Fourier integral operators by Duistermaat-Hörmander

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Contents

Disclaimer

The whole exposition follows [H71, DH72, Chapters I-V]. The indices of sections, theorems, propositions, and equations, etc also coincide with theirs.

Notational index

- $D = \partial/i$.
- $\hat{f}(\xi) = \int e^{-i\langle x,\xi\rangle} f(x)dx$.
- X or Z: an open Euclidean subset or a smooth manifold.
- $T(X)$: the tangent bundle of X.
- $T^*(X)$: the cotangent bundle of X. $T^*(X) \setminus 0$ means that the zero section is removed.
- $N(Y)$: the (co)normal bundle of Y, where Y is a submanifold of X.
- Λ: a closed conic Lagrangean submanifold of $T^*(X) \setminus 0$.
- $\mathcal{D}'(X)$: the space of distributions in X.
- $\mathcal{E}'(X)$: the space of distributions with compact support in X.

1 Oscillatory integrals

Definition (1.1.7, Symbols)**.** Let $m, \rho, \delta \in \mathbb{R}$ with $\rho, \delta \in [0,1]$. Then we denote by $S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ the set of all $a \in C^\infty(X \times \mathbb{R}^N)$ such that for every compact set $K \subset X$ and all multiorders α, β the estimate

$$|\partial_x^\beta \partial_\theta^\alpha a(x,\theta)| \leq C_{\alpha,\beta,K}(1+|\theta|)^{m-\rho|\alpha|+\delta|\beta|} \tag{1.1.1}$$

is valid for all $x \in K$, $\theta \in \mathbb{R}^N$, and some constant $C_{\alpha,\beta,K}$. The elements of $S^m_{\rho,\delta}(X \times \mathbb{R}^N)$ are called symbols of order m and type ρ, δ.

(i). If $\rho + \delta = 1$ we also use the notation S^m_ρ instead of $S^m_{\rho,\delta}$.

(ii). If $\rho = 1$ and $\delta = 0$ we write only S^m and talk about symbols of order m.

(iii). The cone support of a is defined as

$$\operatorname{cone\,supp} a = \{(x, t\theta); (x, \theta) \in \operatorname{supp} a, t \geq 0\}.$$

(iv). If (1.1.1) is only valid for large $|\theta|$, we say that $a \in S^m_{\rho,\delta}$ for large $|\theta|$.

(v). $S^{\infty}_{\rho,\delta} = \cup_{m\in\mathbb{R}} S^m_{\rho,\delta}$ and $S^{-\infty}_{\rho,\delta} = \cap_{m\in\mathbb{R}} S^m_{\rho,\delta}$.

(vi). If $a \in S^m_{\rho,\delta}$ it follows that

$$a^{(\alpha)}_{(\beta)} = \partial^{\alpha}_{\theta}\partial^{\beta}_{x} a \in S^{m-\rho|\alpha|+\delta|\beta|}_{\rho,\delta}.$$

Proposition (1.1.7)**.** *One can similarly define $S^m_{\rho,\delta}(\Gamma)$ if Γ is an open conic set, $S^m_{\rho,\delta}(V)$ if V is a cone bundle over X, and $S^m_{\rho,\delta}(V, W)$ if W is a complex vector bundle over V.*

Example. $S^m_{\rho,\delta}(N(Y), \Omega_{1/2})$ in §2.4 and $S^{m+n/4}_{\rho,\delta}(\Lambda, \Omega_{1/2} \otimes L)$ in §3.2.

1 Oscillatory integrals

From now on let us fix $\rho > 0$ and $\delta < 1$.

Definition (Oscillatory integrals). Let $\kappa \in \mathcal{S}$ and $\kappa(0) = 1$. Define the oscillatory integral as the generalized integral

$$(1.2.1) \qquad I_\phi(au) = \lim_{\varepsilon \to 0} \iint e^{i\phi(x,\theta)} a(x,\theta)\kappa(\varepsilon\theta)u(x)\, dx d\theta, \; u \in C_0^\infty(X),$$

in which $a \in S^m(X \times \mathbb{R}^N)$, and ϕ is real valued and positively homogeneous of degree 1 with respect to θ, $\phi \in C^\infty$ for $\theta \neq 0$, and that ϕ is a phase function, i.e. it has no critical point when $\theta \neq 0$. This assumes that one can perform integration by parts (Lemma 1.2.1) and (1.2.1) can be defined for all $a \in S^\infty_{\rho,\delta}(X \times \mathbb{R}^N)$.

Remark (Non-degenerate phase functions). ϕ is called non-degenerate if at any point in the critical set, (see below) the differentials $d(\partial\phi/\partial\theta_j)$, $j = 1, ..., N$, are linearly independent.

Proposition (1.2.4 and 1.2.5)**.** *Only the behaviour of a on the critical set C "matters", where*

$$C = \{(x,\theta) : x \in X, \theta \in \mathbb{R}^N \setminus \{0\}, \phi'_\theta(x,\theta) = 0\}. \tag{1.2.7}$$

If ϕ is non-degenerate, then C is manifold of dimension $\dim X$.

Theorem (1.4.1, FIOs)**.** *Let X and Y be open sets in $\mathbb{R}^{n_X}$ and $\mathbb{R}^{n_Y}$. Define the FIO*

$$Au(x) = \iint e^{i\phi(x,y,\theta)} a(x,y,\theta) u(y)\, dy d\theta, \;\; u \in C_0^\infty(Y), \;\; x \in X, \tag{1.4.1}$$

in which $a \in S^m(X \times Y \times \mathbb{R}^N)$, and the Schwartz kernel of A

$$K_A(x,y) = \int e^{i\phi(x,y,\theta)} a(x,y,\theta)\, d\theta \tag{1.4.6}$$

is an oscillatory integral.

2 ΨDOs and related classes of distributions

2.1 The calculus of ΨDOs

Definition (ΨDOs). Define a ΨDO $A \in L^m_{\rho,\delta}(X)$ for an open set $X \subset \mathbb{R}^n$ as

$$(2.1.1) \qquad Au(x) = \iint e^{i\langle x-y,\theta\rangle} a(x,y,\theta)u(y)\,dyd\theta,$$

in which $a \in S^m_{\rho,\delta}(X \times X \times \mathbb{R}^n)$.

It is called properly supported if both projections $\operatorname{supp} K_A \to X$ are proper. that is, if

$$\{(x,y) \in \operatorname{supp} K_A; x \in K \text{ or } y \in K\}$$

is compact for every compact set $K \subset X$.

Remark. Given $a_1 \in S^m_{\rho,\delta}(X \times \mathbb{R}^n)$, take in (2.1.1)

$$a(x,y,\theta) = \begin{cases} a_1\left(\frac{x+y}{2},\theta\right) & \text{Weyl quantization;} \\ a_1(x,\theta) & \text{left/standard quantization;} \\ a_1(y,\theta) & \text{right quantization.} \end{cases}$$

Theorem (2.1.1)**.** *If A is a properly supported operator in $L^m_{\rho,\delta}(X)$, $\delta < \rho$, then A can be written in one and only one way in the form*

$$Au(x) = \frac{1}{(2\pi)^n}\int e^{i\langle x,\eta\rangle}\sigma_a(x,\eta)\hat{u}(\eta)\,d\theta, \tag{2.1.5}$$

for $u \in \mathcal{S}(\mathbb{R}^n)$ and $x \in X$. Here $\sigma \in S^m_{\rho,\delta}(X \times \mathbb{R}^n)$ is asymptotically given by

$$\sigma_A(x,\eta) \sim (2\pi)^n \sum_\alpha (iD_\eta)^\alpha D^\alpha_y a(x,y,\eta)/\alpha!|_{x=y}. \tag{2.1.4}$$

The asymptotic expansion of σ_A in terms of $a(x,y,\eta)$ is decreasing in order iff $\delta < \rho$. Recall that ∂_y^α increases the order by $\delta|\alpha|$ while ∂_η^α decreases the order by $\rho|\alpha|$. Furthermore, the asymptotic sum uniquely determines a symbol is Borel's theorem in Proposition 1.1.9.

Note that $\sigma_A \in S^{-\infty}$ iff $K_A \in C^\infty$. Then we have

Corollary (Isomorphism between symbols and ΨDOs)**.** *Theorem 2.1.1 shows that the map $A \to \sigma_A$ defined there together with the map $\sigma_A \to A$ given by*

(2.1.5') $$Au(x) = \frac{1}{(2\pi)^n} \iint e^{i\langle x-y,\eta\rangle} \sigma_A(x,\eta) u(y)\, dy d\theta,$$

leads to an isomorphism

$$L^m_{\rho,\delta}(X)/L^{-\infty}_{\rho,\delta}(X) \to S^m_{\rho,\delta}(X)/S^{-\infty}_{\rho,\delta}(X).$$

We shall call $a(x,\theta) \in S^m_{\rho,\delta}(X \times \mathbb{R}^n)$ a symbol of $A \in L^m_{\rho,\delta}(X)$ when their equivalence classes correspond in this isomorphism.

Theorem (2.1.2 and 2.1.3, Equivalence of phase functions)**.** *Let* ϕ *be a phase function in* $X \times X \times \mathbb{R}^n$ *such that* $\phi(x, y, \theta)$ *is a linear function of* θ *and* $\phi'_\theta(x, y, \theta) = 0$ *is equivalent to* $x = y$*. Every operator of the form (1.4.1) with* $a \in S^m_{\rho,\delta}(X \times X \times \mathbb{R}^n)$ *is then in* $L^m_{\rho,\delta}(X)$ *if* $1 - \rho \leq \delta < \rho$*.*

Moreover, there exists a neighborhood Ω *of* $\Delta(X \times X)$ *(the diagonal in* $X \times X$*) that one can find a* C^∞ *map* $\psi : \Omega \to GL(n, \mathbb{R})$ *such that*

$$\phi(x, y, \psi(x, y)\theta) = \langle x - y, \theta\rangle, \ (x, y) \in \Omega, \tag{2.1.10}$$

and

$$\det \psi(x, x) \det \phi''_{x\theta}(x, y, \theta)|_{y=x} = 1.$$

Consider the change of variables

$$\kappa : X \to X_1$$

as a diffeomorphism between open sets in $\mathbb{R}^n$ with inverse $\kappa_1 = \kappa^{-1}$. Let $A \in L^m_{\rho,\delta}(X)$ and set

$$A_1(u) = (A(u \circ \kappa)) \circ \kappa^{-1}, \quad u \in C_0^\infty(X).$$

This means, if A is of the form (2.1.1), that

$$\begin{aligned} &A_1(u)(x) \\ &= \iint e^{i\langle \kappa_1(x)-y,\theta\rangle} a(\kappa_1(x), y, \theta) u(\kappa(y))\, dy d\theta \\ &= \iint e^{i\langle \kappa_1(x)-\kappa_1(y),\theta\rangle} a(\kappa_1(x), \kappa_1(y), \theta) u(y) |D\kappa_1(y)/Dy|\, dy d\theta. \end{aligned}$$

The phase function $\phi(x, y, \theta) = \langle \kappa_1(x) - \kappa_1(y), \theta\rangle$ satisfies the conditions in Theorem 2.1.2. (It is linear in θ and $\phi'_\theta = 0$ iff $x = y$.)

