Complex Analysis

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Contents

CHAPTER 1

Preliminaries to complex analysis

In this chapter, we first define the basic structure of the complex number system $\mathbb{C}$. This structure is equipped with arithmetic rules, a metric, and a topology, which allows our analysis of functions of the complex numbers.

We then define the complex-valued functions f of a complex variable, that is, $f : \Omega \to \mathbb{C}$ for $\Omega \subset \mathbb{C}$. The main object of study in complex analysis is the properties of holomorphic functions (i.e., complex differentiable functions).

Lastly, we introduce the curves in $\mathbb{C}$ and the integrals of functions along curves. Most of the properties of holomorphic functions are reflected and studied through the integration along curves, such as the Cauchy's theorem in Chapter 2.

1.1. The complex plane

Definition (Complex numbers and complex plane). We use $z = x + iy$, in which $x, y \in \mathbb{R}$ and $i^2 = -1$, to denote a complex number. We say that x and y are the real and imaginary parts of z, denoted by $x = \Re(z)$ and $y = \Im(z)$.

The set of all complex numbers is denoted by $\mathbb{C}$. Each complex number $z = x + iy$ can be identified with a point $(x, y) \in \mathbb{R}^2$. Such identification defines the complex plane.

Remark (Arithmetic operations). The arithmetic operations (addition, subtraction, multiplication, and division) of the complex numbers are obtained simply by treating all numbers as if they were real, and keeping in mind that $i^2 = -1$. For example, let $z_1 = x_1 + iy_1, z_2 = x_2 + iy_2 \in \mathbb{C}$. Then

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2),$$

and

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1).$$

In this way, 0 and 1 are still the additive and multiplicative identities, respectively.

Remark. Let $z = x + iy \in \mathbb{C}$.

- The complex conjugate of z is defined as $\overline{z} = x - iy$. It is obvious that
$$\Re(z) = \frac{z + \overline{z}}{2} \quad \text{and} \quad \Im(z) = \frac{z - \overline{z}}{2i}.$$
- The norm (or absolute value) of z is defined as
$$|z| = \sqrt{x^2 + y^2}.$$
It is obvious that
$$|z|^2 = z\overline{z} \quad \text{and} \quad \frac{1}{z} = \frac{\overline{z}}{|z|^2} \text{ if } z \neq 0.$$
Equipped with $|\cdot|$, $\mathbb{C}$ becomes a metric space.

- Each complex number can be written in polar coordinates as $z = re^{i\theta}$, in which $r = |z|$ and $\tan\theta = y/x$. Here, θ is called the argument of z and is unique up to a multiple of 2π. It is often denoted by $\arg z$.

Definition (Convergence). We say that a sequence of complex numbers $\{z_n\}_{n=1}^{\infty}$ is convergent if there is $z \in \mathbb{C}$ such that $|z_n - z| \to 0$ as $n \to \infty$. That is, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|z_n - z| \leq \varepsilon$ if $n \geq N$. We write

$$\lim_{n\to\infty} z_n = z \quad \text{or} \quad z_n \to z \text{ as } n \to \infty.$$

If $\{z_n\}$ is not convergent, then we say that it is divergent.

Remark. Let $z_n = x_n + iy_n$ and $z = x + iy$. Then $z_n \to z$ iff $x_n \to x$ and $y_n \to y$. That is, $z_n \to z$ iff $\Re(z_n) \to \Re(z)$ and $\Im(z_n) \to \Im(z)$.

Definition (Cauchy sequences). We say that a sequence of complex numbers $\{z_n\}_{n=1}^{\infty}$ is Cauchy if $|z_n - z_m| \to 0$ as $n, m \to \infty$. That is, for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|z_n - z_m| \leq \varepsilon$ if $n, m \geq N$.

Remark.

- It is obvious that a convergent sequence is Cauchy.
- An important fact of the real numbers $\mathbb{R}$ is that $\mathbb{R}$ is complete. This means that a sequence of real numbers is convergent iff it is Cauchy.
- Since $z_n \to z$ iff $\Re(z_n) \to \Re(z)$ and $\Im(z_n) \to \Im(z)$, $\mathbb{C}$ is also complete.

