Intersecting Lagrangian distributions by Melrose-Uhlmann

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Notational index

- X: an n-dim smooth manifold.
- $T(X)$: the tangent bundle of X.
- $T^*(X)$: the cotangent bundle of X.
- $T^*(X) \setminus 0$: the cotangent bundle of X with the zero section removed.
- $N(Y)$: the (co)normal bundle of Y, in which Y is a submanifold of X.
- Δ^*: the diagonal in $T^*(X) \setminus 0 \times T^*(X) \setminus 0$.
- Λ: a closed conic Lagrangian submanifold of $T^*(X) \setminus 0$.
- $D = \partial / i$.
- $\hat{f}$: the Fourier transform of f defined by
$$\hat{f}(\xi) = \int e^{-ix\cdot\xi} f(x)\, dx.$$
- $\mathcal{D}'(X)$: the space of distributions in X.
- $\mathcal{E}'(X)$: the space of distributions with compact support in X.

- $H_{(s)}(X)$: the Sobolev space of order s in X.
- $L^m(X)$: the space of ΨDOs on X of order m.
- $P \in L^m(X)$: a ΨDO with real homogeneous principal symbol p of homogeneous degree m.
- H_p: the Hamilton vector field generated by p.
- $N = \Sigma(P)$: the characteristic variety of a ΨDO P, i.e.

$$\Sigma(P) = \{(x, \xi) \in T^*(X) \setminus 0 : p(x, \xi) = 0\}.$$

- Δ_N: the diagonal in $N \times N$.
- C: the bicharacteristic relation of P, i.e.

$$C = \{((x, \xi), (y, \eta)) \in N \times N : \\ (x, \xi) \text{ and } (y, \eta) \text{ lie on the same bicharacteristic strip}\}.$$

- $I^m(X; \Lambda)$: the space of Lagrangian distributions associated with the Lagrangian $\Lambda \subset X$ and of order m.

0 Duistermaat-Hörmander theory

In [DH72, §6], the existence and regularity of solutions to a pseudo-differential equation

$$Pu = f$$

in a manifold X are investigated.

Let $P \in L^m(X)$ is of real principal type in X. Suppose that X is pseudo-convex with respect to P, i.e. the following equivalent conditions hold.

(i). [DH72, Definition 6.5.1] P has a real homogeneous principal part p and no complete bicharacteristic strip of P stays over a compact set in X.

(ii). [DH72, Theorem 6.3.3] For every compact set $K \subset X$ there is another compact set $K' \subset X$ such that if $u \in \mathcal{E}'(X)$ and $\operatorname{sing\,supp} {}^tPu \subset K$, then $\operatorname{sing\,supp} u \subset K'$.

(iii). [DH72, Theorem 6.3.3] For every compact set $K \subset X$ there is another compact set $K' \subset X$ such that K' contains any interval on a bicharacteristic curve with respect to P having both end points in K.

Then the global parametrices (i.e. inverses) of P are constructed.

Precisely, under the above conditions, the following theorem holds.

Theorem (6.5.3 in [DH72])**.** *For every orientation $C \setminus \Delta_N = C_\nu^+ \cup C_\nu^-$, one can find parametrices E_ν^+ and E_ν^- of P that satisfy*

(i). $\mathrm{WF}'(E_\nu^+) = \Delta^* \cup C_\nu^+$ *and* $\mathrm{WF}'(E_\nu^-) = \Delta^* \cup C_\nu^-$,

(ii). any right or left parametrix E with $\mathrm{WF}'(E)$ *contained in $\Delta^* \cup C_\nu^+$ (resp. $\Delta^* \cup C_\nu^-$) equals E_ν^+ (resp. E_ν^-) modulo C^∞,*

(iii). E_ν^+ and E_ν^- maps $H_{(s)}^{comp}(X)$ continuously to $H_{(s+m-1)}(X)$,

(iv). $E_\nu^+ - E_\nu^- \in I^{1/2-m}(X \times X; C')$, and $E_\nu^+ - E_\nu^-$ is non-characteristic at every point of C'.

Example. The basic example is $P = D_1 = \partial_1/i \in L^1(\mathbb{R}^n)$. Then

- $E_1^+ = iH(x_1 - y_1)\delta(x' - y')$ and $E_1^- = -iH(y_1 - x_1)\delta(x' - y')$.
- $E_1^+ - E_1^- = i\delta(x' - y')$.
- $C = \{(x_1, x', \xi_1, \xi', y_1, y', \eta_1, \eta') : x' = y', \xi' = \eta' \neq 0, \xi_1 = \eta_1 = 0\}$.
- $E_1^+ - E_1^- \in I^{-1/2}(X \times X; C')$.

However, these parametrices E_ν^+ and E_ν^- are not FIOs introduced in [DH72] and the construction is not symbolic. Indeed, the wavefront set of the kernel of such a parametrix is equal to the union of the diagonal (Δ^*) and the Lagrangian manifold obtained by flowing the diagonal out under the Hamilton vector field of p acting inside the characteristic variety (C_ν^+ or C_ν^-). Thus, there are two Lagrangian manifolds intersecting.

Following [MU79, §§1-6], in this note we develop a symbol calculus for distributions associated to a pair of Lagrangian manifolds intersecting cleanly with codimension one, and then use it to give a completely symbolic version of the construction of the parametrices mentioned in above.

The basic example $iH(x_1)\delta(x')$ for $D_1 = \partial_1/i$ provides the prototype of a pair of intersecting Lagrangian manifolds. In this example, we denote

- $\tilde{\Lambda}_0 = T_0^*(\mathbb{R}^n) \setminus 0$,
- $\tilde{\Lambda}_1 = \{(x,\xi) \in T^*(\mathbb{R}^n) \setminus 0 : x' = 0, x_1 \geq 0, \xi_1 = 0\}$,
- $\tilde{\Lambda} = \tilde{\Lambda}_0 \cup \tilde{\Lambda}_1 = N(Y)$, in which $Y = \{x \in \mathbb{R}^n : x' = 0, x_1 \geq 0\}$ is a submanifold of $\mathbb{R}^n$ with boundary,
- $\partial\tilde{\Lambda}_1 = \tilde{\Lambda}_0 \cap \tilde{\Lambda}_1 = \{(0,\xi) : \xi_1 = 0\}$.

1 Lagrangian intersection

Definition (1.1)**.** *Let* $\Lambda_0 \subset T^*(X) \setminus 0$ *be a conic Lagrangian manifold and* $\Lambda_1 \subset T^*(X) \setminus 0$ *be a conic Lagrangian manifold with boundary.*

A pair (Λ_0, Λ_1) *is said to be an intersecting pair of Lagrangian manifolds if*

(i). $\Lambda_0 \cap \Lambda_1 = \partial \Lambda_1$,

(ii). the intersection is clean, i.e.

$$T_\lambda(\Lambda_0) \cap T_\lambda(\Lambda_1) = T(\partial \Lambda_1) \quad \text{for all } \lambda \in \partial \Lambda_1. \tag{1.2}$$

Two such pairs (Λ_0, Λ_1) *and* (Λ_0', Λ_1'), $\Lambda_i' \subset T^*(X') \setminus 0$, *with given base points* $\lambda \in \partial \Lambda_1$ *and* $\lambda' \in \partial \Lambda_1'$ *is said to be locally equivalent if there is a conic neighborhood* V *of* λ *in* $T^*(X) \setminus 0$ *and a homogeneous symplectic transformation* $\chi : V \to T^*(X') \setminus 0$ *such that*

(i). $\chi(\lambda) = \lambda'$,

(ii). $\chi(\Lambda_0 \cap V) \subset \Lambda_0'$,

(iii). $\chi(\Lambda_1 \cap V) \subset \Lambda_1'$.

Remark (c.f. §C.3 in [H85]). Let Y and Z be two two manifolds X. If

$$T_x(Y \cap Z) = T_x(Y) \cap T_x(Z) \quad \text{for all } x \in Y \cap Z,$$

then the intersection is said to be clean. One then has

$$\mathrm{codim}(Y) + \mathrm{codim}(Z) = \mathrm{codim}(Y \cap Z) + e.$$

Here, the non-negative integer e is called the excess of the intersection.

Example.