Therefore, using the theorem, $A_1 \in L^m_{\rho,\delta}(X_1)$. Next we compute its symbol. By Proposition 2.1.3, we obtain

$$\begin{aligned}
&A_1(u)(x)\\
&= \iint e^{i\phi(x,y,\theta)} a(\kappa_1(x), \kappa_1(y), \theta) u(y) |D\kappa_1(y)/Dy| \, dy d\theta\\
&= \iint e^{i\langle x-y, \psi(x,y)^{-1}\theta\rangle} a(\kappa_1(x), \kappa_1(y), \theta) u(y) |D\kappa_1(y)/Dy| \, dy d\theta\\
&= \iint e^{i\langle x-y, \theta\rangle} a(\kappa_1(x), \kappa_1(y), \psi(x,y)\theta) u(y) D(x,y) \, dy d\theta,
\end{aligned}$$

where

$$D(x,y) = |\det \kappa_1'(x)| |\det \psi(x,y)|.$$

Thus

$$D(x,x) = \left| \det \phi''_{x\theta}(x,y,\theta) \det \psi(x,y) \Big|_{y=x} \right| = 1.$$

Taking $a(x,y,\eta) = (2\pi)^{-1}\sigma_A(x,\eta)$ it follows that
(2.1.11)
$$\sigma_{A_1}(x,\eta) = \sum_\alpha (iD_\eta)^\alpha D_y^\alpha \sigma_A(\kappa_1(x), \psi(x,y)\eta) D(x,y)/\alpha!|_{x=y}$$
and
$$\sigma_{A_1}(\kappa(x),\eta) \sim \sum_\beta \frac{1}{\beta!}(iD_\eta)^\beta \sigma_A(x,{}^t\kappa'(x)\eta)\phi_\beta(x,\eta), \tag{2.1.14}$$
in which
$$\phi_\beta(x,\eta) = D_z^\beta e^{i\langle \kappa_x''(z),\eta\rangle}|_{z=x}, \tag{2.1.16}$$
and the first few polynomials are
$$\phi_\beta(x,\eta) = \begin{cases} 1, & \beta = 0; \\ 0, & |\beta| = 1; \\ D_x^\beta i\langle \kappa(x),\eta\rangle, & |\beta| = 2 \text{ or } 3. \end{cases} \tag{2.1.17}$$

The calculus we have given here is exact modulo operators in L^∞ and symbols in $S^{-\infty}$. However, it is complicated by the presence of infinite sums in (2.1.14). Now the terms with $\beta \neq 0$ in these sums are of order $\leq m + 1 - 2\rho$. We can therefore obtain a simpler but cruder calculus if from the isomorphism

$$L^m_{\rho,\delta}(X)/L^{m+1-2\rho}_{\rho,\delta}(X) \to S^m_{\rho,\delta}(X)/S^{m+1-2\rho}_{\rho,\delta}(X).$$

In deriving the symbol of transpose operator ${}^tA$ (or product BA if $B \in L^{m'}_{\rho,\delta}$), one can get the leading term is modulo $m + \delta - \rho$ (or $m + m' + \delta - \rho$). Hence, we weaken the above isomorphism to

$$L^m_{\rho,\delta}(X)/L^{m+\delta-\rho}_{\rho,\delta}(X) \to S^m_{\rho,\delta}(X)/S^{m+\delta-\rho}_{\rho,\delta}(X),$$

since in Theorem 2.1.2 we have assumed that $\rho + \delta \geq 1$ so

$$1 - 2\rho \leq \delta - \rho.$$

Definition (Principal symbols). If $A \in L^m_{\rho,\delta}(X)$ and $a \in S^m_{\rho,\delta}(X \times \mathbb{R}^n)$ we shall call a a principal symbol of A if the residue classes of A and a correspond to each other in this isomorphism.

Remark.

(1). If $a(x, \xi)$ is a principal symbol of A, then $a(x, -\xi)$ is a principal symbol of ${}^tA$.

(2). If $b(x, \xi)$ is a principal symbol of B, then $b(x, \xi)a(x, \xi)$ is a principal symbol of BA.

(3). If A_1 is obtained from A by a change of variables as discussed above, then a principal symbol of A_1 is given by $a(\kappa^{-1}(x), {}^t\kappa'(x)\xi)$.

2.2 The continuity of ΨDOs

Corollary (2.2.3)**.** *Let $A \in L^0_{\rho,\delta}(\mathbb{R}^n)$, $\delta < \rho$, and assume that the kernel of A has compact support in $\mathbb{R}^n \times \mathbb{R}^n$, and that*

$$\limsup_{\eta\to\infty} \sup_x |\sigma_A(x,\eta)| < M.$$

Then one can find another such operator A_1 such that $A_1 - A \in L^{-\infty}$ and

$$\|A_1 u\|_{L^2(\mathbb{R}^n)} \leq M\|u\|_{L^2(\mathbb{R}^n)}, \quad u \in C_0^\infty(\mathbb{R}^n).$$

Corollary (2.2.4)**.** *Let $A \in L^0_{\rho,\delta}(\mathbb{R}^n)$, $\delta < \rho$, and assume that the kernel of A has compact support in $\mathbb{R}^n \times \mathbb{R}^n$, and that $\sigma_A(x,\eta) \to 0$ as $\eta \to \infty$, uniformly with respect to x. Then A is compact in $L^2(\mathbb{R}^n)$.*

2.3 ΨDOs on a manifold

Let X be a smooth paracompact manifold of dimension n.

Definition (ΨDOs on a manifold). Let $1-\rho \le \delta < \rho$. A continuous linear operator $A : C_0^\infty(X) \to C^\infty$ belongs to $L^m_{\rho,\delta}(X)$ iff for each diffeomorphism

$$\kappa : X_\kappa \subset X \to \kappa(X_\kappa) \subset \mathbb{R}^n$$

we have $A_\kappa \in L^m_{\rho,\delta}(\kappa(X_\kappa))$ if

$$(A_\kappa u) \circ \kappa = A(u \circ \kappa), \quad u \in C_0^\infty(\kappa(X_\kappa)).$$

Next we investigate the principal symbols of ΨDOs on a manifold. If $u \in C_0^\infty(\kappa_1(X_{\kappa_1} \cap X_{\kappa_2}))$, then

$$(A_{\kappa_1} u) \circ \kappa_1 = A(u \circ \kappa_1) = A_{\kappa_2}(u \circ \kappa_1 \circ \kappa_2^{-1}) \circ \kappa_2,$$

and therefore

$$(A_{\kappa_1} u) \circ \kappa = A_{\kappa_2}(u \circ \kappa),$$

where

$$\kappa = \kappa_1 \circ \kappa_2^{-1} : \kappa_2(X_{\kappa_1} \cap X_{\kappa_2}) \to \kappa_1(X_{\kappa_2} \cap X_{\kappa_2})$$

is a diffeomorphism. For the symbols we have

$$\sigma_{A_{\kappa_1}}(\kappa(x), \xi) - \sigma_{A_{\kappa_2}}(x, {}^t\kappa'(x)\xi) \in S^{m+\delta-\rho}(\kappa_1(X_{\kappa_2} \cap X_{\kappa_2})),$$

If we regard $x_j X_{\kappa_j} \times \mathbb{R}^n$ as the cotangent space of $\kappa_j X_{\kappa_j}$, then κ maps (x_2, ξ_2) to (x_1, ξ_1) where $x_1 = \kappa(x_2)$ and

$$\langle \kappa'(x_2)t, \xi_1 \rangle = \langle t, \xi_1 \rangle \quad \text{for all } t \in \mathbb{R}^n,$$

thus $\xi_2 = {}^t\kappa'(x_2)\xi_1$. If we keep Proposition 1.1.7 in mind, it follows that if using the map κ_j we pull $\sigma_{A_{\kappa_j}}$ to a function σ_A^j on the cotangent space of X_{κ_j}, then

$$\sigma_A^1 - \sigma_A^2 \in S_{\rho,\delta}^{m+\delta-\rho}(T^*(X_{\kappa_1} \cap X_{\kappa_2})).$$

Using a partition of unity we can therefore piece together an element $\sigma \in S^m_{\rho,\delta}(T^*(X))$ such that

$$\sigma - \sigma^1_A \in S^{m+\delta-\rho}_{\rho,\delta}(T^*(X_{\kappa_1}))$$

for any coordinate system κ_1. We call σ a principal symbol of A.

A more convenient approach can be based on Theorem 2.1.2. We wish to define operators in $L^m_{\rho,\delta}(X)$ directly as FIOs with phase function ϕ and symbol a defined on a real vector bundle E with fiber dimension n over a neighborhood Ω of the diagonal in $X \times X$. We wish ϕ to be linear in the fibers and require that the restriction of ϕ to a fiber is critical at $e \in E$ iff the projection $\pi(e)$ of e on $X \times X$ belongs to the diagonal. The differential of ϕ at such a point can be regarded as a cotangent vector of $X \times X$ at $\pi(e) = (x, x)$ which vanishes on the tangents of the diagonal so it is of the form $(\xi, -\xi)$ where $\xi \in T_x(X)$. The map $E_{(x,x)} \ni e \to \xi \in T_x(X)$ is linear and injective, hence bijective since the dimensions are equal. Thus ϕ defines over the diagonal an isomorphism of E and the cotangent space $T^*(X)$ lifted to $X \times X$ by the projection $(x, y) \to y$, and this isomorphism can be extended to a neighborhood of the diagonal.

On the other hand, if E is defined in this way then we can choose ϕ so that ϕ vanishes over the diagonal and $d\phi = \xi dx - \xi dy$ at (x, x, ξ), where $\xi \in T_x^*(X)$. Indeed, this is possible locally and so globally by means of a partition of unity. In a neighborhood of the diagonal we cannot have any critical points along the fibers so ϕ has the required properties. If ϕ_1 and ϕ_2 are two such functions, then $\phi_1 - \phi_2$ vanishes to the second order over the diagonal and we conclude as in the proof of Proposition 2.1.3 that

$$\phi_2(x, y, \xi) = \phi_1(x, y, \psi(x, y)\xi)$$

over a neighborhood of the diagonal where ψ is a homomorphism $E \to$

E which is the identity over the diagonal. Thus the requirements on E and ϕ determine E and ϕ essentially uniquely.

Now we can define $L^m_{\rho,\delta}(X)$ when $1-\rho \le \delta < \rho$, as the operators which can be written as a sum of an operator with C^∞ kernel and one of the form

$$Au(x) = \frac{1}{(2\pi)^n} \iint e^{i\phi(x,y,\eta)} a(x,y,\eta) u(y)\, dy d\eta,\ \ u \in C_0^\infty(X),$$

where $dyd\eta$ is the invariant element of integration in $T^*(X)$ and $a \in S^m_{\rho,\delta}(X \times X \times \mathbb{R}^n)$ vanishes when (x,y) is outside a suitably small neighborhood of the diagonal. A principal symbol of A is given by $a(x,x,\eta)$.

The equivalence with the first definition is an immediate consequence of Theorem 2.1.2.

2.4 Oscillatory integrals with linear phase function

The above discussion on ΨDOs can be generalized if the phase function is linear in θ, that is, $\phi(x,\theta) = \langle \Phi(x), \theta \rangle$, where Φ'_x is of rank N when $\Phi(x) = 0$. Then the critical set $C = Y \times \mathbb{R}^N$ where $Y = \{x \in Z; \Phi(x) = 0\}$ has codimension N.

Define the distribution of the form

$$\langle A, u \rangle = (2\pi)^{-(n+2N)/4} \iint e^{i\phi(x,\theta)} a(x,\theta) u(x) dx d\theta, \ u \in C_0^\infty(Z), \tag{2.4.1}$$

where $a \in S_{\rho,\delta}^{m+(n-2N)/4}(Z \times \mathbb{R}^n)$.

Note that in ΨDO case, $Z = X \times X$, the phase function is $\langle x-y, \theta \rangle$, and $n = 2N$. Then (2.4.1) reduces to our previous definition of ΨDOs in (2.1.1).

Proposition (2.4.1, Equivalence of phase functions)**.** *If* ϕ_1 *is another linear phase function with the same critical set, then there exists a neighborhood* U *of* Y *that one can find a* C^∞ *map* $\psi : U \to GL(n, \mathbb{R})$ *such that* $\phi_1(x, \theta) = \phi(x, \psi(x)\theta)$*,* $x \in U$*.*

Definition (Conormal distributions, local version)**.** We define $I^m_{\rho,\delta}(Z, Y)$ as the set of all distributions in Z which modulo $C^\infty(Z)$ can be represented in the form (2.4.1) with $a \in S^{m+(n-2N)/4}_{\rho,\delta}(Z \times \mathbb{R}^n)$.

Example. Note that in the case where a is a homogeneous function of θ the corresponding distribution is essentially a homogeneous function of the distance from Z to Y depending smoothly on the nearest point in Y.

By Propositions 2.4.1 and 1.2.4 the definition of $I^m_{\rho,\delta}(Z,Y)$ is independent of the choice of ϕ; moreover, it suffices to have ϕ defined over a neighborhood of Y if one takes a vanishing outside a smaller neighborhood. To assign a principal symbol to the distributions in $I^m_{\rho,\delta}(Z,Y)$, we have the following isomorphism.

$$\begin{aligned}(2.4.2)\qquad & S^{m+(n-2N)/4}_{\rho,\delta}(Y\times\mathbb{R}^n)/S^{m+(n-2N)/4+\delta-\rho}_{\rho,\delta}(Y\times\mathbb{R}^n)\\ &\to I^m_{\rho,\delta}(Z,Y)/I^{m+\delta-\rho}_{\rho,\delta}(Z,Y).\end{aligned}$$

To define conormal distributions globally, we have to examine to what extent (2.4.2) depends on the choice of the phase function ϕ and the local coordinates in Z.

We now digress and discuss distributions, and distribution densities, and densities on a manifold.

Distribution on a manifold

This section follows [H83, §6.3].