Definition (Discs). Let $z_0 \in \mathbb{C}$ and $r > 0$. We define

- the open disc $D_r(z_0) = \{z \in \mathbb{C} : |z - z_0| < r\}$; in particular, we use $\mathbb{D}$ to denote the unit open disc $\mathbb{D} = D_1(0)$,
- the closed disc $\overline{D}_r(z_0) = \{z \in \mathbb{C} : |z - z_0| \leq r\}$,
- the circle $C_r(z_0) = \{z \in \mathbb{C} : |z - z_0| = r\}$, i.e., the boundary of the (open or closed) disc.

Definition (Open sets and closed sets). Let $\Omega \subset \mathbb{C}$.

- We say that $z \in \Omega$ is an interior point of Ω if there exists $r > 0$ such that $D_r(z) \subset \Omega$. The set of interior points of Ω is denoted by $\operatorname{Int}(\Omega)$.
- We say that Ω is open if each $z \in \Omega$ is an interior point of Ω.
- We say that Ω is closed if $\mathbb{C} \setminus \Omega$ is open.
- We say that $z \in \mathbb{C}$ is a limit point of Ω if there is sequence of points $z_n \in \Omega$ and $z_n \neq z$ such that $z_n \to z$ as $n \to \infty$.

Remark. A set Ω is open iff $\Omega = \operatorname{Int}(\Omega)$.

Definition (Closure and boundary). Let $\Omega \subset \mathbb{C}$.

- The closure of Ω, denoted by $\overline{\Omega}$, is the union of Ω and its limit points.
- The boundary of Ω, denoted by $\partial\Omega$, is $\overline{\Omega} \setminus \operatorname{Int}(\Omega)$.

Remark. A set Ω is closed iff $\Omega = \overline{\Omega}$.

Definition (Diameter). Let $\Omega \subset \mathbb{C}$. The diameter of Ω is defined as

$$\operatorname{diam}(\Omega) = \sup_{z,w\in\Omega} |z - w|.$$

We say that Ω is bounded if $\operatorname{diam}(\Omega) < \infty$ and is unbounded otherwise.

Remark. A set $\Omega \subset \mathbb{C}$ is bounded iff $\Omega \subset D_r(0)$ for some $r > 0$.

Definition (Compact sets). Let $\Omega \subset \mathbb{C}$.

- We say that a collection of open sets $\{U_\alpha\}$ is an open cover of Ω if $\Omega \subset \cup_\alpha U_\alpha$.
- We say that Ω is compact if every open cover of Ω has a finite subcover.

Remark. Let $\Omega \subset \mathbb{C}$. The following statements are equivalent.

- Ω is compact.
- Ω is closed and bounded.
- Every sequence of points in Ω has a subsequence that converges to a point in Ω.

Definition (Connected sets).

- We say that an open set $\Omega \subset \mathbb{C}$ is connected if $\Omega = \Omega_1 \cup \Omega_2$ with two disjoint open sets Ω_1 and Ω_2 implies that one of them is empty.
- We say that a closed set $\Omega \subset \mathbb{C}$ is connected if $\Omega = \Omega_1 \cup \Omega_2$ with two disjoint closed sets Ω_1 and Ω_2 implies that one of them is empty.

Remark. An open set is connected iff any two points in the set can be joined by a curve which is contained in the set (i.e., it it path-connected). See Section 1.4 for the precise definition of curves.

$\boxed{\textbf{Homework Assignment}}$.

1-1. Let $z_n = x_n + iy_n$ and $z = x + iy$. Prove that $z_n \to z$ iff $x_n \to x$ and $y_n \to y$.
1-2. Sketch the following sets.
(a). $\Re(z) = c$, in which $c \in \mathbb{R}$ is a constant. More generally, $\Re(z) > c$ and $\Re(z) < c$.
(b). $\Im(z) = c$, in which $c \in \mathbb{R}$ is a constant. More generally, $\Im(z) > c$ and $\Im(z) < c$.
(c). $|z - z_1| = |z - z_2|$, in which $z_1, z_2 \in \mathbb{C}$.

1.2. Functions on the complex plane

Definition. Let $\Omega \subset \mathbb{C}$ and $f : \Omega \to \mathbb{C}$. We say that f is continuous at $z_0 \in \Omega$ if for every $\varepsilon > 0$, there is $\delta > 0$ such that $|f(z) - f(z_0)| \le \varepsilon$ if $|z - z_0| < \delta$ and $z \in \Omega$. We say that f is continuous on Ω if f is continuous at every $z \in \Omega$.