1. When $e = 0$, Y and Z are said to intersect transversally. In this case
   - $\mathrm{codim}(Y) + \mathrm{codim}(Z) = \mathrm{codim}(Y \cap Z)$,
   - $\dim(Y) + \dim(Z) - \dim(Y \cap Z) = \dim(X)$,
   - $N_x(Y) \cap N_x(Z) = \{0\}$.
2. The intersecting pair (Λ_0, Λ_1) is a special clean intersecting case when the excess

$$e = n - 1 \quad \text{and} \quad \Lambda_0 \cap \Lambda_1 = \partial \Lambda_1.$$

Theorem (Local coordinates for clean intersections, c.f. Theorem 21.2.10 in [H85]). *Let $Y, Z \subset X$ be two conic Lagrangian submanifolds intersecting cleanly with excess e at $\gamma \in X$. Then one can choose homogeneous symplectic coordinates (x, ξ) at γ such that*

(i). the coordinates of $\gamma = (0, \varepsilon_1)$, in which $\varepsilon_1 = (1, 0, ..., 0)$,

(ii). near $(0, \epsilon_1)$,

$$Y = \{(0, \xi)\} \quad \text{and} \quad Z = \{(0, x'', \xi', 0)\},$$

in which

$$x' = (x_1, ..., x_e) \quad \text{and} \quad x'' = (x_{e+1}, ..., x_n).$$

That is, Y is locally the conormal bundle of a point and Z the conormal bundle of a manifold of codimension e through it. If they intersect transversally, i.e. $e = 0$, then they are both locally conormal bundles of points.

Note that if two manifolds intersecting cleanly, then their conormal bundles of manifolds must also intersect cleanly.

Corollary (c.f. Corollary 21.2.11 in [H85])**.** *Let Y and Z be two Lagrangian subspaces of a symplectic vector space X. Then we can find a Lagrangian subspace W that is transversal to both Y and Z.*

Proof. By [H85, Theorem 21.2.10], one can choose local coordinates such that $Y = \{(0, \xi)\}$ and $Z = \{(0, x'', \xi', 0)\}$. Set

$$W = \{(x, \xi) : x'' = \xi'', \xi' = 0\}.$$

Then W is transversal to both Y and Z by simply noticing that two Lagrangian subspaces are transversal iff their intersection is zero. □

Proposition (1.3)**.** *All intersecting pairs of Lagrangian manifolds in manifolds of a fixed dimension are locally equivalent to the following pair.*

- $x = (x_1, x')$, in which $x_1 \in \mathbb{R}$ and $x' = (x_2, ..., x_n) \in \mathbb{R}^{n-1}$.
- $\tilde{\Lambda}_0 = T_0^*(\mathbb{R}^n) \setminus 0$.
- $\tilde{\Lambda}_1 = \{(x, \xi) \in T^*(\mathbb{R}^n) \setminus 0 : x' = 0, x_1 \geq 0, \xi_1 = 0\}$.
- $\tilde{\Lambda} = \tilde{\Lambda}_0 \cup \tilde{\Lambda}_1 = N(Y)$, in which $Y = \{x \in \mathbb{R}^n : x' = 0, x_1 \geq 0\}$.
- $\tilde{\Lambda}_0 \cap \tilde{\Lambda}_1 = \partial\tilde{\Lambda}_1 = \{(0, \xi) : \xi_1 = 0\}$.

Proof of Proposition 1.3.

Step 1. Let (Λ_0, Λ_1) be an intersecting pair and $\lambda \in \partial\Lambda_1$. By definition of locally equivalence of intersecting Lagrangian manifolds, it suffices to show that there is a conic neighborhood V of λ in $T^*(X) \setminus 0$ and a homogeneous symplectic transformation

$$\chi : V \to T^*(\mathbb{R}^n) \setminus 0 \tag{1.4}$$

such that

(i). $\chi(\lambda) = \tilde{\lambda} = (0, \tilde{\xi})$, in which $\tilde{\xi} = (0, ..., 0, 1)$,

(ii). $\chi(\Lambda_0 \cap V) \subset \tilde{\Lambda}_0'$,

(iii). $\chi(\Lambda_1 \cap V) \subset \tilde{\Lambda}_1'$.

Choose a homogeneous symplectic transformation χ such that

$$\chi(\lambda) = \tilde{\lambda}, \quad \chi(\Lambda_0 \cap V) \subset \tilde{\Lambda}_0, \quad \text{and} \quad \chi(\partial\Lambda_1 \cap V) \subset \partial\tilde{\Lambda}_1.$$

Similarly, choose a homogeneous symplectic transformation $\bar{\chi}$ such that

$$\bar{\chi}(\lambda) = \tilde{\lambda}, \quad \bar{\chi}(\Lambda_1 \cap V) \subset \tilde{\Lambda}_1, \quad \text{and} \quad \bar{\chi}(\partial\Lambda_1 \cap V) \subset \partial\tilde{\Lambda}_1.$$

Step 2. Write $X_1 = \chi^*(x_1)$ and $\bar{\Xi}_1 = \bar{\chi}^*(\xi_1)$. We claim

$$\{\bar{\Xi}_1, X_1\} \neq 0.$$

Proof of the claim. Notice that

$$\bar{\chi} \circ \chi^{-1} : T^*(\mathbb{R}^n) \to T^*(\mathbb{R}^n)$$

is a symplectic transformation so

$$\{\bar{\Xi}_1, X_1\} = \{(\bar{\chi}^{-1})^*\xi_1, (\chi^{-1})^*x_1\} = (\bar{\chi}^{-1})^*\{\xi_1, (\bar{\chi} \circ \chi^{-1})^*x_1\} \neq 0.$$

The claim follows if we show that $(\bar{\chi} \circ \chi^{-1})^*x_1 \in \mathrm{span}\{x_1\}$.

From the choice of χ and $\bar{\chi}$, we have that

$$(\bar{\chi} \circ \chi^{-1})^*\partial\tilde{\Lambda}_1 = \partial\tilde{\Lambda}_1 = \mathrm{span}\{\xi_2, ..., \xi_n\}.$$

Denote $\tilde{\Lambda}_2 = \mathrm{span}\{x_2, ..., x_n\}$. Then

$$(\bar{\chi} \circ \chi^{-1})^*\tilde{\Lambda}_2 = \tilde{\Lambda}_2$$

follows because symplectic transformation transforms conjugate pairs to conjugate pairs. Furthermore,

$$(\bar{\chi} \circ \chi^{-1})^*(\partial\tilde{\Lambda}_1)^{\perp} = (\partial\tilde{\Lambda}_1)^{\perp} = \mathrm{span}\{x_1, ..., x_n, \xi_1\}.$$

Also

$$(\bar{\chi} \circ \chi^{-1})^* \tilde{\Lambda}_2^{\perp} = \tilde{\Lambda}_2^{\perp} = \mathrm{span}\{x_1, \xi_1, ..., \xi_n\},$$

where $\perp$ denotes the symplectic orthogonal completion.

From the fact that

$$x_1 \in (\partial \tilde{\Lambda}_1)^{\perp} \cap \tilde{\Lambda}_2^{\perp} = \mathrm{span}\{x_1, \xi_1\},$$

we arrive at

$$(\bar{\chi} \circ \chi^{-1})^* x_1 \in \mathrm{span}\{x_1, \xi_1\}.$$

The proof of the claim is complete if we show

$$(\bar{\chi} \circ \chi^{-1})^* x_1 \notin \mathrm{span}\{\xi_1\},$$

or equivalently $(\chi^{-1})^* x_1$ and $(\bar{\chi}^{-1})^* \xi_1$ are linearly independent in X. This is apparent since the nonzero vectors

$$(\chi^{-1})^* x_1 \in T_\lambda(\Lambda_1) \setminus T_\lambda(\partial \Lambda) \quad \text{and} \quad (\bar{\chi}^{-1})^* \xi_1 \in T_\lambda(\Lambda_0) \setminus T_\lambda(\partial \Lambda),$$

and the intersecting $\Lambda_0 \cap \Lambda_1 = \partial \Lambda_1$ is clean.

Therefore, we can assume

$$\{\bar{\Xi}_1, X_1\} > 0.$$

□

Step 3. Since $\{\bar{\Xi}_1, X_1\} > 0$, we can define Ξ_1 such that

$$H_{X_1}\Xi_1 = -\{\Xi_1, X_1\} = -1 \quad \text{and} \quad \Xi_1 = 0 \text{ on } \{\bar{\Xi}_1 = 0\}.$$

Darboux's theorem asserts that one can find C^∞ functions $X_j, \Xi_j, j = 2, ..., n$ such that

$$\sigma : \varepsilon \to (X(\varepsilon), \Xi(\varepsilon))$$

is a homogeneous symplectic transformation to $T^*(\mathbb{R}^n)$ with $\sigma(\lambda) = \tilde{\lambda}$. Let

$$M = \{\varepsilon \in V : X_1(\varepsilon) = \Xi_1(\varepsilon) = 0\},$$

where V is a suitably small open conic neighborhood of λ, and σ maps M to $T^*(\mathbb{R}^{n-1})$. Then $\partial\Lambda_1 \subset M$ is Lagrangian. So there exist functions $x_2, ..., x_n$ and $\xi_2, ..., \xi_n$ on M for which

$$\bar{\sigma} : \varepsilon \to (x_2(\varepsilon), ..., x_n(\varepsilon), \xi_2(\varepsilon), ..., \xi_n(\varepsilon))$$

is a homogeneous symplectic map such that $\bar{\sigma}(\lambda) = \tilde{\lambda}$.