Definition (Distributions on a manifold). Let X be a smooth manifold. In every coordinate system

$$\kappa_1 : X_{\kappa_1} \subset X \to \kappa_1(X_{\kappa_1}) \subset \mathbb{R}^n$$

we are given a distribution $u_{\kappa_1} \in \mathcal{D}'(\kappa(X_{\kappa_1}))$ such that

$$u_{\kappa_2} = (\kappa_1 \circ \kappa_2^{-1})^* u_{\kappa_1} \quad \text{in } \kappa_2(X_{\kappa_1} \cap X_{\kappa_2}),$$

we call the system u_κ a distribution u in X. The set of all distributions in X is denoted by $\mathcal{D}'(X)$.

One may wonder why we did not define $\mathcal{D}'(X)$ as the space of continuous linear forms on $C_0^\infty(X)$. The reason for this is that if $f \in C(X)$ and $\phi \in C_0^\infty(X)$ we have no invariant way of integrating $f\phi$ in order to identity f with such a linear form.

Distribution densities on a manifold

However, we would have obtained something rather close to $\mathcal{D}'(X)$. In fact, let u be a continuous linear form on $C_0^\infty(X)$. Then u defines a distribution $u_{\kappa_1} \in \mathcal{D}'(\kappa_1(X_{\kappa_1}))$ by

$$u_{\kappa_1}(\phi) = u(\phi \circ \kappa_1), \quad \phi \in C_0^\infty(\kappa_1(X_{\kappa_1})).$$

(We define $\phi \circ \kappa_1 = 0$ outside $\kappa_1(X_{\kappa_1})$.) If $\phi \in C_0^\infty(\kappa_2(X_{\kappa_1} \cap X_{\kappa_2}))$ then

$$u_{\kappa_2}(\phi) = u(\phi \circ \kappa_2) = u(\phi \circ \kappa^{-1} \circ \kappa_1) = u_{\kappa_1}(\phi \circ \kappa^{-1})$$

where

$$\kappa = \kappa_1 \circ \kappa_2^{-1} : \kappa_2(X_{\kappa_1} \cap X_{\kappa_2}) \to \kappa_1(X_{\kappa_1} \cap X_{\kappa_2}).$$

Therefore, if u_{κ_1} and u_{κ_2} are smooth functions, then for every $\phi \in C_0^\infty(\kappa_2(X_{\kappa_1} \cap X_{\kappa_2}))$

$$\begin{aligned} &\int_{\kappa_2(X_{\kappa_1} \cap X_{\kappa_2})} u_{\kappa_2}(x)\phi(x)\,dx \\ = &\int_{\kappa_1(X_{\kappa_1} \cap X_{\kappa_2})} u_{\kappa_1}(y)\phi(\kappa^{-1}(y))\,dy \\ = &\int_{\kappa_2(X_{\kappa_1} \cap X_{\kappa_2})} (u_{\kappa_1} \circ \kappa)(x)\phi(x)|\det \kappa'(x)|\,dx \\ = &\int_{\kappa_2(X_{\kappa_1} \cap X_{\kappa_2})} \kappa^* u_{\kappa_1}(x)|\det \kappa'(x)|\phi(x)\,dx. \end{aligned}$$

Hence,

$$u_{\kappa_2} = |\det \kappa'|\kappa^* u_{\kappa_1} \quad \text{in } \kappa_2(X_{\kappa_1} \cap X_{\kappa_2})$$

because the definition of pullback of distributions should coincide with pullback of continuous functions.

Conversely, assume given distributions u_κ in $\kappa(X_\kappa)$ satisfying the above equation for all κ in the atlas. Choose a partition of unity $1 = \sum \kappa_j$ with $\kappa_j \in C_0^\infty(\kappa_j(X_{\kappa_j}))$. Then

$$U(\phi) = \sum \langle u_{\kappa_j}, (\kappa_j \phi) \circ \kappa_j^{-1} \rangle, \quad \phi \in C_0^\infty(X)$$

is a continuous linear form on $C_0^\infty(X)$.

The continuous linear form u on $C_0^\infty(X)$ can thus be identified with the system $u_\kappa \in \mathcal{D}'(\kappa(X_{\kappa_j}))$ satisfying

$$u_{\kappa_j} = |\det(\kappa_i \circ \kappa_j^{-1})'|(\kappa_i \circ \kappa_j^{-1})^* u_{\kappa_i} \quad \text{in } \kappa_j(X_{\kappa_i} \cap X_{\kappa_j}).$$

They are called distribution densities. Compare with the definition of distributions on X and its transition under change of variables.

Densities on a manifold

A density in a manifold X is a measure which in a local coordinate patch with local coordinates $x_1, ..., x_n$ ($n = \dim(X)$) can be written in the form

$$a(x)dx_1 \cdots dx_n.$$

If we have an overlapping coordinate patch with local coordinates $\tilde{x}_1, ..., \tilde{x}_n$ the measure can also be expressed in the form

$$\tilde{a}(\tilde{x})d\tilde{x}_1 \cdots d\tilde{x}_n,$$

so we have the transformation law

$$\tilde{a}(\tilde{x}) = a(x)|Dx/D\tilde{x}|$$

in the overlap. More generally, a density of order α on X is defined if for each choice of local coordinates we have a function $a(x)$ of the local coordinates which obeys the transformation law

$$\tilde{a}(\tilde{x}) = a(x)|Dx/D\tilde{x}|^{\alpha}.$$

In particular, smooth densities of order 0 are smooth functions. Densities of order α can of course be regarded as sections of a line bundle Ω_α on Y, defined by the transition functions $|Dx/D\tilde{x}|^{\alpha}$, and we have

$$\Omega_\alpha \otimes \Omega_\beta = \Omega_{\alpha+\beta}.$$

If u, v are densities of order α and $1-\alpha$ and the tensor product uv has compact support, then uv is a measure with compact support so $\int_X uv$ is well defined.

Definition (Distribution densities). We define the space of distributions with values in Ω_α as the dual space of $C_0^\infty(X, \Omega_{1-\alpha})$.

Hence, one can also define the distribution space $\mathcal{D}'(X)$ as the dual space of $C_0^\infty(X, \Omega_1)$, i.e. the space of continuous linear forms on C_0^∞ densities.

We shall return to conormal distributions and examine to what extent (2.4.2) depends on the choice of the phase function ϕ and the local coordinates in Z. To begin with we keep the local coordinates in Z but replace the phase function ϕ by another ϕ_1. According to Proposition 2.4.1 we may assume that $\phi_1(x,\theta) = \phi(x, \psi(x)\theta)$, $x \in U$, where U is a neighborhood of Y and ψ a smooth map $U \to GL(N,\mathbb{R})$. A substitution of variables now gives

$$\iint e^{i\phi(x,\theta)} a(x,\theta)u(x)\,dxd\theta = \iint e^{\phi_1(x,\theta)} a_1(x,\theta)u(x)\,dxd\theta,$$

where

$$a_1(x,\theta) = a(x,\psi(x)\theta)|\det\psi(x)|. \tag{2.4.3}$$

To put this transformation law in a more natural form we first note that the map

$$Y \times \mathbb{R}^N \ni (x,\theta) \to (x, \phi'_x(x,\theta)) = (x, \Phi(x)\theta)$$

is a bijection to the normal bundle $N(Y)$ of Y in $T^*(Z)$, which is linear along the fibers.

We can therefore regard a as a function on $N(Y)$, and similarly for a_1. If $x \in Y$ and

$$(x, \phi'_x(x,\theta)) = (x,\xi) = (x, \phi'_{1x}(x,\theta_1))$$

we must have $\psi(x)\theta_1 = \theta$ so that

$$a_1(x,\theta_1) = a(x,\psi(x)\theta)|\det\psi(x)| = a(x,\theta)|\det\psi(x)|.$$

Regarded as functions on $N(Y)$ the functions a and a_1 therefore differ only by the factor $|\det\psi(x)|$. To take care of this factor we shall consider the measures defined in Y and in $N(Y)$ by the choice of ϕ.

Writing $\phi(x,\theta) = \langle\Phi(x),\theta\rangle$ we know that the map $x \to \Phi(x)$ is of rank N when $\Phi(x) = 0$. The composition $\delta(\Phi)$ where δ is the Dirac measure in $\mathbb{R}^N$ is then a well defined measure with support in Y. If $y_1, \ldots, y_{n-N}$ are local coordinates on Y and we extend them to C^∞

functions in a neighborhood of Y, then the measure is equal to

$$|D(y,\Phi)/Dx|^{-1}\, dy_1 \cdots dy_{n-N}.$$

Thus the measure is a density on Y. Using the Lebesgue measure in $\mathbb{R}^N$ we have on $Y \times \mathbb{R}^N$ a density given by

$$d_\phi = \delta(\Phi) d\theta_1 \cdots d\theta_N,$$

or in terms of local coordinates $y_1, ..., y_{n-N}$ on Y

$$(2.4.4) \qquad |D(y,\Phi)/Dx|^{-1}\, dy_1 \cdots dy_{n-N} d\theta_1 \cdots d\theta_N.$$

This we shall map to a density on the normal bundle $N(Y)$ using the inverse of the map

$$\kappa_\phi : Y \times \mathbb{R}^N \ni (y,\theta) \to (y, {}^t\Phi'_y \theta).$$

We wish to compare d_ϕ with the density d_{ϕ_1}, constructed from the phase function ϕ_1, that is, from $\Phi_1 =^t \psi\Phi$. In local coordinates d_{ϕ_1} is given by

$$(2.4.5) \qquad |D(y,\Phi_1)/Dx|^{-1}\, dy_1\cdots dy_{n-N}d\theta_1\cdots d\theta_N$$
$$= |\det\psi|^{-1}|D(y,\Phi)/Dx|^{-1}\, dy_1\cdots dy_{n-N}d\theta_1\cdots d\theta_N,$$

and d_{ϕ_1}, should be mapped to a density on $N(Y)$ using the inverse of the map

$$\kappa_{\phi_1} : Y\times\mathbb{R}^N \ni (y,\theta)\to(y,{}^t\,\Phi'_{1y}\theta) = (y,{}^t\,\Phi'_y\psi\theta).$$

Now $\kappa = \kappa_{\phi_1}^{-1}\circ\kappa_\phi$ is the map

$$(y,\theta)\to(y,\psi^{-1}\theta),$$

so

$$\kappa^* d_{\phi_1} = |\det\psi|^{-2} d_\phi.$$

If we recall (2.4.3), which with our present notations can be written

$$\kappa^* a_1 = |\det\psi| a,$$

we conclude that

$$\kappa^* a_1 \sqrt{d_{\phi_1}} = a\sqrt{d_\phi}.$$

Thus $a_1\sqrt{d_{\phi_1}}$ and $a\sqrt{d_\phi}$ define the same element in $S^{m+n/4}_{\rho,\delta}(N(Y),\Omega_{1/2})$. That the order here becomes independent of N is another partial justification for the normalizations that have been made.

Next we consider the effect of a change of variables. Thus let $Z \to \tilde{Z}$ be a diffeomorphism between open sets in $\mathbb{R}^n$. Writing $x = x(\tilde{x})$ we transform (2.4.1) to

$$\langle A, u\rangle = (2\pi)^{-(n+2N)/4} \iint e^{i\phi(x,\theta)} a(x,\theta)u(x)dxd\theta \tag{2.4.1}$$

$$(2\pi)^{-(n+2N)/4} \iint e^{i\tilde{\phi}(\tilde{x},\theta)} \tilde{a}(\tilde{x},\theta)\tilde{u}(\tilde{x})d\tilde{x}d\theta = \langle \tilde{A}, \tilde{u}\rangle.$$

Here

$$\tilde{u}(\tilde{x}) = u(x)|Dx/D\tilde{x}|^{1/2},$$

$$\tilde{\phi}(\tilde{x}, \theta) = \phi(x(\tilde{x}), \theta),$$

and

$$\tilde{a}(\tilde{x}, \theta) = a(x(\tilde{x}), \theta)|Dx/D\tilde{x}|^{1/2}.$$

that is, we regard u as a density of order 1/2 which means that A is also transformed to $\tilde{A}$ as a density of order 1/2.

Let $y_1, ..., y_N$ be local coordinates on

$$Y = \{x \in Z; \phi'_\theta(x, \theta) = 0\},$$

considered as functions in Z, and let $\tilde{y}_1, ..., \tilde{y}_N$ be the corresponding functions in $\tilde{Z}$ which are thus local coordinates on

$$\tilde{Y} = \{\tilde{x} \in \tilde{Z}; \tilde{\phi}'_\theta(\tilde{x}, \theta) = 0\}.$$

Clearly (x, θ) and $(\tilde{x}, \theta)$ define points in $N(Y)$ and $N(\tilde{Y})$ which correspond under the isomorphism between $T^*(Z)$ and $T^*(Z)$. Now

$$\frac{D(y,\Phi)}{Dx} = \frac{D(\tilde{y},\tilde{\Phi})}{D\tilde{x}}\frac{D\tilde{x}}{Dx}.$$

Then

$$a(x,\theta)\left|\frac{D(y,\Phi)}{Dx}\right|^{-\frac{1}{2}} = \tilde{a}(\tilde{x},\theta)\left|\frac{D(\tilde{y},\tilde{\Phi})}{D\tilde{x}}\right|^{-\frac{1}{2}}.$$

Thus our construction is also invariant under changes of variables in Z. Putting together with the invariance under changes of phase functions, we finally arrived at the intrinsic definition of conormal distributions on a manifold, i.e. global version of (2.4.1).