Remark. f is continuous at z_0 iff for every $\{z_n\} \subset \Omega$ such that $z_n \to z_0$, we have that $f(z_n) \to f(z_0)$.

THEOREM 1.1. *A continuous function on a compact set is bounded and attains its maximum and minimum. Here,*

(i). we say that a function $f : \Omega \to \mathbb{C}$ is bounded if its range $f(\Omega)$ is a bounded set in $\mathbb{C}$,
(ii). we say that a function $f : \Omega \to \mathbb{C}$ attains its maximum (minimum) in Ω if there is $z_0 \in \Omega$ such that $|f(z)| \le |f(z_0)|$ ($|f(z)| \ge |f(z_0)|$) for all $z \in \Omega$.

Definition (Holomorphic functions). Let $\Omega \subset \mathbb{C}$ and $f : \Omega \to \mathbb{C}$. We say that f is holomorphic (or complex differentiable) at $z_0 \in \Omega$ if

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h}$$

exists. We then denote the limit $f'(z_0)$ or $\partial_z f(z_0)$ and call it the derivative of f at z_0.

We say f is holomorphic on Ω if it is holomorphic at every $z \in \Omega$. A function $f : \mathbb{C} \to \mathbb{C}$ that is differentiable everywhere is called an entire function.

THEOREM 1.2. *Let $\Omega \subset \mathbb{C}$ and $f, g : \Omega \to \mathbb{C}$ be holomorphic on Ω. Then*

(i). $af + bg$ is holomorphic on Ω, in which $a, b \in \mathbb{C}$ are constants, and $(af + bg)' = af' + bg'$.

(ii). fg is holomorphic on Ω, and $(fg)' = f'g + fg'$.
(iii). f/g is holomorphic at $z \in \Omega$ if $g(z) \neq 0$, and

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}.$$

Let $\Omega, U \subset \mathbb{C}$. Suppose that $f : \Omega \to U$ and $g : U \to \mathbb{C}$ are holomorphic. Then $g \circ f(z) = g(f(z))$ is holomorphic on Ω, and the chain rule applies:

$$(g \circ f)'(z) = g'(f(z))f'(z).$$

Example.
(1). Let $p(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial. Then p is holomorphic on $\mathbb{C}$.
(2). Let $f = p/q$ be a rational function, i.e., p and q are polynomials. Then f is holomorphic at $z \in \mathbb{C}$ if $q(z) \neq 0$.
(3). The function $f(z) = \overline{z}$ is not holomorphic anywhere.

The definition of $f'(z)$ of $f : \Omega \to \mathbb{C}$ with $\Omega \subset \mathbb{C}$ is similar to the one of $u'(x)$ of $u : \Omega \to \mathbb{R}$ with $\Omega \subset \mathbb{R}$. Moreover, the definition of the holomorphic functions of one complex variable is similar to the one of the differentiable functions of one real variable.

Example. Let $u : \mathbb{R} \to \mathbb{R}$ be defined by $u(x) = x^{\frac{4}{3}}$. Then u is differentiable on $\mathbb{R}$ but is not twice differentiable at $x = 0$. Similarly, $u(x, y) = x^{\frac{4}{3}} + y^{\frac{4}{3}}$ is differentiable but is not twice differentiable at $(0, 0)$.

However, holomorphic functions have much stronger properties than the differentiable functions of one real variable. For example, a holomorphic function is infinitely differentiable and is in fact analytic, even if only the first derivative is required in the definition. (See Section 1.3 for analytic functions.) The study of these properties of holomorphic functions is a major topic of complex analysis. We now explain some immediate restrictions of the holomorphic functions.