Extending $x_1, ..., x_2, \xi_1, ..., \xi_n$ to a neighborhood of Λ using

$$H_{x_i}x_j = H_{\xi_i}x_j = H_{x_i}\xi_j = H_{\xi_i}\xi_j = 0 \quad \text{for } i \neq j$$

gives the local coordinates. $\square$

2 Basic examples

In this section, we study the basic example $P = D_1 = \partial_1/i$ in $\mathbb{R}^n$.

Definition (2.1)**.** *Denote $I^m_{comp}(\mathbb{R}^n; \tilde{\Lambda}_0, \tilde{\Lambda}_1)$ the subspace of $\mathcal{E}'(\mathbb{R}^n)$ that consists of the distributions u which can be written in the form $u = u_1 + u_2$, in which $u_2 \in C_0^\infty(\mathbb{R}^n)$ and*

$$u_1(x) = \int_0^\infty \int_{\mathbb{R}^n} e^{i[(x_1-s)\cdot\xi_1 + x'\cdot\xi']} a(s,x,\xi)\, d\xi ds, \tag{2.2}$$

where $a \in S^{m+1/2-n/4}_{cl}(\mathbb{R}^{n+1} \times \mathbb{R}^n)$ is a classical symbol with compact cone-support.

Remark.

(i). (2.2) is well defined as an oscillatory integral, even prior to the s-integration, and as such it depends continuously on a in the topology of $S^{m'}$ for any $m' > m + 1/2 - n/4$.

(ii). One can homogenize (2.2) by introducing

$$s = \frac{\sigma}{\sqrt{|\xi|^2 + \sigma^2}}.$$

Then

$$u_1(x) = \int_0^\infty \int_{\mathbb{R}^n} \exp\left[i \frac{\sigma}{\sqrt{|\xi|^2 + \sigma^2}} \cdot \xi_1 + ix \cdot \xi \right] \tilde{a}(\sigma, x, \xi)\, d\xi d\sigma,$$

in which

$$\tilde{a}(\sigma, x, \xi) = a\left(\frac{\sigma}{\sqrt{|\xi|^2 + \sigma^2}}, x, \xi \right) \cdot \frac{|\xi|^2}{(|\xi|^2 + \sigma^2)^{\frac{3}{2}}}$$

is an element in $S_{cl}^{m-1/2-n/4}(\mathbb{R}^{n+1} \times \mathbb{R}^n)$.

(iii). If $a = 1$, then

$$u_1(x) = (2\pi)^n H(x_1)\delta(x'),$$

in which H is the Heaviside function.

Proposition (2.3)**.** *If* $u \in I^m_{comp}(\mathbb{R}^n; \tilde{\Lambda}_0, \tilde{\Lambda}_1),$ *then*

$$\mathrm{WF}(u) \subset \tilde{\Lambda}. \tag{2.4}$$

If $B \in L^0(\mathbb{R}^n),$ *then*

(i). If $\mathrm{WF}(B) \cap \tilde{\Lambda}_0 = \emptyset,$ *then* $Bu \in I^m(\mathbb{R}^n; \tilde{\Lambda}_1)$*.*

(ii). If $\mathrm{WF}(B) \cap \tilde{\Lambda}_1 = \emptyset,$ *then* $Bu \in I^{m-1/2}(\mathbb{R}^n; \tilde{\Lambda}_0)$*.*

Example. Let

$$a = 1 \in S^0_{cl}(\mathbb{R}^{n+1} \times \mathbb{R}^n).$$

Then

$$m = \frac{n}{4} - \frac{1}{2} \quad \text{and} \quad u(x) = (2\pi)^n H(x_1)\delta(x').$$

Furthermore, for some zeroth order ΨDO χ,

(i). $\chi u \in I^{n/4-1/2}(\mathbb{R}^n; \tilde{\Lambda}_1)$ for χ microlocally supported outside of $\tilde{\Lambda}_0$,

(ii). $\chi u \in I^{n/4-1}(\mathbb{R}^n; \tilde{\Lambda}_0)$ for χ microlocally supported outside of $\tilde{\Lambda}_1$.

Proof of Proposition 2.3.

Step 1. Write

$$\tilde{u}(s,x) = \int_{\mathbb{R}^n} e^{i[(x_1-s)\cdot\xi_1 + x'\cdot\xi']} a(s,x,\xi)\, d\xi.$$

Then

$$\mathrm{WF}(\tilde{u}) \subset \{(s,x,\sigma,\xi) : x_1 = s \geq 0, x' = 0, \sigma = -\xi_1\}.$$

Consider the projection to the second component:

$$\pi : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n, \quad \pi(s,x) = x.$$

Then for each $v \in C_0^\infty(\mathbb{R}^n)$,

$$\begin{aligned}
& \langle \pi_*(H(s)\tilde{u}(s,x)), v\rangle_{L^2(\mathbb{R}^n)} \\
= \; & \langle H(s)\tilde{u}(s,x), \pi^*(v)\rangle_{L^2(\mathbb{R}\times\mathbb{R}^n)} \\
= \; & \int_{\mathbb{R}} \int_{\mathbb{R}^n} H(s)\tilde{u}(s,x) \cdot \pi^* v(s,x)\, dx ds \\
= \; & \int_{\mathbb{R}^n} \int_0^\infty \int_{\mathbb{R}^n} e^{i[(x_1-s)\cdot\xi_1 + x'\cdot\xi']} a(s,x,\xi) d\xi ds \cdot v(x)\, dx \\
= \; & \langle u_1, v\rangle_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

It then follows that

$$u_1(x) = \pi_*(H(s)\tilde{u}(s,x)).$$

By the push-forward of wavefront of distributions (c.f. [D96, Proposition 1.3.4]),

$$\begin{aligned} \mathrm{WF}(\pi_* u) \subset \quad & \{(y,\eta) \in T^*(Y) \setminus 0 : \\ & y = \pi(x) : y = \pi(x), (x,^t D\pi_x \eta) \in \mathrm{WF}(u) \text{ for some } x \in X\} \end{aligned}$$

for a smooth mapping $\pi : X \to Y$. Therefore, observe that

$$\mathrm{WF}(H(s)\tilde{u}(s,x)) \subset [\mathrm{WF}(H(s)) + \mathrm{WF}(\tilde{u}(s,x))] \cup \mathrm{WF}(H(s)) \cup \mathrm{WF}(\tilde{u}(s,x)),$$

$${}^t D\pi_x = \begin{bmatrix} 0 \\ Id_{n\times n} \end{bmatrix}_{(n+1)\times n}$$

and

$$\mathrm{WF}(u_1) \subset \pi_* \mathrm{WF}(H(s)\tilde{u}(s,x)).$$

We now have that

$$\mathrm{WF}(u) = \mathrm{WF}(u_1) \subset \tilde{\Lambda}.$$

Here, we also used the facts that

$$\pi_*(\mathrm{WF}(H(s)) + \mathrm{WF}(\tilde{u}(s,x))) = \tilde{\Lambda}_0 \cup \tilde{\Lambda}_1,$$

$$\pi_*(\mathrm{WF}(H(s))) = \emptyset,$$

and

$$\pi_*(\mathrm{WF}(\tilde{u}(s,x))) = \tilde{\Lambda}_1.$$

Step 2. Let B be a properly supported zeroth order classical ΨDO. Then

$$B[e^{ix\cdot\xi}a(x,\xi)] = e^{ix\cdot\xi}(\mathcal{B}a)$$

defines a continuous linear map

$$\mathcal{B} : S^m_{cl} \to S^m_{cl},$$

preserving compactness of cone-supports.

In particular, B can be applied under the integral sign in (2.2) showing that Bu_1 is of the same form with a replaced by $\mathcal{B}a$.

Step 3. If

$$\mathrm{WF}(B) \cap \tilde{\Lambda}_0 = \emptyset,$$

we can then assume, by discarding a smoothing operator, that, for some $\varepsilon > 0$, $\mathcal{B}a = 0$ in $|x| \leq \varepsilon$.