Those distribution densities are elements in $\Omega_{1/2}(Z)$, that is, acting on $C_0^\infty(Z,\Omega_{1/2}(Z))$. Their symbols live on $N(Y)$ and are elements in $\Omega_{1/2}(N(Y))$.

Theorem (2.4.2, Conormal distributions on a manifold)**.** *Let Z be a manifold and Y a closed submanifold. Let $I^m_{\rho,\delta}(Z,Y)$ where $1-\rho \le \delta < \rho$ be the set of all distribution densities of order $1/2$ on Z which are in $C^\infty(Z \setminus Y)$ and in a neighborhood of any point in Y can be expressed in the form (2.4.1) where $a \in S^{m+(n-2N)/4}_{\rho,\delta}$ and ϕ is a linear phase function which is critical along the fibers over Y and only there. Then the restriction of a to these points gives rise to an isomorphism*

$$S^{m+n/4}_{\rho,\delta}(N(Y),\Omega_{1/2}(N(Y)))/S^{m+n/4+\delta-\rho}_{\rho,\delta}(N(Y),\Omega_{1/2}(N(Y))) \\ \to I^m_{\rho,\delta}(Z,Y;\Omega_{1/2}(Z))/I^{m+\delta-\rho}_{\rho,\delta}(Z,Y;\Omega_{1/2}(Z)). \tag{2.4.8}$$

We shall say that a is a principal symbol of the distribution $A \in \mathcal{D}'(Z,\Omega_{1/2})$ if their residue classes correspond under this isomorphism.

Remark (ΨDOs viewed as conormal distribution densities)**.** For a ΨDO in $L^m_{\rho,\delta}(X)$, $Z = X \times X = \{(x, y); x, y \in X\}$ and $Y = \Delta(X \times X)$. If the phase function ϕ is a linear and is critical along the fibers over Y and only there, then ϕ is uniquely determined as $\langle x - y, \theta \rangle$.

Moreover, the 1 densities in (i.e., the volume form), thus the half densities, on $N(\Delta)$ are canonically defined by lifting from the canonical volume form on $T^*(X)$, that is,

$$(d\xi \wedge dx)^n/n!$$

Therefore we do not have to impose $\Omega_{1/2}(N(\Delta))$ in the invariance and isomorphism, and one can regard the symbols as living on $T^*(X)$ (instead of living on $N(\Delta)$) by the diffeomorphism between $T^*(X)$ and $N(\Delta)$. As the simplest example of local canonical graph, this will be further justified in §4.1.

3 Distributions defined by oscillatory integrals

3.1 Equivalence of non-degenerate phase functions

Recall that an oscillatory integral is defined by

$$I_\phi(au) = \iint e^{i\phi(x,\theta)} a(x,\theta) u(x)\, dx d\theta, \ u \in C_0^\infty(X),$$

where ϕ is a non-degenerate phase function satisfying

(1). it is real valued and positively homogeneous of degree 1 with respect to θ,

(2). it is C^∞ for $\theta \neq 0$, and has no critical point when $\theta \neq 0$,

(3). the differentials $d(\partial\phi/\partial\theta_j), j = 1, ..., N$, are linearly independent on the critical set

$$C = \{(x,\theta); \phi'_\theta(x,\theta) = 0\}.$$

(3). The non-degenerate condition in (3) guarantees that C is a smooth manifold of dimension $\dim(X)$. Consider the map

$$C \ni (x, \theta) \to (x, \phi'_x) \in \Gamma \subset T^*(X) \setminus 0, \tag{3.1.1}$$

where 0 stands for the zero section, is a local diffeomorphism to a smooth submanifold Λ.

(2). Λ does not contain the zero section is from condition (2) above that ϕ has no critical point when $\theta \neq 0$.

(1). Λ is conic since ϕ is positively homogeneous of degree 1 with respect to θ in condition (1).

Furthermore, Λ is Lagrangean:

$$\sum \xi_j dx_j = 0 \quad \text{on } \Lambda. \tag{3.1.3}$$

This is because $\phi'_x\, dx = d\phi - \phi'_\theta\, d\theta = 0$ from Euler's homogeneity relation that $\phi = \theta \cdot \phi'_\theta = 0$ on C.

Definition (Lagrangean submanifolds). $\Lambda \subset T^*(X)$ is called Lagrangean if the canonical two form

$$\sum d\xi_j \wedge dx_j = 0 \quad \text{on } \Lambda. \tag{3.1.4}$$

One can easily show that if Λ is a conic Lagrangean submanifold, then the stronger condition (3.1.3) holds.

Example. Let $H(\xi)$ be a homogeneous smooth function of ξ of degree 1 in a cone $\Gamma \subset \mathbb{R}^n$, and define

$$\phi(x, \xi) = \langle x, \xi \rangle - H(\xi).$$

Then the condition $\phi'_\xi = 0$ means that $x = H'(\xi)$, so ϕ is non-degenerate and

$$\Lambda = \{(H'(\xi, \xi), \xi \in \Gamma\}.$$

This example of conic Lagrangean manifold actually covers the general case.

Theorem (3.1.3)**.** *Let $\Lambda \subset T^*(X)$ be a conic Lagrangean manifold. For every $\lambda_0 \in \Lambda$ with the local coordinates $x_1, ..., x_n$ at $\pi\lambda_0 \in X$ suitably chosen one can find a function H which is homogeneous of degree 1 in an open cone Γ in $\mathbb{R}^n$ such that $\phi(x,\xi) = \sum_{j=1}^n x_j\xi_j - H(\xi)$ the Lagrangean manifold defined by ϕ is a neighborhood of λ in Λ.*

Next is an important relation between a Lagrangean manifold and any non-degenerate phase function defining it.

Theorem (3.1.4)**.** *Let ϕ be a non-degenerate phase function in a conic neighborhood $(x_0, \theta_0) \in X \times \mathbb{R}^N$ with $\phi'_\theta(x_0, \theta_0) = 0$, and set $\xi_0 = \phi'_x(x_0, \theta_0)$ so that (x_0, ξ_0) belongs to the corresponding Lagrangean manifold Λ. Then we have*

$$N - \operatorname{rank} \phi''_{\theta\theta}(x_0, \theta_0) = n - \operatorname{rank} d\pi_\Lambda(x_0, \xi_0), \tag{3.1.5}$$

where π is the projection $T^*(X) \to X$ and π_Λ is restriction to Λ.

Theorem (3.1.6, Equivalence of phase functions). *Let ϕ and $\tilde{\phi}$ be non-degenerate phase functions in conic neighbourhoods of $(x_0, \theta_0) \in X \times (\mathbb{R}^N \setminus 0)$ and $(x_0, \tilde{\theta}_0) \in X \times (\mathbb{R}^{\tilde{N}} \setminus 0)$ respectively. Then the functions ϕ and $\tilde{\phi}$ are equivalent in some conic neighbourhoods of these points, under a diffeomorphism mapping (x_0, θ_0) to $(x_0, \tilde{\theta}_0)$, iff*

(i). The elements of Lagrangean manifolds defined by ϕ and $\tilde{\phi}$ at (x_0, θ_0) and $(x_0, \tilde{\theta}_0)$ are the same.

(ii). $N = \tilde{N}$.

(iii). $\phi''_{\theta\theta}(x_0, \theta_0)$ and $\tilde{\phi}''_{\theta\theta}(x_0, \tilde{\theta}_0)$ have the same signature.

Note that $\phi''_{\theta\theta}(x_0, \xi_0)$ can have any signature compatible with Theorem 3.1.4 when we only know the corresponding Lagrangean manifold Λ. Only in one case do we get a perfect analogue of Proposition 2.4.1.

Corollary (3.1.8)**.** *Let ϕ_j be a non-degenerate phase function at (x_0, θ_j) where*

$$\phi'_{1\theta}(x_0, \theta_1) = 0 \quad \textit{and} \quad \phi'_{2\theta}(x_0, \theta_2) = 0,$$

and

$$\phi''_{1\theta\theta}(x_0, \theta_1) = 0 \quad \textit{and} \quad \phi''_{2\theta\theta}(x_0, \theta_2) = 0.$$

Then it follows that ϕ_1 and ϕ_2 are equivalent at (x_0, θ_1) and (x_0, θ_2) iff the corresponding germs of Lagrange manifolds are the same.

We only need to verify the three conditions in Theorem 3.1.6.

(iii). Trivially true.

(ii). From Theorem 3.1.4, $N_1 = n - \operatorname{rank} d\pi_\Lambda(x_0, \xi_0) = N_2$.

(i). Yeah, we only need this: (x_0, θ_1) and (x_0, θ_2) define the same element on Λ iff the corresponding germs of Lagrangean manifolds are the same.

Given a Lagrangean manifold, locally there are a lot of phase functions defining it, the above discussion concludes the equivalence between these phase functions if they define the same Lagrangean manifold.

On the other side, one can change the number of θ variables without chaining the corresponding Lagrangean manifold.

Remark (Increase the number of θ variables)**.** Let $\phi(x, \theta)$ be a non-degenerate phase function in a conic neighborhood of (x_0, θ_0), let $\sigma \in \mathbb{R}^k$ and introduce

$$\phi_1(x, \theta, \sigma) = \phi(x, \theta) + A(\sigma, \sigma)/|\theta|$$

where A is a non-singular quadratic form in $\mathbb{R}^k$. This function is homogeneous of degree 1 in a conic neighborhood of $(x_0, \theta_0, 0)$ in $X \times \mathbb{R}N + k$. The equations

$$\partial\phi_1/\partial\theta = \partial\phi_1/\partial\sigma = 0$$

mean that $\sigma = 0$ and that $\partial\phi/\partial\theta = 0$, so it is clear that ϕ_1 is a non-degenerate phase function defining the same Lagrangean manifold as ϕ. Thus we can always increase the number of θ variables as much as we like.

However, it seems more meaningful to see how (and how far) we can decrease the number of θ variables.

Remark (Decrease the number of θ variables)**.** Let $\phi(x, \theta)$ be a non-degenerate phase function in a conic neighborhood of $(x_0, \theta_0) \in C$, (defined by (3.1.2)), we can decrease the fiber dimension by k units if

$$\operatorname{rank} \phi''_{\theta\theta}(x_0, \theta_0) = k.$$

One notices that $k < N$, which means that one can not eliminate all the θ variables. This is because from the assumption on the phase function ϕ, ϕ'_θ is homogeneous of degree 0, and

$$\phi''_{\theta\theta}(x_0, \theta_0)\theta_0 = \phi'_\theta(x_0, \theta_0) = 0$$

for $\theta_0 \neq 0$, hence

$$\operatorname{rank} \phi''_{\theta\theta}(x_0, \theta_0) < N.$$

We hereby give another explanation of why we can not eliminate all the θ variables. Recall that

$$n - \operatorname{rank} d\pi_\Lambda(x_0, \xi_0) = N - \operatorname{rank} \phi''_{\theta\theta}(x_0, \theta_0), \tag{3.1.5}$$

and $\Lambda \subset T^*(X)$ is a conic Lagrangean manifold, with dimension of the germ in ξ at least one.

But $\dim \Lambda = \dim X = n$ so the projection $d\pi_\Lambda(x_0, \xi_0)$ will definitely lose rank (by the dimension of the germ at (x_0, ξ_0)). Hence,

$$N - \operatorname{rank} \phi''_{\theta\theta}(x_0, \theta_0) > 0.$$

It is in fact the nature of Λ being conic that we can not locally define it by a non-degenerate phase function with no θ variables.

Remark (Non-conic Lagrangean manifolds)**.** In semiclassical Lagrangean theory, however, one allows non-conic Lagrangean manifold, then there are non-degenerate phase function with no θ variables. E.g.

$$a(x)e^{i\phi(x)/h}$$

is associated with the non-conic Lagrangean manifold

$$\Lambda = \{(x,0) : x \in \operatorname{supp} a, \phi'_x = 0\} \subset T^*(\mathbb{R}^n)$$

and the projection

$$d\pi_\Lambda : \Lambda \to \mathbb{R}^n$$

has full rank n in this case. Also, this Lagrangean manifold can not be written in the form of

$$\{(H'(\xi),\xi)\}$$

in any local coordinates.

