at each $x_0 \in \mathbb{Q} \setminus \{0\}$.

Question 4.12. Let $f : D \to \mathbb{R}$ be continuous at $x_0 \in D$. Show that for every sequence $\{x_n\}_{n=1}^{\infty} \subset D$ such that $x_n \to x_0$, we have that $f(x_n) \to f(x_0)$.