Choose $\mu \in C^\infty(\mathbb{R})$ with

$$\mu(s) = \begin{cases} 1 & \text{if } s \geq \frac{1}{2}\varepsilon, \\ 0 & \text{if } s < \frac{1}{4}\varepsilon. \end{cases}$$

Then the Lagrangian distribution defined by

$$v_1(x) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i[(x_1 - s)\cdot\xi_1 + x'\cdot\xi']} \mu(s) \mathcal{B}a(s, x, \xi)\, d\xi ds$$

is an element in $I^m(\mathbb{R}^n; \tilde{\Lambda}_1)$.

To show that Bu is also in this space we shall verify that

$$Bu(x) - v_1(x) = \int_{\mathbb{R}} \int_{\mathbb{R}^n} e^{i[(x_1 - s)\cdot\xi_1 + x'\cdot\xi']} [1 - \mu(s)] \mathcal{B}a(s, x, \xi)\, d\xi ds \in C^\infty. \quad (2.5)$$

In fact, the operator

$$L = \frac{i[(x_1 - s)\partial_{\xi_1} + x'\partial_{\xi'}]}{(x_1 - s)^2 + |x'|^2}$$

satisfies that

$$L^t e^{i[(x_1 - s)\cdot\xi_1 + x'\cdot\xi']} = e^{i[(x_1 - s)\cdot\xi_1 + x'\cdot\xi']},$$

and has smooth coefficients on

$$\operatorname{supp}(1-\mu)\mathcal{B}a \subset \{(s,x,\xi) : s < \frac{1}{4}\varepsilon, |x| > \varepsilon\}.$$

By integration by parts it follows that $Bu - v_1 \in C^\infty$.

Step 4. If

$$\mathrm{WF}(B) \cap \tilde{\Lambda}_0 = \emptyset,$$

we can then assume $\mathcal{B}a = 0$ if

$$|x'|^2 + \frac{\xi_1^2}{|\xi|^2} \leq \varepsilon^2 \quad \text{and} \quad x_1 > -\epsilon.$$

It follows that the operator

$$M = -\frac{i(\xi_1\partial_s/|\xi|^2 - x'\partial_{\xi'})}{|x'|^2 + \xi_1^2/|\xi|^2}$$

satisfies that

$$M^t e^{i[(x_1-s)\cdot\xi_1+x'\cdot\xi']} = e^{i[(x_1-s)\cdot\xi_1+x'\cdot\xi']}$$

and has smooth coefficients on $\operatorname{supp}\mathcal{B}a$ provided $x_1 > -\varepsilon$.

Integration by parts yields

$$Bu = \int_0^\infty \int_{\mathbb{R}^n} e^{i[(x_1-s)\cdot\xi_1+x'\cdot\xi']} M[\mathcal{B}a(s,x,\xi)]\, d\xi ds \tag{2.6}$$

$$+ \int_{\mathbb{R}^n} e^{ix\cdot\xi} \cdot \frac{\xi_1 \mathcal{B}a(0,x,\xi)}{|\xi_1|^2+|x'|^2|\xi|^2}\, d\xi.$$

Here, the second term is a distribution in $I^{m-1/2}(\mathbb{R}^n;\tilde{\Lambda}_0)$ since the symbol

$$\frac{\xi_1 \mathcal{B}a(0,x,\xi)}{|\xi_1|^2+|x'|^2|\xi|^2} \in S^{m-1/2-n/4},$$

and first term is smooth by taking iterations.

Notice that the boundary terms from integration with respect to s is of lower order, therefore not affecting the principal symbol in $I^{m-1/2}(\mathbb{R}^n;\tilde{\Lambda}_0)$. $\square$

Remark (2.7). Ignoring non-vanishing densities and Maslov contributions we find the principal symbol of $u \in I^m_{comp}(\mathbb{R}^n; \tilde{\Lambda}_0, \tilde{\Lambda}_1)$ as a Lagrangian distribution near $\tilde{\Lambda}_0 \setminus \partial\tilde{\Lambda}_1$ and $\tilde{\Lambda}_1 \setminus \partial\tilde{\Lambda}_1$ as

$$\sigma^{(1)}(u) = a(x_1, (x_1, 0), (0, \xi')) \quad \text{on } \tilde{\Lambda}_1 \setminus \partial\tilde{\Lambda}_1, \tag{2.8}$$

and

$$\sigma^{(0)}(u) = -ia(0, 0, \xi)/\xi_1 \quad \text{on } \tilde{\Lambda}_0 \setminus \partial\tilde{\Lambda}_1. \tag{2.9}$$

Observe that $\sigma^{(1)}(u)$ is smooth up to $\partial\tilde{\Lambda}_1$ and that

$$\xi_1\sigma^{(0)}(u) = -i\sigma^{(1)}(u) \quad \text{on } \partial\tilde{\Lambda}_1, \tag{2.10}$$

which gives additional restriction on the symbol pair

$$\sigma(u) = (\sigma^{(1)}(u), \sigma^{(0)}(u)),$$

and this shall be fully investigated together with defining half densities and Maslov bundle on $(\tilde{\Lambda}_0, \tilde{\Lambda}_1)$ in §4.

3 The space $I^m(X;\Lambda_0,\Lambda_1)$

Suppose that (Λ_0,Λ_1) is an intersecting pair of Lagrangian manifolds in X. Given $\lambda \in \Lambda_0 \cap \Lambda_1$, we can, according to Proposition 1.3, find a local parametrization χ of the intersecting pair as $(\tilde{\Lambda}_0,\tilde{\Lambda}_1)$, which allows us to define $I^m(X;\Lambda_0,\Lambda_1)$.

Definition (3.1). $I^m(X;\Lambda_0,\Lambda_1)$ *consists of those distributional half densities, u, on X which are modelled microlocally by*

$$u - u_0 - u_1 - \sum_j F_j v_j \in C^\infty, \quad \textit{in which}$$

(i). $u_0 \in I^{m-1/2}(X;\Lambda_0)$ and $u_1 \in I^m(X;\Lambda_1 \setminus \partial\Lambda_1)$.

(ii). $\{V_j\}$ is a countable covering of $\partial\Lambda_1$ by parametrization

$$\chi_j : V_j \to T^*(\mathbb{R}^n) \quad \textit{and} \quad \partial\Lambda_1 \subset \cup_j V_j.$$

(iii). $\{F_j\}$ is a family of FIOs that F_j is associated to χ_j^{-1}.

(iv). The distributions $v_j \in I^m_{comp}(\mathbb{R}^n;\tilde{\Lambda}_0,\tilde{\Lambda}_1)$ such that $F_j v_j$ have locally finite supports.

Remark. Immediately following the definition, let $u \in I^m(\mathbb{R}^n; \tilde{\Lambda}_0, \tilde{\Lambda}_1)$ and F be a properly supported zeroth order FIO associated with a canonical transformation that leaving both $\tilde{\Lambda}_0$ and $\tilde{\Lambda}_1$ invariant. Then $Fu \in I^m(\mathbb{R}^n; \tilde{\Lambda}_0, \tilde{\Lambda}_1)$.

In fact, the following stronger result holds.

Proposition (3.2). *Let*

$$F : \mathcal{E}'(\mathbb{R}^n) \to \mathcal{E}'(\mathbb{R}^n)$$

be an FIO of order m' associated with a local canonical graph Γ such that

$$\Gamma \circ \tilde{\Lambda}_0 \subset \tilde{\Lambda}_0 \quad \textit{and} \quad \Gamma \circ \tilde{\Lambda}_1 \subset \tilde{\Lambda}_1.$$

Then

$$F : I^m_{comp}(\mathbb{R}^n; \tilde{\Lambda}_0, \tilde{\Lambda}_1) \to I^{m+m'}_{comp}(\mathbb{R}^n; \tilde{\Lambda}_0, \tilde{\Lambda}_1). \tag{3.3}$$

Proof of Proposition 3.2.

Step 1. Suppose that $u \in I^m_{comp}(\mathbb{R}^n; \tilde{\Lambda}_0, \tilde{\Lambda}_1)$ and can be written as

$$u(y) = \int_0^\infty \int_{\mathbb{R}^n} e^{i[(y_1 - s)\cdot\eta_1 + y'\cdot\eta']} a(s, y, \eta)\, d\eta ds,$$

where $a \in S_{cl}^{m+1/2-n'/4}$. Suppose that

$$Fv(x) = \int_{\mathbb{R}^{2n}} \int_{\mathbb{R}^n} e^{i\phi(x,y,\xi,\eta)} b(x,y,\xi,\eta) v(y)\, dy d\xi d\eta.$$

Using [H71], we may assume

$$\phi(x,y,\xi,\eta) = x \cdot \xi - y \cdot \eta - H(\xi,\eta)$$

is a non-degenerate phase function defining

$$\Gamma = \{(H'_\xi, \xi, H'_\eta, \eta)\} = \{(y,\eta) = \chi(x,\xi)\}$$

for some canonical transformation χ.