Example. Let $\Lambda = \{(0, y_2; \eta_1, 0); y_2, \eta_1 \in \mathbb{R}\} \subset T^*(\mathbb{R}^2)$ be a (conic) Lagrangean manifold. Consider the transformation:

$$(y_1, y_2; \eta_1, \eta_2) \to (x_1, x_2; \xi_1, \xi_2)$$

such that

$$\begin{cases} x_1 = y_1 + y_2^2/2, & \eta_1 = \xi_1; \\ x_2 = y_2, & \eta_2 = y_2\xi_1 + \xi_2. \end{cases}$$

Under the new coordinates x_1, x_2,

$$\Lambda = \left\{ \left(\frac{\xi_2^2}{2\xi_1^2}, -\frac{\xi_2}{\xi_1}; \xi_1, \xi_2 \right) \right\} = \{H'_{\xi_1}, H'_{\xi_2}; \xi_1, \xi_2\}$$

if we take

$$H(\xi_1, \xi_2) = -\frac{\xi_2^2}{2\xi_1}.$$

3.2 Invariance under change of phase functions and global definition

The FIO now is defined for $u \in C_0^\infty(X)$ as

$$\langle A, u \rangle = \frac{1}{(2\pi)^{(n+2N)/4}} \iint e^{i\phi(x,\theta)} a(x,\theta) u(x)\, dx d\theta. \tag{3.2.1}$$

Theorem (3.2.1, Invariant definition of the principal symbol)**.** *Let* $\phi(x,\theta)$ *and* $\tilde{\phi}(x,\tilde{\theta})$ *be non-degenerate phase functions in neighbourhoods of* $(x_0,\theta_0) \in X \times (\mathbb{R}^N \setminus 0)$ *and* $(x_0,\tilde{\theta}_0) \in X \times (\mathbb{R}^{\tilde{N}} \setminus 0)$ *which define the same elements of Lagrange manifold* Λ *there:* $\phi'_x(x_0,\theta_0) = \tilde{\phi}'_x(x_0,\tilde{\theta}_0) = \xi_0$*. Then*

(i). The difference

$$\sigma = \operatorname{sgn} \phi''_{\theta\theta}(x,\theta) - \operatorname{sgn} \tilde{\phi}''_{\tilde{\theta}\tilde{\theta}}(x,\tilde{\theta}), \tag{3.2.10}$$

$$\phi'_\theta = \tilde{\phi}'_{\tilde{\theta}} = 0,\ \ \phi'_x = \tilde{\phi}'_x = \xi \in T^*_x(X)$$

is constant in a neighborhood of (x_0, ξ_0) *in* Λ.

(ii). Every distribution which can be defined by (3.2.1) with $a \in S_\rho^{\mu+(n-2N)/4}$, $\rho > 1/2$, *and* $\operatorname{cone\,supp} \phi$ *in a sufficiently small conic neighborhood of* (x_0, θ_0) *can also be written in the same form with* ϕ *replaced by* $\tilde{\phi}$ *and* a *replaced by a function* $\tilde{a} \in S_\rho^{\mu+(n-2\tilde{N})/4}$ *with* $\operatorname{cone\,supp} \tilde{\phi}$ *in a small conic neighborhood of* $(x_0, \tilde{\theta}_0)$, *so that (3.2.11):*

$$(\exp \pi i\sigma/4)a(x,\theta)\sqrt{d_C} - a(x,\tilde{\theta})\sqrt{d_{\tilde{C}}} \in S_\rho^{\mu+n/4+1-2\rho}(\Lambda, \Omega_{1/2}),$$

the two terms being of course in $S_\rho^{\mu+n/4}(\Lambda, \Omega_{1/2})$.

Definition (3.2.2, FIOs). We define $I^m_\rho(X,\Lambda)$ as the set of all $A \in \mathcal{D}'(X,\Omega_{1/2})$ that $A = \sum A_j$ with the supports of A_j locally finite and

$$\langle A_j, u\rangle$$

$$= \frac{1}{(2\pi)^{(n+2N_j)/4}} \iint e^{i(\phi_j(x,\theta)-\pi N_j/4)} a_j(x,\theta)u(x)\,dxd\theta \tag{3.2.14}$$

for $u \in C^\infty(X)$, where dx is the Lebesgue measure with respect to the local coordinates in X'_j, $\theta \in \mathbb{R}^{N_j}$, and $a_j \in S^{m+(n-2N_j)/4}_\rho(\mathbb{R}^n \times \mathbb{R}^{N_j})$.

Theorem (3.2.5)**.** *With the help of Keller-Maslov bundle L, there is a natural isomorphism:*

$$S^{m+n/4}_\rho(\Lambda,\Omega_{1/2}\otimes L)/S^{m+n/4+1-2\rho}_\rho(\Lambda,\Omega_{1/2}\otimes L)$$

$$\to I^m_\rho(X,\Lambda)/I^{m+1-2\rho}_\rho(X,\Lambda). \tag{3.2.16}$$

Theorem (3.2.6)**.** *Let* $A \in I^m_\rho(X,\Lambda)$ *and* $a \in S_\rho^{m+n/4}(\Lambda, \Omega_{1/2} \otimes L)$ *be a principal symbol. Then*

$$WF(A) \subset \Lambda$$

and

$$a \in S^{m+n/4+1-2\rho}(\Lambda, \Omega_{1/2} \otimes L)$$

in $\Lambda \setminus WF(A)$.

4 A calculus for some classes of FIOs

4.1 Operators associated with a canonical relation

Theorem (4.1.1)**.** *Every element of* $I^m_\rho(X \times Y, \Lambda)$ *(for* $\rho > 1/2$*) is a continuous map from* $C^k_0(Y)$ *to* $\mathcal{D}'^k(X)$ *if*

$$m - k\rho < -3(n_X + n_Y)/4. \tag{4.1.1}$$

If Λ *does not intersect* $T^*(X) \times 0_Y$ *(resp.* $0_X \times T^*(Y)$*) where* 0_Y *(resp.* 0_X*) is the zero section in* $T^*(Y)$ *(resp.* $T^*(X)$*) then every element of* $I^m_\rho(X \times Y, \Lambda)$ *is a continuous map from* $C^k_0(Y)$ *to* $C^j(X)$ *if*

$$m + j - k\rho < -3(n_X + n_Y)/4. \tag{4.1.2}$$

The kernels of all operators in $I^m_\rho(X \times Y, \Lambda)$ *are in* C^∞ *outside the projection of* Λ *in* $X \times Y$*.*

Note that locally A is defined by a symbol of order $m+(n-2N)/4 = m - N/4$, (1.4.3) and (1.4.5) in Theorem 1.4.1 imply the above conditions.

Definition (4.1.2, homogeneous canonical relations). A closed conic submanifold C of $T^*(X \times Y)$ is called a homogeneous canonical relation from $T^*(Y)$ to $T^*(X)$ if C is contained in $(T^*(X)\setminus 0)\times(T^*(Y)\setminus 0)$ and is Lagrangean with respect to $\sigma_X - \sigma_Y$, that is, C' (the image of C under the map which is identity on $T^*(X)$ and multiplication by -1 in the fibers of $T^*(Y)$) is Lagrangean with respect to $\sigma_{X\times Y} = \sigma_X + \sigma_Y$.

Definition (4.1.5, Local canonical graphs). A homogeneous canonical relation is called a local canonical graph if the projection $C \to T^*(Y)$ and consequently the projection $C \to T^*(X)$ is a local diffeomorphism so that C is locally the graph of a canonical transformation. This implies that $n_X = n_Y = n$.

Under this condition, the densities (thus the half densities) on C are intrinsically defined by lifting the standard density $\sigma_X^n/n!$ in $T^*(X)$ or $\sigma_Y^n/n!$ in $T^*(Y)$ by the projections, and the isomorphism in (3.2.16) states

$$S_\rho^m(C,L)/S^{m+1-2\rho}(C,L)$$

$$\to I_\rho^m(X\times Y,C')/I_\rho^{m+1-2\rho}(X\times Y,C'). \tag{4.1.7}$$

Example (ΨDOs). The simplest examples of the above two types are the Ψ**DOs** in X. They are related to $\Lambda = N(\Delta)$ where Δ is the diagonal in $X\times X$, and $C=\Lambda'$ is the the graph of the identity map (thus a canonical transformation) $(T^*(X)\setminus 0)\to(T^*(X)\setminus 0)$.

Example (4.1.6, Cauchy problem for the wave equation). The phase function

$$\phi(x,y,\theta)=\langle x'-y',\theta\rangle+(x_n-y_n)|\theta|,$$

where $x = (x', x_n)$ and $y = (y', y_n)$ are in $\mathbb{R}^n$ and $0 \neq \theta \in \mathbb{R}^{n-1}$, then the critical set

$$C_\phi = \{(x, y, \theta) : \phi'_\theta(x, y, \theta) = x' - y' + (x_n - y_n)\theta/|\theta| = 0\},$$

and the Lagrangean submanifold on $T^*(\mathbb{R}^n \times \mathbb{R}^n)$ is a homogeneous canonical relation from $T^*(Y)$ to $T^*(X)$ as

$$C = \{(x, \xi, y, \eta) : \xi = \phi'_x, \eta = -\phi'_y, \phi'_\theta(x, y, \theta) = 0\}.$$

But it is not a local canonical graph simply because $\phi'_\theta = 0$ does not provide a bijection from $T^*(\mathbb{R}^n)$ to $T^*(\mathbb{R}^n)$ (thus no canonical transformations between them). However, if we regard x_n and y_n as (time) parameters, then $\{(x', \phi'_{x'}, y', -\phi_{y'})\} = \{(x', \theta, y', \theta)\}$ is the graph of the canonical transformation $y' = x' + (x_n - y_n)\theta/|\theta|$, (One of course needs to check $d\eta' \wedge dy' = d\xi' \wedge dx'$.) therefore defines a local canonical graph.

4.2 Adjoints and products

Theorem (4.2.1, Adjoints)**.** *Let* $\rho > 1/2$*.* $A \in I^m_\rho(X \times Y, C')$*, where* C *is a homogeneous canonical relation from* $T^*(Y)$ *to* $T^*(X)$*, is defined as*

$$\langle A, u\rangle = (2\pi)^{-(n_X+n_Y+2N)/4} \iiint e^{i\phi(x,y,\theta)} a(x,y,\theta)u(x,y)\, dxdyd\theta,$$

viewed as a continuous map $C_0^\infty(Y, \Omega_{1/2}) \to \mathcal{D}'(X, \Omega_{1/2})$*, the adjoint* A^* *satisfies*

$$(Au, v) = (u, A^*v),\ u \in C_0^\infty(Y, \Omega_{1/2}),\ v \in C_0^\infty(X, \Omega_{1/2}).$$

Then $A^* \in I^m_\rho(Y \times X, C'_s)$ *where* C_s *is the inverse image of* C *under the map*

$$s : T^*(Y) \times T^*(X) \to T^*(X) \times T^*(Y)$$

interchanging the two factors. Furthermore, if

$$a \in I_\rho^{m+(n_X+n_Y)/4}(C, \Omega_{1/2} \otimes L_C)$$

is a principal symbol of A, then

$$s^*\overline{a} \in I_\rho^{m+(n_X+n_Y)/4}(C_s, \Omega_{1/2} \otimes L_{C_s})$$

is a principal symbol of A^*.

Theorem (4.2.2 and 4.2.3, Products)**.** *Let $\rho > 1/2$. Let C_1 and C_2 be homogeneous canonical relations from $T^*(Y)$ to $T^*(X)$ and from $T^*(Z)$ to $T^*(Y)$ respectively, assume that $C_1 \times C_2$ intersects the diagonal in $T^*(X) \times T^*(Y) \times T^*(Y) \times T^*(Z)$ transversally and that the projection from the intersection to $T^*(X) \times T^*(Z)$ is proper, thus gives a homogeneous canonical relation $C_1 \circ C_2$ from $T^*(Z)$ to $T^*(X)$.*

If $A_1 \in I_\rho^{m_1}(X \times Y, C_1')$, $A_2 \in I_\rho^{m_2}(Y \times Z, C_2')$ are properly supported, it follows that

$$A_1 A_2 \in I_\rho^{m_1+m_2}(X \times Z, (C_1 \circ C_2)').$$

Furthermore, if a_1 and a_2 are principal symbols of A_1 and A_2, then $a_1 \times a_2$ is a principal symbol of $A_1 A_2$.

4.3 L^2 estimates

Theorem (4.3.1, L^2 continuity and compactness under local canonical graphs)**.** *If $A \in I^0_\rho(X \times Y, C')$ and properly supported, C is a graph of a canonical transformation $T^*(Y) \to T^*(X)$, then A is bounded from $L^2_c(Y, \Omega_{1/2})$ to $L^2_c(X, \Omega_{1/2})$, from $L^2_{\text{loc}}(Y, \Omega_{1/2})$ to $L^2_{\text{loc}}(X, \Omega_{1/2})$. A is compact iff a principal symbol tends to 0 at ∞ on C over compact subsets of $X \times Y$.*

Outline of the proof for Theorem 4.3.1.