From Proposition 4.1.3, Γ is locally canonical graph iff

$$D(\phi) = \begin{bmatrix} \phi''_{\xi\xi} & \phi''_{\xi\eta} & \phi''_{\xi x} \\ \phi''_{y\xi} & \phi''_{y\eta} & \phi''_{yx} \end{bmatrix} = \begin{bmatrix} -H''_{\xi\xi} & -H''_{\xi\eta} & Id \\ 0 & Id & 0 \end{bmatrix} \neq 0 \quad \text{i.e. } H''_{\xi\eta} \neq 0.$$

Thus, we can reduce the number of ξ, η variables by n and write

$$Fv(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i\phi(x,y,\theta)} b(x,y,\theta) v(y)\, dy d\theta.$$

Step 2. Now

$$
\begin{aligned}
&Fu(x)\\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\phi(x,y,\theta)} b(x,y,\xi,\eta)\left[\int_0^\infty \int_{\mathbb{R}^n} e^{i[(y_1-s)\cdot\eta_1+y'\cdot\eta']} a(s,y,\eta) d\eta ds\right] dy d\theta\\
&= \int_0^\infty \int_{\mathbb{R}^n} \left[\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i\psi(x,y,s,\theta,\eta)} ba\, dy d\eta\right] d\theta ds
\end{aligned}
$$

with $\psi(x,y,s,\theta,\eta) = \phi(x,y,\theta) + (y_1 - s)\cdot\eta_1 + y'\cdot\eta'$. Notice that

$$d_\eta\psi = 0 \Leftrightarrow y_1 = s, y_2 = \cdots = y_n = 0,$$

$$d_y\psi = 0 \Leftrightarrow \eta = -\phi'_\theta,$$

$$d^2_{(y,\eta)}\psi = \begin{bmatrix} 0 & Id \\ Id & 0 \end{bmatrix}, \quad \text{i.e. } \psi \text{ is non-degenerate.}$$

Applying stationary phase to the inner integral, the main contribution is from

$$Fu(x) = \int_0^\infty \int_{\mathbb{R}^n} e^{i\phi(x,(s,0),\theta)} c(s,x,\theta)\, d\theta ds, \tag{3.4}$$

in which $\bar\phi(x,s,\theta) = \phi(x,(s,0),\theta)$ is non-degenerate.

Step 3. We can always decompose F by using a microlocal partition of unity. So assume that in the region of interest

$$\Gamma \circ \tilde{\Lambda}_0 = \tilde{\Lambda}_0 \quad \text{and} \quad \Gamma \circ \tilde{\Lambda}_1 = \tilde{\Lambda}_1,$$

which means that

$$\chi \tilde{\Lambda}_0 = \tilde{\Lambda}_0 \quad \text{and} \quad \chi \tilde{\Lambda}_1 = \tilde{\Lambda}_1.$$

Notice also that $\chi \tilde{\Lambda}_0 = \tilde{\Lambda}_0 = \{x = 0\}$ implies

$$\bar{\phi}'_\theta = 0, s = 0 \text{ (i.e., } y = 0) \Leftrightarrow x = 0.$$

It then follows that

$$\frac{\partial \bar{\phi}}{\partial \theta} = \sum_i c_{ji} \left[x_i - s\alpha_i(x, s, \theta) \right],$$

where the matrix c is invertible since ϕ is non-degenerate.

Replacing θ_i by $\sum_j c_{ji}\theta_j$ reduces $\bar{\phi}$ to the form

$$\bar{\phi}(x, s, \theta) = \theta \cdot x - s\alpha(x, s, \theta). \tag{3.7}$$

Write

$$\alpha = \alpha(0, 0, \theta) + x \cdot \beta(x, s, \theta) + s\gamma(x, s, \theta)$$

and

$$\theta_i - s\beta_i(x, s, \theta)$$

as new θ variables. It then allows us to assume that (3.7) is

$$\bar{\phi}(x, s, \theta) = \theta \cdot x - s\alpha(\theta) + s^2\gamma(x, s, \theta). \tag{3.8}$$

Furthermore, $\Gamma \circ \tilde{\Lambda}_1 = \tilde{\Lambda}_1 = \{x_1 \geq 0, x' = 0, \xi_1 = 0\}$ implies

$$\bar{\phi}'_\theta = 0, \bar{\phi}'_s = 0 \text{ (i.e., } \phi'_y = 0) \Leftrightarrow x' = 0, \bar{\phi}'_{x_1} = 0.$$

It then follows that

$$x_i - s\frac{\partial \alpha}{\partial \theta_i} + s^2\frac{\partial \gamma}{\partial \theta_i} = 0,\ -\alpha(\theta) + \frac{\partial(s^2\gamma)}{\partial s} = 0. \tag{3.9}$$

So it forces $x' = 0$ and $x_1 \geq 0$ when $s = 0$. Hence,

$$\frac{\partial \alpha}{\partial \theta_1} > 0.$$

Thus on the surface C defined by (3.9), s and $\theta' = (\theta_2, ..., \theta_n)$ can be taken as coordinates.

Moreover, $x_i = 0$ on C for $i \geq 2$ so differentiating with respect to s and setting $s = 0$ in (3.9), we have

$$\frac{\partial \alpha}{\partial \theta'} = 0 \quad \text{when } \alpha = 0.$$

Since $\partial\alpha/\partial\theta_1 \neq 0$,

$$\alpha(\theta) = \theta_1 - \rho(\theta')\beta(\theta)$$

with ρ homogeneous of degree one. But $\partial\rho/\partial\theta' = 0$ so

$$\rho = 0 \text{ and } \alpha(\theta) = \theta_1\beta(\theta).$$

Replacing s by $s\beta(\theta)$ and relabelling gives

$$\bar{\phi}(x, s, \theta) = \theta \cdot x - s\theta_1 + s^2\gamma(x, s, \theta).$$

Step 4. In fact, repeating this process implies for any $k \in \mathbb{N}$,

$$\bar{\phi}(x, s, \theta) = \theta \cdot x - s\theta_1 + s^k\gamma(x, s, \theta).$$

That is, $\bar{\phi}$ differs from the linear phase function

$$-s\xi_1 + x \cdot \xi,\ \ s > 0$$

by a term vanishes to higher order in s. Then we apply the technique in [H71, §3.1] to prove that they are equivalent. $\square$

4 Symbol maps

Proposition (4.1)**.** *If $u \in I^m(X; \Lambda_0, \Lambda_1)$, then*

$$\mathrm{WF}(u) \subset \Lambda = \Lambda_0 \cup \Lambda_1.$$

If $B \in L^0(\mathbb{R}^n)$ is a properly supported classical ΨDO, then

(i).

$$\mathrm{WF}(B) \cap \Lambda_0 = \emptyset \Rightarrow Bu \in I^m(X; \Lambda_1 \setminus \partial\Lambda_1). \tag{4.2}$$

(ii).

$$\mathrm{WF}(B) \cap \Lambda_1 = \emptyset \Rightarrow Bu \in I^{m-1/2}(X; \Lambda_0 \setminus \partial\Lambda_1). \tag{4.3}$$

Remark.

(i). The proof follows from the invariance in Propositions 3.2 and corresponding results in $\mathbb{R}^n$ in Proposition 2.3.

(ii). The principal symbols of u near Λ_1 and Λ_0 are

$$\sigma^{(1)}(u)(\lambda') = \frac{\sigma_m(Bu)(\lambda')}{\sigma_0(B)(\lambda')} \quad \text{on } \Lambda_1 \setminus \partial\Lambda_1, \tag{4.4}$$

in which $\sigma_0(B)(\lambda') \neq 0$ for $\lambda \in \Lambda_1 \setminus \partial\Lambda_1$ and λ' is near λ. This is independent of B observing the standard trivialisation of $\Omega^{1/2}$ and L over the diagonal, and

$$\sigma^{(0)}(u)(\lambda') = \frac{\sigma_{m-1/2}(Bu)(\lambda')}{\sigma_0(B)(\lambda')} \quad \text{on } \Lambda_0 \setminus \partial\Lambda_1, \tag{4.5}$$

in which $\sigma_0(B)(\lambda) \neq 0$ for $\lambda \in \Lambda_0 \setminus \partial\Lambda_1$ and λ' is near λ.

(iii). We have the symbol map

$$\sigma : I^m(X; \Lambda_0, \Lambda_1) \to \tag{4.6}$$

$$C^\infty_{m+n/4}(\Lambda_1 \setminus \partial\Lambda_1; \Omega^{1/2} \otimes L_1) \otimes C^\infty_{m+n/4-1/2}(\Lambda_0 \setminus \partial\Lambda_1; \Omega^{1/2} \otimes L_0),$$

where C^∞_k is the space of C^∞ sections which are homogeneous of degree k, is apparently not surjective. We shall investigate the image under this map.

(iv). Recall (2.10) that

$$\lim_{x_1\to 0}\sigma^{(1)}(u)=\lim_{\xi_1\to 0}\sigma^{(0)}(u),$$

and

$$\xi_1\sigma^{(0)}(u)=-i\sigma^{(1)}(u)\quad\text{on }\partial\tilde{\Lambda}_1,\tag{2.10}$$

which shows that the mapping in (4.6) is not surjective.

Consider only the half densities on $\tilde{\Lambda}_0$, $\tilde{\Lambda}_1$, and $\tilde{\Lambda}_0\cap\tilde{\Lambda}_1=\partial\tilde{\Lambda}_1$:

$$\sigma^{(1)}(u)=a(x_1,(x_1,0),(0,\xi'))|dx_1\wedge d\xi_2\wedge\cdots\wedge d\xi_n|^{\frac{1}{2}},$$

and

$$\sigma^{(0)}(u)=-i\xi_1^{-1}a(0,0,(\xi_1,\xi'))|d\xi_1\wedge d\xi_2\wedge\cdots\wedge d\xi_n|^{\frac{1}{2}}.$$

Then the above correspondence between $\sigma^{(1)}(u)$ and $\sigma^{(0)}(u)$ is intrinsically defined, that is, independent of the choice of local coordinates.

We further denote such correspondence as

$$\sigma^{(1)}(u)=R'\sigma^{(0)}(u).$$

We remark that the homogeneous degree of $\sigma^{(0)}(u)$ is 1/2 lower than that of $\sigma^{(1)}(u)$. (-1 from ξ_1^{-1}, $+1/2$ from $|d\xi_n|^{1/2}$.)

Lemma (4.9'). *Suppose $\lambda_0 \in \partial\Lambda_1$. One can then choose $n-1$ functions $h_2, ..., h_n$ which have independent differentials on $\partial\Lambda_1$ near λ_0. In addition, pick f such that*

(i). f vanishes on Λ_0 with homogeneous degree 0,

(ii). f is positive on $\Lambda_1 \setminus \partial\Lambda_1$,

(iii). f has differential non-zero on Λ_1 near λ_0,

and g such that

(i). g vanishes on Λ_1 with homogeneous degree 1,

(ii). g has differential non-zero on Λ_0 near λ_0,

(iii). $\{f, g\}(\lambda_0) < 0$.

We can then find the map R' such that for each

$$a \in C^\infty_{m-1/2}(\Lambda_0 \setminus \partial\Lambda_1; \Omega^{1/2}(\Lambda_0)),$$

with $ga \in C^\infty(\Lambda_0; \Omega^{1/2}(\Lambda_0))$, we have that

$$b = R'a \in C^\infty_m(\partial\Lambda_1; \Omega^{1/2}(\Lambda_1)).$$

Furthermore, the definition of R' is independent of $h_2, ..., h_n$, f, and g.

Proof of Lemma 4.9'.

Step 1. Choosing $h_2, ..., h_n$, f, and g can be done similarly by the construction in Proposition 1.3 because the intersection is clean. Therefore, one can regard $g, h_2, ..., h_n$ as local coordinates on Λ_0 near λ_0, and $f, h_2, ..., h_n$ as local coordinates on Λ_1 near λ_0. The basic example is

$$\{h_2, ..., h_n\} = \{\xi_2, ..., \xi_n\}, \quad f = x_1, \quad \text{and} \quad g_1 = \xi_1.$$

Step 2. Choose $a \in C^\infty_{m-1/2}(\Lambda_0 \setminus \partial\Lambda_1; \Omega^{1/2}(\Lambda_0))$ that

$$ga = r|dg \wedge dh_2 \wedge \cdots \wedge dh_n|^{1/2} \in C^\infty(\Lambda_0; \Omega^{1/2}(\Lambda_0)).$$

This means that r is smooth in Λ_0 and

$$a = g^{-1} r|dg \wedge dh_2 \wedge \cdots \wedge dh_n|^{1/2}.$$

Define

$$b = r|df \wedge dh_2 \wedge \cdots \wedge dh_n|^{1/2}(\{g, f\})^{-1/2} = R'a. \tag{4.7}$$

Then

$$b \in C^\infty_m(\partial\Lambda_1; \Omega^{1/2}(\Lambda_1))$$

restricted to $\partial\Lambda_1$.

Step 3. The map R' is clearly independent of the choice of $h_2, ..., h_n$. We have to show that it is also independent of the choice of f and g.

Choose $n-1$ functions $k_2, ..., k_n$ which vanishes on $\Lambda = \Lambda_0 \cup \Lambda_1$ and have independent differentials on near λ_0 (with homogeneous degree 0). Then another choice of f_1 would be represented as

$$f_1 = \alpha f + \sum \beta_j k_j$$

with $\alpha > 0$ on $\partial\Lambda_1$ because both f and f_1 are positive on $\Lambda_1 \setminus \partial\Lambda_1$.

Note that H_g is tangent to Λ_1. Then at $\partial\Lambda_1$ on Λ_1, we deduce that

$$\{f_1, g\} = \alpha\{f, g\} < 0,$$

and

$$df_1 \wedge dh_2 \wedge \cdots \wedge dh_n = \alpha df \wedge dh_2 \wedge \cdots \wedge dh_n.$$

Moreover, (4.7) is kept invariant under this change.

One can similarly change g and verify that (4.7) is well-defined depending only on the intersecting pair. $\square$

Lemma (4.9)**.** *For each* $\lambda \in \partial\Lambda_1$, *the fibres* $(L_0)_\lambda$ *and* $(L_1)_\lambda$ *are naturally isomorphic. Thus there is a well-defined Maslov bundle* L *over* $\Lambda = \Lambda_0 \cup \Lambda_1$.

Remark. Lemma 4.9' and Lemma 4.9 together induce a map

$$R : C^\infty_{m-1/2}(\Lambda_0 \setminus \partial\Lambda_1; \Omega^{1/2}(\Lambda_0) \otimes L_0) \to C^\infty_m(\partial\Lambda_1; \Omega^{1/2}(\Lambda_1) \otimes L_1)$$

as a tensor product of R' with the Maslov bundle for

$$a \in C^\infty_{m-1/2}(\Lambda_0 \setminus \partial\Lambda_1; \Omega^{1/2}(\Lambda_0) \otimes L_0)$$

such that

$$ga \in C^\infty(\Lambda_0; \Omega^{1/2}(\Lambda_0)).$$

Definition (4.11)**.** *We define*

$$S_m(\Lambda; \Omega^{1/2} \otimes L_1) \subset$$
$$C^\infty_{m+n/4}(\Lambda_1; \Omega^{1/2} \otimes L_1) \otimes C^\infty_{m+n/4-1/2}(\Lambda_0 \setminus \partial\Lambda_1; \Omega^{1/2} \otimes L_0)$$

as the subspace consisting of those sections (b, a) *such that the following conditions are satisfied.*

(i). Let g *be with homogeneous degree* 1*. Then*

$$ga \text{ is } C^\infty \text{ on } \Lambda_0 \text{ whenever } g \text{ vanishes on } \partial\Lambda_1. \tag{4.10}$$

(ii). The projection

$$supp(a) \cup supp(b) \to X$$

is proper.

(iii).

$$b|\partial\Lambda_1 = e^{i\pi/4}(2\pi)^{1/4} Ra. \tag{4.12}$$

Theorem (4.13)**.** *The sequence*

$$0 \hookrightarrow I^{m-1}(X;\Lambda_0,\Lambda_1) \hookrightarrow I^m(X;\Lambda_0,\Lambda_1) \xrightarrow{\sigma} S_{m+n/4}(\Lambda;\Omega^{1/2}\otimes L) \to 0.$$

is exact.

Definition (Exact sequences)**.** *A sequence*

$$G_0 \xrightarrow{f_1} G_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} G_n$$

of groups and group homomorphisms is called exact if the image of each homomorphism is equal to the kernel of the next, i.e.

$$\mathrm{Im}(f_k) = \ker(f_{k+1}).$$

A short exact sequence of abelian groups

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

forces f to be one-to-one and g to be onto, moreover,

$$C \simeq B/f(A).$$

Proof of Theorem 4.13.