(1). By the adjoint and product theorems of FIOs, under local canonical graph condition, A^*A is a ΨDO with principal symbol $|a|^2$ on $T^*(Y)$ if a is a principal symbol of A. Furthermore, a and $|a|^2$ are both of degree 0 because the order of FIO equals the the degree of the principal symbol under local canonical graphs from (4.1.7).

(2). The L^2 continuity of A and A^* follows that of A^*A, therefore the classical results on L^2 estimates for ΨDOs with order/degree 0 (cf. §2.2) imply that of A.

(3). The theorem is still valid if $A \in I_\rho^m$ when $m \leq 0$.

□

Theorem (4.3.2, L^2 continuity under homogeneous canonical relations)**.** *If $A \in I_\rho^m(X \times Y, C')$ and properly supported, C is a homogeneous canonical relation, and $C \to X$, $C \to Y$ have surjective differentials, then A is L^2 bounded from $L_c^2(Y, \Omega_{1/2})$ to $L_{\text{loc}}^2(X, \Omega_{1/2})$ provided that*

$$m \leq (2k - n_X - n_Y)/4.$$

Here, the differentials of the projections $C \to T^*(X)$ and $C \to T^*(Y)$ have rank at least $k + n_X$ and $k + n_Y$.

Outline of the proof for Theorem 4.3.2.

(1). Locally we split the x and y coordinates into (x', x'') and (y', y'') respectively: x' and y' have k coordinates, x'' has $(n_X - k)$ coordinates, and y'' has $(n_Y - k)$ coordinates.

Moreover, if we denote X_1 and Y_1 are the x'-space and y'-space, then C defines a canonical graph $C_1 : T^*(Y_1) \to T^*(X_1)$.

(2). For a fixed pair (x'', y''), write $A_{(x'',y'')}$ as an FIO of the form

$$\begin{aligned} &A_{(x'',y'')}u(\cdot, y'')(x') \\ &= \frac{1}{(2\pi)^{(2k+2N)/4}} \iint e^{i(\phi(x',x'',y',y'',\theta)-\pi N/4)} \\ &\qquad a(x', x'', y', y'', \theta)u(y', y'')\, dy', \end{aligned}$$

with a symbol $a \in S^{m+(n_X+n_Y-2N)/4}(\mathbb{R}^{n_X+n_Y} \times \mathbb{R}^N)$.

To find the order of $A_{(x'',y'')}$, recall that in the definition of FIO in (3.2.14), the degree of the principal symbol is $(n-2N)/4$ larger than the order of the FIO, therefore $A_{(x'',y'')}$ is an FIO in $I_\rho^{m_1}(X_1 \times Y_1, C_1)$ where

$$\begin{aligned} m_1 &= m + \frac{n_X + n_Y - 2N}{4} - \frac{\dim X_1 + \dim Y_1 - 2N}{4} \\ &= m - \frac{2k - n_X - n_Y}{4}. \end{aligned}$$

(3). If $m_1 \leq 0$, that is, $m \leq (2k - n_x - n_Y)/4$, then $A_{(x'',y'')}$ is an L^2 continuous FIO from $L^2_c(Y_1, \Omega_{1/2})$ to $L^2_{\text{loc}}(X_1, \Omega_{1/2})$ in the view of the above theorem for FIOs under local canonical graphs.

(4). Write

$$Au(x) = \int [A_{(x'',y'')} u(\cdot, y'')(x')]\, dy'',$$

in which the L^2 norms of $A_{(x'',y'')}$ are uniformly bounded as

(x'', y'') varies, and the main theorem follows by integrating in these two variables.

(5). The procedure described in (i)–(iv) works if

$$C \to X,\ C \to Y \tag{4.1.8}$$

have surjective differentials, and the rank of the differentials of the projections $C \to T^*(X)$ and $C \to T^*(Y)$ are $\geq k + n_X$ and $\geq k + n_Y$. This is essentially Theorem 4.1.9.

(6). Condition (4.1.8) can be relaxed substantially, see Theorem 4.4.4 in [D96] for details.

□

Remark. The above L^2 continuity results can be generalized to $H_{(s)} \to H_{(s-m)}$ for FIOs in I^m_ρ, see e.g. [H85, §25.3].

5 Additional results on the calculus

Following §IV, we continue investigating the calculus of FIOs and accomplish these goals:

1. Generalize asymptotic expansion theorem to FIOs in $I_\rho^m(X,\Lambda)$.
2. Generalize definitions of characteristic and ellipticity to FIOs in $I_\rho^m(X,\Lambda)$, then prove the inverse theorem for elliptic FIOs in $I_\rho^m(X\times Y,C')$ under local canonical graphs.
3. Define and investigate the subprincipal symbols of ΨDOs.
4. Under a special case, compute the principal symbol of PA for $P\in L_\rho^m(X)$ and $A\in I_\rho^m(X\times Y,C')$ using the subprincipal symbol of P.
5. Give Sobolev characterization of Lagrangean distributions in $I_\rho^m(X,\Lambda)$.

5 Additional results on the calculus

5.1 Preliminaries

Proposition (5.1.1)**.** *Let* $A_k \in I_\rho^{m_k}(X,\Lambda)$*,* $k = 0, 1, 2, \ldots$ *where* $m_k \to -\infty$ *as* $k \to \infty$*. Set* $m'_k = \max_{j \geq k} m_j$*. Then one can find* $A \in I_\rho^{m_0}(X,\Lambda)$ *such that*

$$A - \sum_{j<k} A_j \in I_\rho^{m'_k}(X,\Lambda), \quad k=1,2,\ldots \tag{5.1.1}$$

A is uniquely determined modulo $C^\infty(X)$ *and has the same property relative to any rearrangement of the series* $\sum A_j$*; we write* $A \sim \sum A_j$*.*

Proof of Proposition 5.1.1.

(1). Recall $I_\rho^m(X,\Lambda)$ in Definition 3.2.2, this proposition is essentially local.

(2). Each A_k can be split into a sum of $\sum A_{ki}$ modulo C^∞ since the supports of these distributions are locally finite.

(3). For a fixed i, the problem is reduced to the case for symbol classes in $\mathbb{R}^n \times \mathbb{R}^{N_i}$. (See Proposition 1.1.9.)

□

Definition (Characteristic set). For $A \in I_\rho^m(X, \Lambda)$ is non-characteristic at $\lambda \in \Lambda$ iff any principal symbol $a \in S^{m+n/4}(\Lambda, \Omega_{1/2} \otimes L)$ has a reciprocal $B \in S^{-m-n/4}(\Lambda, \Omega_{-1/2} \otimes L^{-1})$ in a conic neighborhood of λ. The set of characteristic points is denoted by $\gamma(A)$. A is called elliptic if $\gamma(A) = \emptyset$.

Proposition (5.1.2)**.** *Let C be a bijective homogeneous canonical transformation from $T^*(Y)\backslash 0$ onto $T^*(X)\backslash 0$, thus $\dim X = \dim Y$, and assume that*

$$A \in I^m_\rho(X \times Y, C')$$

is elliptic and properly supported. Then there exists a properly supported elliptic FIO

$$B \in I^{-m}_\rho(Y \times X, (C^{-1})')$$

which is a left and right parametrix, that is,

$$BA - \mathrm{Id} \quad \text{and} \quad AB - \mathrm{Id}$$

have C^∞ kernels. Any other parametrix for A differs from B by an operator with C^∞ kernel.

Proof of Proposition 5.1.2.

(1). We can regard a principal symbol a of A as an element of $S^m_\rho(C, L_C)$ since C is a local canonical graph.

(2). Ellipticity of A guarantees that there exists $b \in S^{-m}_\rho(C, L_C^{-1})$ such that $ba = 1$ outside a large enough sphere. Then if

$$B_0 \in I^{-m}_\rho(Y \times X, (C^{-1})')$$

has a principal symbol b, then $AB_0 = \mathrm{Id} - R_1$ and $B_0 A = \mathrm{Id} - R_2$ where R_1 and R_2 are ΨDOs with degree -1 from the product theorem. (See Theorems 4.2.2 and 4.2.3.)

(3). The problem is reduced to finding the right parametrix of $\mathrm{Id} - R_1$ and the left parametrix of $\mathrm{Id} - R_2$. In fact, they have two sided parametrices $\sim \sum_0^\infty R_1^k$ and $\sim \sum_0^\infty R_2^k$. (See Proposition 2.5.1.)

□

Remark (Local notions and results). An FIO originally defined in $I_\rho^m(X,\Lambda)$ can be restricted to $I_\rho^m(X,K)$ by pulling back of a C^∞ mapping. Here $K\subset\Lambda$ is a closed conic subset of Λ. An analogous isomorphism follows as

$$S_\rho^{m+n/4}(K,\Omega_{1/2}\otimes L)/S_\rho^{m+n/4+1-2\rho}(K,\Omega_{1/2}\otimes L)$$
$$\to I_\rho^m(X,K)/I_\rho^{m+1-2\rho}(X,K)$$

where $S_\rho^{m+n/4}(K,\Omega_{1/2}\otimes L)$ denotes the set of

$$a\in S_\rho^{m+n/4}(\Lambda,\Omega_{1/2}\otimes L)$$

such that $a\in S^{-\infty}$ on $\Lambda\setminus K$. Note that this definition and isomorphism depend on Λ though it does not appear in the equation.

Now let C be the graph of a homogeneous canonical transformation from a conic neighborhood of $(y_0,\eta_0)\in T^*(Y)\setminus 0$ to a conic neighborhood of $(x_0, \xi_0) \in T^*(Y) \setminus 0$ with $c_0 = ((x_0, \xi_0), (y_0, \eta_0)) \in C$.

Let $A \in I^m_\rho(X \times Y, K')$ be a closed conic subset of C which is non-characteristic at c_0. Then by Proposition 5.1.2, one can find

$$B \in I^{-m}_\rho(Y \times X, (K^{-1})')$$

which is the left and right parametrix of A, i.e.,

$$BA - \mathrm{Id}_Y \quad \text{and} \quad AB - \mathrm{Id}_X$$

have C^∞ kernels, and therefore,

$$(x_0, \xi_0) \notin WF(AB - \mathrm{Id}_X),$$

and

$$(y_0, \eta_0) \notin WF(BA - \mathrm{Id}_Y).$$

5.2 The subprincipal symbol of a ΨDO

Now back to (2.1.14) and (2.1.17) and continue to work on the expansion to $|\alpha| = 1$, assume the expansions are $a = a^0 + a^1$ and $a_\kappa = a_\kappa^0 + a_\kappa^1$. Then the principal symbols $a_\kappa^0(\kappa(x), \xi) = a^0(x, {}^t\kappa'(x)\xi)$, and if we define the subprincipal symbols

$$a^{1s}(x, \xi) = a^1(x, \xi) - \frac{1}{2i} \sum \frac{\partial^2 a^0(x, \xi)}{\partial x_i \partial \xi_j},$$

a computation (See pp. 83 in §18.1 of [H85].) gives

$$a_\kappa^{1s}(x, \xi) = a^{1s}(x, \xi) - \frac{1}{2} \sum \frac{\partial a^0(x, \xi)}{\partial \xi_j} \cdot \frac{D_j |\kappa'(x)|}{|\kappa'(x)|},$$

which is invariant at the points in $T^*(X)$ where a^0 vanishes of second order. It is also invariant under measure preserving changes of variables, and we shall see later a complete invariant.

To explain the above reasoning more clearly and cleanly, we utilize the half densities to take care the case when $|\alpha| = 1, 2$. The following proposition is essentially local.

Proposition (5.2.1)**.** *Let P be a ΨDO in X, considered as an operator between densities of order $1/2$. If $P \in L^m_\rho$ and for some choice of local coordinates $p(x,\xi)$ denotes the full symbol, then*

$$p - \frac{1}{2i} \sum \frac{\partial^2 p}{\partial x_i \partial \xi_j} \in S^m_\rho \tag{5.2.7}$$

is modulo $S^{m+2(1-2\rho)}$ independent of the choice of local coordinates.

Remark. Consider a ΨDO $P \in L^m(X)$ with a homogeneous principal symbol p. Then there exists a C^∞ homogeneous function p of degree m on $T^*(X) \setminus 0$ such that for any system of local coordinates the full symbol of P is of the from $p+r$ where $r \in S^{m-1}$. Clearly p is then uniquely determined by P, and the above proposition asserts

$$r - \frac{1}{2i}\sum \frac{\partial^2 p}{\partial x_i \partial \xi_j} \in S^{m-1} \tag{5.2.8}$$

is uniquely determined modulo S^{m-2}. We can therefore choose $c \in S^{m-1}(T^*(X))$ which agrees with (5.2.8) modulo S^{m-2}, called a sub-principal symbol of P, and of course we can choose c homogeneous of degree $m-1$.

Proof of Proposition 5.2.1.

(1). For a local coordinates $x_1, ..., x_n$ in an open set X_1, consider another system of coordinates $\varphi_1, ..., \varphi_n$ and set

$$\varphi(x,\theta) = \sum \varphi_j(x)\theta_j.$$

Note that $\varphi = \kappa$ in previous computations.

Choose a half density $w \in C_0^\infty(X_1)$ and consider $e^{-i\varphi}P(we^{i\varphi})$ as a function on $T^*(X_1)$.

(2). Observe (2.1.14) above,
(5.2.1)
$$e^{-i\varphi}P(we^{i\varphi}) \sim \sum p^{(\alpha)}(x, \varphi'_x) D_z^\alpha (w(z)e^{i\rho(x,z,\theta)})/\alpha!|_{z=x}$$

where

$$\rho(x, z, \theta) = \varphi(z, \theta) - \varphi(x, \theta) - \langle z - x, \varphi'_x(x, \theta)\rangle.$$

(3). Expand the summation above to $|\alpha| = 2$, recall the values of $D_z^\alpha e^{i\rho(x,z,\theta)}|_{z=x}$ in (2.1.17):

(a). $\alpha = 0$: $D_z^\alpha(w(z)e^{i\rho(x,z,\theta)})|_{z=x} = w$.

(b). $|\alpha| = 1$: $\sum D_z^{(j)}(w(z)e^{i\rho(x,z,\theta)})|_{z=x} = \sum D_j w$.

(c). $|\alpha| = 2$:

$$\begin{aligned}
&\sum D_z^{(jk)}(we^{i\rho(x,z,\theta)})|_{z=x} \\
&= \sum w(x) D_x^{(jk)} i\varphi(x,\theta) + \sum D_x^{(jk)} w(x) \\
&= \sum \frac{w(x)}{i} \frac{\partial^2 \varphi(x)}{\partial x_j \partial x_k} \qquad \bmod \ S^0.
\end{aligned}$$

Therefore,

$$e^{-i\varphi} P(we^{i\varphi}) = p(x, \varphi'_x) w(x) + \sum p^{(j)}(x, \varphi'_x) D_j w(x) + \frac{1}{2i} \sum p^{(jk)}(x, \varphi'_x) w(x) \frac{\partial^2 \varphi}{\partial x_j \partial x_k} \qquad \bmod \ S^{m+2(1-2\rho)}, \tag{5.2.2}$$

in which

$$\sum p^{(jk)}(x, \varphi'_x) \frac{\partial^2 \varphi}{\partial x_j \partial x_k} = \sum \frac{\partial p^{(j)}(x, \varphi'_x)}{\partial x_j} - \sum \frac{\partial^2 p(x, \varphi'_x)}{\partial x_j \partial \xi_k}.$$

(4). Rewrite (5.2.2) as

$$e^{-i\varphi}P(we^{i\varphi}) = \left[p(x,\varphi'_x) - \frac{1}{2i}\sum \frac{\partial^2 p}{\partial x_i \partial \xi_j}\right] w(x) \tag{5.2.6}$$

$$-i\sum\left[p^{(j)}(x,\varphi'_x)\frac{\partial w(x)}{\partial x_j} + \frac{1}{2}\frac{\partial p^{(j)}(x,\varphi'_x)}{\partial x_j}w(x)\right] \quad \mod\ S^{m+2(1-2\rho)},$$

and we only need to show the second summation is module $S^{m+2(1-2\rho)}$ independent of the choice of local coordinates $x_1, ..., x_n$ to finish the proof.

(5). In fact,

$$\sum p^{(j)}(x,\varphi'_x)\frac{\partial w(x)}{\partial x_j} + \frac{1}{2}\frac{\partial p^{(j)}(x,\varphi'_x)}{\partial x_j}w(x) = \mathcal{L}_v w,$$

where v is the vector field $(p^{(1)}(x,\varphi'_x), ..., p^{(n)}(x,\varphi'_x))$ in X_1 defined by the function $p(x,\xi)$ on $T^*(X)$ and $\mathcal{L}_v w$ is the Lie derivative of w with respect of v, and it is independent of the choice of local coordinates.

□

Remark (Lie derivatives of densities, cf. pp. 22 in §25.2 of [H85]). Let v be a real C^∞ vector field on a manifold X. Then v generates a local one parameter group of C^∞ maps φ^t in X, defined by

$$\frac{d\varphi^t(x)}{dt} = v(\varphi^t(x)),\ \varphi^0(x) = x,\ x \in X.$$

If $a \in \Omega^\alpha(X)$, then we define the **Lie derivative** $\mathcal{L}_v a$ along v by

$$\mathcal{L}_v a = \frac{d}{dt}(\varphi^t)^* a|_{t=0}. \tag{5.2.4}$$

Let $x_1, ..., x_n$ be local coordinates in X and write $a = u|dx|^\alpha$. Then

$$(\varphi^t)^* a = u_t|dx|^\alpha,$$

in which

$$u_t(x) = u(\varphi^t(x)) \left| \frac{\partial \varphi^t(x)}{\partial x} \right|^\alpha .$$

Hence,

$$\begin{aligned}
&\mathcal{L}_v(u|dx|^\alpha) \\
&= \frac{d}{dt}(\varphi^t)^* a|_{t=0} \\
&= |dx|^\alpha \left[\frac{d}{dt} u(\varphi^t(x)) \left| \frac{\partial \varphi^t(x)}{\partial x} \right|^\alpha \right]_{t=0} + |dx|^\alpha \left[u(\varphi^t(x)) \frac{d}{dt} \left| \frac{\partial \varphi^t(x)}{\partial x} \right|^\alpha \right]_{t=0} \\
&= \left(\sum v_j \frac{\partial u}{\partial x_j} + \alpha \cdot div\, v \cdot u \right) |dx|^\alpha ,
\end{aligned}$$

where we compute that the derivative of the Jacobian is $Tr(\partial v_j / \partial x_k) = div\, v$ when $t = 0$.

We take this as the definition of the Lie derivative if v is a complex vector field. it is clear that the definition is independent of the choice of local coordinates since this is true when v is real.

5.3 Products with vanishing principal symbol

Throughout this section, we denote

- $P \in L^m_\rho(X)$ is a properly supported ΨDO with a homogeneous principal symbol p and a subprincipal symbol c.
- H_p is the Hamilton field of p, that is,
$$\sum \left(\frac{\partial p}{\partial \xi_j} \cdot \frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j} \cdot \frac{\partial}{\partial \xi_j} \right),$$
and $\mathcal{L}_{H_p}$ is the corresponding Lie derivative.
- $C : T^*(Y) \setminus 0 \to T^*(X) \setminus 0$ is a homogeneous canonical relation.
- $A \in I^{m'}_\rho(X \times Y, C')$, and $a \in S^{m'+(n_X+n_Y)/4}_\rho(C', \Omega_{1/2} \otimes L)$ is a principal symbol of A.
- $n = n_X + n_Y$.

Theorem (5.3.1)**.** *If* $p = 0$ *on the projection of* C *on* $T^*(X) \setminus 0$*, then* $PA \in I_\rho^{m+m'-\rho}(X \times Y, C')$ *has*

$$i^{-1}\mathcal{L}_{H_p}a + ca \tag{5.3.1}$$

as a principal symbol. Here H_p *is lifted to a function on* $(T^*(X) \setminus 0) \times (T^*(Y) \setminus 0)$ *via the projection onto the first factor. The vector field* $\mathcal{L}_{H_p}$ *is tangent to* C *so (5.3.1) is well defined.*

Remark. Note that since the Keller-Maslov bundle L is not involved in differentiations because the transition functions are constants, then we define $\mathcal{L}_{H_p}a$ by using local trivializations of L which only differ by a constant factor and therefore do not affect the definition.

Proof of Theorem 5.3.1.

(1). As in the proof of Theorem 5.2.1, we argue locally. Recall Example 3.1.2 and Theorem 3.1.3, we can choose local coordinates in X and Y so that the phase function for A is

$$\varphi(x, y, \xi, \eta) = \langle x, \xi\rangle + \langle y, \eta\rangle - H(\xi, \eta),$$

that is, the Lagrangean submanifold C is locally paramatrized by (ξ, η). Here, H is defined in a conic neighborhood of

$$(\xi_0, \eta_0) \in (\mathbb{R}^{n_X} \setminus 0) \times (\mathbb{R}^{n_Y} \setminus 0).$$

Therefore, the map

$$(\xi, \eta) \to (H'_\xi, H'_\eta, \xi, \eta),$$

is a local parametrization of the critical set

$$C_\varphi = \{\varphi'_\xi = \varphi'_\eta = 0\},$$

and the density on C_φ defined by the pull back of the Dirac measure in $\mathbb{R}^{n_X+n_Y}$ with the map

$$(x, y, \xi, \eta) \to (x - H'_\xi, y - H'_\eta)$$

coincides with the Lebesgue measure in (ξ, η). (One can take a simple example $\rho^*(\delta)$ where $\rho(x, \xi) = x - f(\xi)$.)

(2). Under this local coordinates
(5.3.2)

$$Au(x) = \frac{1}{(2\pi)^{3(n_X+n_Y)/4}} \iiint e^{i[\langle x,\xi\rangle + \langle y,\eta\rangle - H(\xi,\eta)]} a_0(\xi, \eta) u(y) dy d\xi d\eta$$

for $u \in C_0^\infty$, in which

$$a_0 \in S^{m'-(n_X+n_Y)/4}(\mathbb{R}^{n_X+n_Y} \times \mathbb{R}^{n_X+n_Y})$$

has a support in a conic neighborhood of (ξ_0, η_0) where H is C^∞.

This means locally the principal symbol a can be represented as

$$a_0|d\xi|^{\frac{1}{2}}|d\eta|^{\frac{1}{2}}.$$

Rewrite Au as

$$\frac{1}{(2\pi)^{3(n_X+n_Y)/4}}\iint e^{i[\langle x,\xi\rangle - H(\xi,\eta)]}a_0(\xi,\eta)\hat{u}(-\eta)dyd\xi d\eta$$
$$= \frac{1}{(2\pi)^{3(n_X+n_Y)/4}}\int e^{i\langle x,\xi\rangle}\left[\int e^{-iH(\xi,\eta)}a_0(\xi,\eta)\hat{u}(-\eta)d\eta\right]d\xi.$$

We see that the Fourier transform of Au is

$$\frac{(2\pi)^{n_X}}{(2\pi)^{3(n_X+n_Y)/4}}\int e^{-iH(\xi,\eta)}a_0(\xi,\eta)\hat{u}(-\eta)d\eta,$$

so

$$PAu(x) = \frac{1}{(2\pi)^{3(n_X+n_Y)/4}}$$

(5.3.3)

$$\iiint e^{i[\langle x,\xi\rangle+\langle y,\eta\rangle-H(\xi,\eta)]}\left[p(x,\xi)+r(x,\xi)\right]a_0(\xi,\eta)u(y)dyd\xi d\eta,$$

in which $p+r$ is the full symbol of P and p is a principal symbol. Since $p(x,\xi)=0$ on C we can write by Taylor's formula

(5.3.4)

$$p(x,\xi) = \sum p_j(x,\xi,\eta)\left(x_j - \frac{\partial H}{\partial \xi_j}\right) = \sum p_j(x,\xi,\eta)\cdot\frac{\partial\varphi(x,y,\xi,\eta)}{\partial\xi_j},$$

where $p_j \in C^\infty$ and is homogeneous of degree m with respect to (ξ,η).