Step 1. In the view of the invariance proved in Proposition 3.2, we only need to verify in $\mathbb{R}^n$.

Step 2. We show that σ is surjective. Choose any pair

$$(b,a) \in C^\infty_{m+n/4}(\tilde{\Lambda}_1; \Omega^{1/2} \otimes L_1) \otimes C^\infty_{m+n/4-1/2}(\tilde{\Lambda}_0 \setminus \partial\tilde{\Lambda}_1; \Omega^{1/2} \otimes L_0)$$

such that the conditions in Definition 4.11 are satisfied. Then one can find

$$c(s,x,\xi) \in S^{m+1/2-n/4}_{cl}(\mathbb{R}^{n+1} \times \mathbb{R}^n)$$

such that

$$b = c(x_1,(x_1,0),(0,\xi'))|dx_1 \wedge d\xi_2 \wedge \cdots \wedge d\xi_n|^{\frac{1}{2}}, \; x_1 \geq 0,$$

and

$$a = -i\xi_1^{-1}c(0,0,(\xi_1,0))|d\xi_1 \wedge d\xi_2 \wedge \cdots \wedge d\xi_n|^{\frac{1}{2}}.$$

Therefore, the distribution $u \in I^m(\mathbb{R}^n; \tilde{\Lambda}_0, \tilde{\Lambda}_1)$ defined by

$$u(x) = \int_0^\infty \int_{\mathbb{R}^n} e^{i[(x_1-s)\cdot\xi_1 + x'\cdot\xi']} c(s,x,\xi) d\xi ds \quad \mod C^\infty$$

has the property that

$$\sigma(u) = (b,a).$$

Step 3. We show that

$$I^{m-1}(X;\Lambda_0,\Lambda_1)\subset\ker(\sigma).$$

That is, if

$$u\in I^{m-1}(X;\Lambda_0,\Lambda_1)\subset I^m(X;\Lambda_0,\Lambda_1),$$

then the symbol pair $\sigma(u)=(b,a)=0$, in which

$$(b,a)\in C^\infty_{m+n/4}(\Lambda_1;\Omega^{1/2}\otimes L_1)\otimes C^\infty_{m+n/4-1/2}(\Lambda_0\setminus\partial\Lambda_1;\Omega^{1/2}\otimes L_0).$$

From Proposition 4.1, for some $B\in L^0(X)$ as a properly supported ΨDO, if $\mathrm{WF}(B)\cap\Lambda_0=\emptyset$, then pick $\sigma_0(B)\neq 0$ near λ, using the standard trivialisation of $\Omega^{1/2}$ and L over the diagonal, the principal symbol of u near Λ_1 is

$$\frac{\sigma_m(Bu)(\lambda)}{\sigma_0(B)(\lambda)}\in C^\infty_{m+n/4-1}(\Lambda_1;\Omega^{1/2}\otimes L_1)\quad\text{on }\Lambda_1.$$

Therefore,

$$\sigma_m(Bu)(\lambda)=0\quad\text{and}\quad b=0.$$

If $\mathrm{WF}(B) \cap \Lambda_1 = \emptyset$, then pick $\sigma_0(B) \neq 0$ near λ, and the principal symbol of u near $\Lambda_0 \setminus \partial\Lambda_1$ is

$$\sigma^{(0)}(u)(\lambda) = \frac{\sigma_{m-1/2}(Bu)(\lambda)}{\sigma_0(B)(\lambda)} \in C^\infty_{m+n/4-3/2}(\Lambda_0 \setminus \partial\Lambda_1; \Omega^{1/2} \otimes L_0) \quad \text{on } \Lambda_0 \setminus \partial\Lambda_1.$$

Therefore,

$$\sigma_{m-1}(Bu)(\lambda) = 0 \quad \text{and} \quad a = 0.$$

Step 4. We show that

$$\ker(\sigma) \subset I^{m-1}(\mathbb{R}^n; \tilde{\Lambda}_0, \tilde{\Lambda}_1),$$

which implies that

$$\sigma : I^m(\mathbb{R}^n; \tilde{\Lambda}_0, \tilde{\Lambda}_1)/I^{m-1}(\mathbb{R}^n; \tilde{\Lambda}_0, \tilde{\Lambda}_1) \to S_{m+n/4}(\tilde{\Lambda}; \Omega^{1/2} \otimes L)$$

is injective.

Let $u \in I^m(\mathbb{R}^n; \tilde{\Lambda}_0, \tilde{\Lambda}_1)$ defined by

$$u(x) = \int_0^\infty \int_{\mathbb{R}^n} e^{i[(x_1-s)\cdot\xi_1 + x'\cdot\xi']} c(s, x, \xi)\, d\xi ds \quad \mod C^\infty. \tag{2.2}$$

Here,

$$c(s,x,\xi) \in S_{cl}^{m+1/2-n/4}(\mathbb{R}^{n+1} \times \mathbb{R}^n),$$

and $\sigma(u) = (b,a) = 0$. We need to prove that

$$u \in I^{m-1}(\mathbb{R}^n; \tilde{\Lambda}_0, \tilde{\Lambda}_1).$$

Since c vanishes on $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_0 \setminus \partial\tilde{\Lambda}_1$, i.e.

$$c(x_1,(x_1,0),(0,\xi')) = 0, \quad x_1 \geq 0,$$

and

$$c(0,0,(0,\xi')) = 0,$$

we have that

$$c(s,x,\xi) = \sum_{j=2}^{n} x_j d_j(s,x,\xi) + (x_1 - s)e(s,x,\xi) + \xi_1 f(s,x,\xi).$$

Here, d_j's and e are of homogeneous degree $m+1/2-n/4$, and f is of homogeneous degree $m-1/2-n/4$ and satisfies

$$f = 0 \text{ if } s = x_1 \geq 0 \text{ and } x' = 0.$$

Therefore, (2.2) breaks into three integrals, and the first two are
$$\int_0^\infty \int_{\mathbb{R}^n} e^{i[(x_1-s)\cdot\xi_1+x'\cdot\xi']} x_j d_j(s,x,\xi)\,d\xi ds = \int_0^\infty \int_{\mathbb{R}^n} e^{i[(x_1-s)\cdot\xi_1+x'\cdot\xi']} D_j d_j(s,x,\xi)\,d\xi ds,$$
and
$$\begin{aligned} &\int_0^\infty \int_{\mathbb{R}^n} e^{i[(x_1-s)\cdot\xi_1+x'\cdot\xi']} (x_1-s)e(s,x,\xi)\,d\xi ds \\ = &\int_0^\infty \int_{\mathbb{R}^n} e^{i[(x_1-s)\cdot\xi_1+x'\cdot\xi']} D_j e(s,x,\xi)\,d\xi ds, \end{aligned}$$
which are both in $I^{m-1}(\mathbb{R}^n;\tilde{\Lambda}_0,\tilde{\Lambda}_1)$. We concentrate on
$$\int_0^\infty \int_{\mathbb{R}^n} e^{i[(x_1-s)\cdot\xi_1+x'\cdot\xi']} \xi_1 f(s,x,\xi)\,d\xi ds.$$
We can write
$$f(s,x,\xi) = (x_1-s)g(s,x,\xi) + \sum_{j=2}^n x_j h_j(s,x,\xi).$$
Similar integration by parts reduces the order of $\xi_1 f(s,x,\xi)$ by one and yields that the integral is also in $I^{m-1}(\mathbb{R}^n;\tilde{\Lambda}_0,\tilde{\Lambda}_1)$. The proof is complete. $\square$

5 Symbol calculus

Recall that

$$I^{m-1/2}(X;\Lambda_0) \subset I^m(X;\Lambda_0,\Lambda_1), \tag{5.1}$$

and that $u \in I^m(X;\Lambda_0,\Lambda_1)$ can be written

$$u = u_0 + u_1 + \sum F_j v_j$$

such that $u_0 \in I^{m-1/2}(X;\Lambda_0)$, $u_1 \in I^m(X;\Lambda_1 \setminus \partial\Lambda_1)$, and $v_j \in I^m(\mathbb{R}^n;\tilde{\Lambda}_0,\tilde{\Lambda}_1)$. Then we have the sequence

$$\begin{aligned} 0 \hookrightarrow I^{m-1/2}(X;\Lambda_0) + I^{m-1}(X;\Lambda_0,\Lambda_1) \hookrightarrow \\ I^m(X;\Lambda_0,\Lambda_1) \to S_{m+n/4}(\Lambda_1;\Omega^{1/2}\otimes L) \to 0. \end{aligned} \tag{5.2}$$

Here, $S_{m+n/4}(\Lambda_1;\Omega^{1/2}\otimes L) = C^\infty_{m+n/2}(\Lambda_1;\Omega^{1/2}\otimes L)$.