(3). Integrate by parts on ξ_j in (5.3.3) yields

$$\begin{aligned}&\int_{\mathbb{R}} e^{i\varphi}\cdot p_j\cdot\frac{\partial\varphi}{\partial\xi_j}\cdot a_0 d\xi_j\\ &=\int_{\mathbb{R}}\frac{\partial e^{i\varphi}}{i\partial\xi_j}\cdot p_j\cdot a_0 d\xi_j\\ &=-\int_{\mathbb{R}} e^{i\varphi}\cdot\frac{\partial(p_j\cdot a_0)}{i\partial\xi_j}d\xi_j.\end{aligned}$$

Therefore,

$$PAu(x)=\frac{1}{(2\pi)^{3(n_X+n_Y)/4}}$$

(5.3.5)

$$\iiint e^{i\varphi(x,y,\xi,\eta)}\left[ra_0-\frac{1}{i}\sum\frac{\partial(p_j(x,\xi,\eta)a_0(\xi,\eta))}{\partial\xi_j}\right]u(y)dyd\xi d\eta,$$

and since the principal symbol of A is

$$a_0(\xi,\eta)|d\xi|^{\frac{1}{2}}|d\eta|^{\frac{1}{2}},$$

the principal symbol of PA is

$$\left[ra_0 - \sum D_{\xi_j}(p_j a_0)\right]_{x=\partial H/\partial\xi} |d\xi|^{\frac{1}{2}}|d\eta|^{\frac{1}{2}}. \tag{5.3.6}$$

(4). Now

$$H_p = \sum\left(\frac{\partial p}{\partial\xi_j}\cdot\frac{\partial}{\partial x_j} - \frac{\partial p}{\partial x_j}\cdot\frac{\partial}{\partial\xi_j}\right)$$

is tangential to C. Functions on C are restrictions of functions of the form $F(\xi,\eta)$, to in terms of the parameters (ξ,η) on C this vector field has the form

$$\sum -\frac{\partial p}{\partial x_j}\bigg|_{x=\partial H/\partial\xi}\cdot\frac{\partial}{\partial\xi_j} = \sum -p_j\cdot\frac{\partial}{\partial\xi_j}.$$

Recall the formula of Lie derivative of half densities in §5.2:

$$\mathcal{L}_{H_p}(a_0|d\xi|^{\frac{1}{2}}|d\eta|^{\frac{1}{2}}) = \left[\sum -p_j \cdot \frac{\partial a_0}{\partial \xi_j} - \frac{1}{2}\frac{\partial p_j}{\partial \xi_j} a_0\right] |d\xi|^{\frac{1}{2}}|d\eta|^{\frac{1}{2}}.$$

Plugging back to (5.3.6),

$$\begin{aligned}
&\left[r a_0 - \sum D_{\xi_j}(p_j a_0)\right]_{x=\partial H/\partial \xi} |d\xi|^{\frac{1}{2}}|d\eta|^{\frac{1}{2}} \\
= &\left[r a_0 - \sum a_0 D_{\xi_j} p_j - \sum p_j D_{\xi_j} a_0\right]_{x=\partial H/\partial \xi} |d\xi|^{\frac{1}{2}}|d\eta|^{\frac{1}{2}} \\
= &\left(r - \frac{1}{2i}\sum \frac{\partial^2 p}{\partial x_j \partial \xi_j}\right) a_0|d\xi|^{\frac{1}{2}}|d\eta|^{\frac{1}{2}} + i^{-1}\mathcal{L}_{H_p}(a_0|d\xi|^{\frac{1}{2}}|d\eta|^{\frac{1}{2}}),
\end{aligned}$$

noting that $p_j = \partial p/\partial x_j$ when $x = H'_{\xi_j}$. The theorem is completed if we recall the definition of the subprincipal c of P, and the order of PA follows naturally.

□

Theorem (5.3.2)**.** *Let $\rho > 2/3$. Assume that for every $\mu \in \mathbb{R}$,*

$$H_p S^{\mu}(C) \supset S^{\mu+m-1}(C). \tag{5.3.10}$$

For every $B \in I_{\rho}^{m+m'-1}(X \times Y, C')$ one can then find $A \in I_{\rho}^{m'}(X \times Y)$ such that

$$PA - B$$

has a C^{∞} kernel. If b is the principal symbol of B and a is any solution of

$$b = i^{-1}\mathcal{L}_{H_p} a + ca \tag{5.3.9}$$

belonging to $S^{m'+n/4}(C, L \otimes \Omega_{1/2})$ one can choose A with principal symbol in the class of a modulo $S^{m'+n/4+2-3\rho}(C, L \otimes \Omega_{1/2})$.

Proof of Theorem 5.3.2.

(1). We want to find the solution to (5.3.9). Write $a = a_0 w$ and $b = b_0 w$ where a_0 and b_0 are scalar symbols. Then

$$\mathcal{L}_{H_p}(a_0 w) = \left(H_p a_0 - \frac{1}{2} \frac{\partial^2 p}{\partial x_j \partial \xi_j} a_0 \right) w,$$

from proof of Theorem 5.3.1. Then (5.3.9) is reduced to

$$i^{-1}\mathcal{L}_{H_p} a_0 + c' a_0 = b_0$$

for some $c' \in S^{m-1}$. The hypothesis guarantees the existence of $\gamma \in S^0$ such that

$$H_p \gamma = c'.$$

One also has $e^{\pm i\gamma} \in S^0$ by Proposition 1.1.8. If we denote $a_0 = ie^{-i\gamma} a_1$ and $b_0 = e^{-i\gamma} b_1$, then the equation is further reduced to

$$i^{-1}\mathcal{L}_{H_p}(ie^{-i\gamma} a_1) + H_p\gamma \cdot ie^{-i\gamma} a_1 = e^{-i\gamma} H_p a_1 = e^{-i\gamma} b_1.$$

That is,

$$H_p a_1 = b_1,$$

where $a_1 \in S^{\mu}$ and $b_1 \in S^{\mu+m-1}$, and this has a solution according to the hypothesis.

(2). Let $A_0 \in I_{\rho}^{m'}(X \times Y, C')$ with a principal symbol satisfying (5.3.9), and set

$$B_1 = B - PA_0,$$

then B and PA_0 are both in $I_{\rho}^{m+m'-1}(X \times Y, C')$ with the same principal symbols, therefore

$$B_1 \in I_{\rho}^{\mu}(X \times Y, C'),$$

where

$$\mu = m + m' - \rho - (2\rho - 1) = m + m' - 1 - (3\rho - 2).$$

Iterating the argument in (1), we obtain sequences

$$A_j \in I_\rho^{m'-j(3\rho-2)}(X \times Y, C')$$

and

$$B_j \in I_\rho^{m+m'-1-j(3\rho-2)}(X \times Y, C')$$

such that

$$B_0 = B$$

and

(5.3.11) $$B_{j+1} = B_j - PA_j, \ j = 0, 1, 2, ...$$

Let

$$A = \sum A_j$$

then adding (5.3.11) yields

$$P(A_0 + \cdots + A_j) = B_0 - B_{j+1},$$

which says $PA - B_0 \in I_\rho^{-\infty}(X \times Y, C') = C^\infty(X \times Y)$ and therefore proves the theorem.

□

Remark (1). A similar results is valid for $AP = B$ for this is equivalent to $P^*A^* = B^*$.

5.4 The smoothness of elements in $I_\rho^m(X, \Lambda)$

Theorem (5.4.1)**.** *$I_\rho^m(X,\Lambda) \subset H_{(s)}$ iff $m + n/4 + s < 0$. Moreover, if $u \in I_\rho^m(X,\Lambda)$ and u has some non-characteristic point, then it follows that $u \notin H_{(s)}$ when $m + n/4 + s \geq 0$.*

Remark. If we regard $\Lambda \subset T^*(X) \times T^*(\{0\})$ as a canonical relation from $T^*(\{0\})$ to $T^*(X)$ and u as a multiplicative operator from $C_0^\infty(\{0\})$ to $\mathcal{D}'(X)$, the projection $\Lambda \to X$ is generally not surjective. (For if it is, then $u \in C^\infty$ and there is nothing to prove.) Therefore, we can not apply Theorem 4.3.2 here to prove the L^2 boundedness of u. One instead has to create a more suitable Y (namely, $\mathbb{R}^n$) to transform the question to a simpler setting.

Definition (Hilbert–Schmidt operators, cf. Definition 19.1.11 in [H85])**.** If H_1 and H_2 are Hilbert spaces then the space $\mathcal{L}_2(H_1, H_2)$ of Hilbert-

Schmidt operators from H_1 to H_2 consists of all $T \in \mathcal{L}_2(H_1, H_2)$ such that

$$\|T\|_2^2 = \sum \|Te_i\|^2$$

is finite, if $\{e_i\}$ is a complete orthonormal system in H_1.

Proposition (Hilbert–Schmidt integral operators)**.** *A Hilbert–Schmidt kernel is a function* $k\colon (X, d\mu) \otimes (Y, d\nu) \to \mathbb{C}$ *with*

$$\|k\|_{L^2}^2 = \int_X \int_Y |k(x,y)|^2 d\mu(x) d\nu(y) < \infty,$$

and the associated Hilbert–Schmidt integral operator is the operator $K\colon L^2(Y) \to L^2(X)$ *given by*

$$Ku(x) = \int_Y k(x,y) u(y) d\nu(y).$$

Then K *is a Hilbert–Schmidt operator with Hilbert–Schmidt norm*

$$\|K\|_2 = \|k\|_{L^2}.$$

Corollary (5.4.2)**.** *An FIO from* $\mathcal{D}'(Y)$ *to* $\mathcal{D}'(X)$ *is a Hilbert–Schmidt integral operator if it is of order* $< -(n_X + n_Y)/4$ *and the kernel has compact support.*

Proof of Theorem 5.4.1.

(1). It is sufficient to prove the results for $s = 0$. For if

$$I_\rho^m(X, \Lambda) \subset H_{(0)}$$

if $m+n/4 < 0$, then for any $u \in I_\rho^{m'}(X, \Lambda)$ with $m'+n/4+s < 0$, one can find an elliptic ΨDO $B \in L^s(X)$ such that

$$v = Bu \in I_\rho^{m'+s}$$

satisfying $(m' + s) + n/4 < 0$. Applying the hypothesis,

$$v \in H_{(0)},$$

which implies

$$u = B^{-1}v \in H_{(s)}$$

noting that $B^{-1} \in L^{-s}(X)$ since B is elliptic. The other result is valid by a similar argument.

(2). It is sufficient to prove the results if $u \in I^m_\rho(X, \Lambda)$ and $WF(u)$ is in a small neighborhood of $(x_0, \xi_0) \in \Lambda$ for u is C^∞ in $\Lambda \backslash WF(u)$. That is, we want to show

(a). $u \in H_{(0)} = L^2_{\text{loc}}$ iff $m + n/4 < 0$,

(b). $u \notin L^2_{\text{loc}}$ if $m + n/4 \geq 0$ and (x_0, ξ_0) is a non-characteristic point.

(3). It is sufficient to prove the results for the case in terms of

$$u(x) = \frac{1}{(2\pi)^{3n/4}} \int e^{i\langle x,\theta\rangle} a(\theta) d\theta.$$

where $a \in S^{m-n/4}$. Since u is rapidly decreasing at infinity,

$$u \in L^2_{\text{loc}} \Leftrightarrow u \in L^2 \Leftrightarrow a \in L^2.$$

(a). By Parseval's formula,

$$a \in L^2_{\text{loc}} \Leftrightarrow 2(m - n/4) < -n \Leftrightarrow m + n/4 < 0.$$

(b). There is a non-characteristic point, and then $a^{-1} \in S^{-m+n/4}$ implies

$$\frac{1}{|a(\theta)|} \leq (1+|\theta|)^{-m+n/4},$$

therefore,

$$|a(\theta)| \geq (1+|\theta|)^{m-n/4} \geq (1+|\theta|)^{-n/2} \Rightarrow a \notin L^2.$$

(4). We transform the question to the one in $\mathbb{R}^n$: Λ is defined by $x = H'(\xi)$ on suitably local coordinates. It follows that the homogeneous canonical transformation

$$C : (x, \xi) \to (x - H'(\xi), \xi)$$

maps Λ into the fiber in $T^*(\mathbb{R}^n) \setminus 0$ over $0 \in \mathbb{R}^n$.

Let $K \subset C$ be a closed and conic neighborhood of $(x_0, \xi_0, y_0, \eta_0)$. Here (y_0, η_0) is the image of (x_0, ξ_0) (in fact, $y_0 = 0$).

Choose $A \in I^0_\rho(X \times \mathbb{R}^n, K')$ and $B \in I^0_\rho(\mathbb{R}^n \times X, (K^{-1})')$ be two elliptic operators such that they are parametrices to each other, therefore according to (5.1.2) in §5.1

$$(x_0, \xi_0) \notin WF(AB - I_X).$$

We see that $ABu - u \in C^\infty$, hence by the L^2 continuity of FIOs with order 0,

$$u \in L^2_{\text{loc}} \Rightarrow Bu \in L^2_{\text{loc}} \Rightarrow ABu \in L^2_{\text{loc}} \Rightarrow u \in L^2_{\text{loc}}.$$

Now $Bu \in I^m_\rho(\mathbb{R}^n, ((K^{-1}) \circ \Lambda)')$, one only needs to notice that the Lagrangean is $T^*_0(\mathbb{R}^n)$ and Bu is of the special form

$$\frac{1}{(2\pi)^{3n/4}} \int e^{i\langle x,\theta\rangle} a(\theta) d\theta$$

where $a \in S^{m-n/4}$.

□

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