One can also carry out the symbol calculus for such distributions as we did for Lagrangian distribution and FIOs, e.g.

$$F : I^m(X;\Lambda_0,\Lambda_1) \to I^{m+k}(X;C\circ\Lambda_0,C\circ\Lambda_1)$$

for some $F \in I^k(X\times Y, C)$ associated with a canonical relation $C \subset T^*(X)\times T^*(Y)$ which meets Λ_0 and Λ_1 transversally.

Borrowing from Theorem 5.3.1 in [DH72], we have

Proposition (5.4)**.** *Let $P \in L^m(X)$ be a properly supported classical ΨDO such that $p = \sigma(P)$ vanishes on the part Λ_1 of an intersecting pair (Λ_0, Λ_1) of Lagrangians. Then $u \in I^k(X; \Lambda_0, \Lambda_1)$ implies that*

$$Pu = f + g.$$

Here,

(i). $f \in I^{m+k-1/2}(X; \Lambda_0)$,

(ii). $g \in I^{m+k-1}(X; \Lambda_0, \Lambda_1)$,

(iii). $\sigma(g)|\Lambda_1 = (-i\mathcal{L}_{H_p} + c)\sigma(u)|\Lambda_1$, in which $\mathcal{L}_{H_p}$ is the Lie action of H_p and c is the subprincipal symbol of P.

Proposition (5.5). *Let*

$$u_j \in I^{m-j}(X; \Lambda_0, \Lambda_1) \text{ for } j = 0, 1, ...$$

with $\cup_j \mathrm{WF}(u_j)$ *a proper conic subset of* $T^*(X) \setminus 0$.

Then there exists $u \in I^m(X; \Lambda_0, \Lambda_1)$ *such that for every* N *there is an* N' *for which*

$$u - \sum_{j=0}^{N'} \in C^N(X).$$

We write

$$u \sim \sum_j u_j,$$

in which u *is uniquely determined modulo* $C^\infty(X)$.

6 Operators of real principal type

Let $P \in L^m(Y)$ be of real principal type and Y is pseudo-convex with respect to P. We want to find the parametrix of P, that is,

$$P(y, D_y)E_e(y, y') = \delta(y, y') \mod C^\infty(Y \times Y),$$

where P is the lift to $Y \times Y$ under the first projection $X = Y \times Y \to Y$ of a ΨDO on Y. Write

$$\Lambda_0 = \{(y, \eta, y, -\eta) \in T^*(Y) \setminus 0 \times T^*(Y) \setminus 0\}.$$

Then

$$f = \delta(y, y') \in I^0(X; \Lambda_0)$$

and the question is reduced to finding the solution to

$$Pu = f \mod C^\infty(X).$$

For each choice of orientation Λ_1^e on $C \setminus \Delta_N$, we construct a solution in the following proposition that

$$E_e \in I^{-m+1/2}(X; \Lambda_0, \Lambda_1^e),$$

and we recover the right parametrices in Theorem 6.5.3 of [DH72].

Let $\Lambda_0 \subset X$ be an embedded conic Lagrangian submanifold, we want to build an intersecting pair (Λ_0, Λ_1) for the equation

$$Pu = f \in I^k(X, \Lambda_0).$$

Assume

$$H_p \text{ is nowhere tangent to } \Lambda_0 \text{ on } \Lambda_0 \cap N, \tag{6.1}$$

which implies N is a conic hypersurface near $\partial\Lambda_1 = \Lambda_0 \cap N$ and $\mathcal{H} = \mathrm{span}\{H_p\}$ is a line bundle on $\partial\Lambda_1$. We also require

$$\mathcal{H} \text{ has an orientation } e_j \text{ on each component } (\partial\Lambda_1)_j \text{ of } \partial\Lambda_1, \tag{6.2}$$

where $e_j = \pm 1$ as the orientation is that of $\pm H_p$. We also need the non-return condition (6.3), which is trivially satisfied in the product case $X = Y \times Y$:

"Given a relatively open cone $\gamma \in (\partial\Lambda_1)_j$, lying over a compact set, there exists $\gamma' \subset N$ with $\gamma \subset \gamma'$ such that any integral curve of $e_k H_p$ starting on $(\partial\Lambda_1)_k$ which means γ' starts in γ, and $k = j$."

(6.1)-(6.3) yield that the union over j of the forward flow-outs of the $(\partial\Lambda_1)_j$ under $e_j H_p$ is an embedded Lagrangian submanifold $\Lambda_1^e \subset T^*(X) \setminus 0$ with boundary (c.f. Guillemin and Melrose, *The Poisson summation formula for manifolds with boundary.* Adv. in Math. 32 (1979), no. 3, 204–232.)

Proposition (6.6)**.** *If P and Λ_0 satisfy (6.1)-(6.3), then given $f \in I^k(X;\Lambda_0)$ there is $u \in I^{k-m+1/2}(X;\Lambda_0,\Lambda_1^e)$ such that $Pu - f \in C^\infty(X)$.*

Proof of Proposition 6.6.

Step 1. First we assume $u_0 \in I^{k-m+1/2}(X;\Lambda_0,\Lambda_1^e)$. Proposition 5.4 yields that

$$Pu_0 = f_0 + g_0,$$

Here,

(i). $f_0 \in I^k(X;\Lambda_0)$,

(ii). $g_0 \in I^{k-1/2}(X;\Lambda_0,\Lambda_1^e)$,

(iii). $\sigma(f_0) = \sigma_m(P)\sigma(u_0)|\Lambda_0$.

It then follows that

$$\sigma(u_0) = p^{-1}\sigma(f) \quad \text{on } \Lambda_0$$

by letting $f_0 = f$. Moreover, Theorem 4.13 says

$$\sigma(u_0)|\partial\Lambda_1 = e^{i\pi/4}(2\pi)^{1/4}R(p^{-1}\sigma(f)). \tag{6.7}$$

Proposition 5.4 also yields that

$$\sigma(g_0)|\Lambda_1^e = (-i\mathcal{L}_{H_p} + c)\sigma(u_0)|\Lambda_1^e.$$

Hence, $\sigma(g_0) = 0$ on Λ_1 reduces the above equation to a transport equation for $\sigma(u_0)$ with initial condition given by (6.7),

$$(\mathcal{L}_{H_p} + ic)\sigma(u_0) = 0 \text{ on } \Lambda_1^e. \tag{6.8}$$

The existence and uniqueness of solutions to (6.8) are guaranteed by the conditions (6.1)-(6.3). That is, we find u_0 such that

$$Pu_0 - f = g_0 \in I^{k-1/2}(X; \Lambda_0, \Lambda_1^e),$$

in which $\sigma(g_0) = 0$ on Λ_1^e. Furthermore, write

$$Pu_0 - f = f_1 + g_1 \in I^{k-1}(X; \Lambda_0) + I^{k-3/2}(X; \Lambda_0, \Lambda_1^e).$$

Then similarly we find u_1 such that

$$-Pu_1 - f_1 - g_1 = f_2 + g_2 \in I^{k-2}(X; \Lambda_0) + I^{k-5/2}(X; \Lambda_0, \Lambda_1^e).$$

That is,

$$P(u_0 + u_1) - f \in I^{k-2}(X; \Lambda_0) + I^{k-5/2}(X; \Lambda_0, \Lambda_1^e).$$

Step 2. Repeat the procedure in Step 1, one can find

$$u_j \in I^{k-m+1/2-j}(X;\Lambda_0,\Lambda_1^e),$$

and

$$f_j \in I^{k-j}(X;\Lambda_0) \quad \text{and} \quad g_j \in I^{k-1/2-j}(X;\Lambda_0,\Lambda_1^e)$$

such that

$$\sigma(u_j) = p^{-1}\sigma(f_j) \quad \text{on } \Lambda_0, \tag{6.9}$$

$$(\mathcal{L}_{H_p} + ic)\sigma(u_j) = i\sigma(g_j) \quad \text{on } \Lambda_1^e \tag{6.10}$$

with the initial condition given by

$$\sigma(u_j)|\partial\Lambda_1 = e^{i\pi/4}(2\pi)^{1/4}R(p^{-1}\sigma(f_j)).$$

Therefore,

$$P\left(\sum_{j=0}^{N} u_j\right) - f \in I^{k-N-1}(X;\Lambda_0) + I^{k-N-1-1/2}(X;\Lambda_0,\Lambda_1^e),$$

and $u \sim \sum_j u_j$ solves the equation $Pu = f$ modulo C^∞. $\square$

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