Write $z = x + iy$, $h = h_1 + ih_2$, and $f(z) = u(x, y) + iv(x, y)$. That is, u and v are real-valued functions of two real variables. Suppose that f is holomorphic at $z_0 = x_0 + iy_0$. Then

$$\begin{aligned} & \lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \\ = \ & \lim_{h \to 0} \frac{u(x_0 + h_1, y_0 + h_2) - u(x_0, y_0)}{h} + i \lim_{h \to 0} \frac{v(x_0 + h_1, y_0 + h_2) - v(x_0, y_0)}{h}. \end{aligned}$$

In particular, taking $h_2 = 0$ and $h_1 \to 0$, i.e., $h \to 0$ along the real line, we have that

$$\begin{aligned} & \lim_{h_1 \to 0} \frac{f(z_0 + h_1) - f(z_0)}{h_1} \\ = \ & \lim_{h_1 \to 0} \frac{u(x_0 + h_1, y_0) - u(x_0, y_0)}{h_1} + i \lim_{h_1 \to 0} \frac{v(x_0 + h_1, y_0) - v(x_0, y_0)}{h_1} \\ = \ & \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0). \end{aligned}$$

Taking $h_1 = 0$ and $h_2 \to 0$, i.e., $h \to 0$ along the imaginary line, we have that

$$\begin{aligned} & \lim_{h_2 \to 0} \frac{f(z_0 + ih_2) - f(z_0)}{ih_2} \\ = \ & \lim_{h_2 \to 0} \frac{u(x_0, y_0 + h_2) - u(x_0, y_0)}{ih_2} + i \lim_{h_2 \to 0} \frac{v(x_0, y_0 + h_2) - v(x_0, y_0)}{ih_2} \end{aligned}$$

$$= \frac{1}{i}\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0).$$

Therefore, u and v are both differentiable. Furthermore, since the above two limits must be equal,

$$\frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0) = \frac{1}{i}\frac{\partial u}{\partial y}(x_0, y_0) + \frac{\partial v}{\partial y}(x_0, y_0),$$

which implies that

$$\begin{cases} \frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \\ \frac{\partial u}{\partial y}(x_0, y_0) = -\frac{\partial v}{\partial x}(x_0, y_0). \end{cases} \tag{1.1}$$

These two equations are called the Cauchy-Riemann equations, which provide the restrictions of the real and imaginary components of a holomorphic function. Indeed, the converse is also true: If u and v are continuously differentiable and (1.1) holds at z_0, then f is holomorphic at z_0.

PROPOSITION 1.3. *Suppose that f is holomorphic at z_0. Write*

$$\frac{\partial f}{\partial x}(z_0) = \lim_{h_1 \to 0} \frac{f(z_0 + h_1) - f(z_0)}{h_1} \quad \text{and} \quad \frac{\partial f}{\partial y}(z_0) = \lim_{h_2 \to 0} \frac{f(z_0 + ih_2) - f(z_0)}{h_2}.$$

Then

$$f'(z_0) = \frac{\partial f}{\partial z}(z_0) = \frac{1}{2}\left(\frac{\partial}{\partial x} + \frac{1}{i}\frac{\partial}{\partial y}\right) f(z_0),$$

and

$$\frac{\partial f}{\partial \overline{z}}(z_0) := \frac{1}{2}\left(\frac{\partial}{\partial x} - \frac{1}{i}\frac{\partial}{\partial y}\right) f(z_0) = 0.$$

Remark. The above definition can also be understood as follows. Since $z = x + iy$ and $\overline{z} = x - iy$, we have that

$$x = \frac{z + \overline{z}}{2} \quad \text{and} \quad y = \frac{z - \overline{z}}{2i}.$$

Therefore, by the chain rule,

$$\partial_z = \frac{1}{2}\left(\partial_x + \frac{1}{i}\partial_y\right) \quad \text{and} \quad \partial_{\overline{z}} = \frac{1}{2}\left(\partial_x - \frac{1}{i}\partial_y\right).$$

Homework Assignment.

1-3. Let $w = se^{i\varphi}$ with $s \geq 0$ and $\varphi \in \mathbb{R}$.

(a). Suppose that $z^n = w$ for some $n \in \mathbb{N}$. Find all the solutions of z. (This in particular shows that $w^{\frac{1}{n}}$ does not define a single-valued function on $\mathbb{C}$ for $n \geq 2$.)

(b). Suppose that $e^z = w$. Find all the solutions of z. (This in particular shows that $\log w$ does not define a single-valued function on $\mathbb{C}$.)

1-4. Let $\Omega \subset \mathbb{C}$. We say $u : \Omega \to \mathbb{R}$ is harmonic if

$$\Delta u(x, y) := \frac{\partial^2 u(x, y)}{\partial x^2} + \frac{\partial^2 u(x, y)}{\partial y^2} = 0 \quad \text{for all } (x, y) \in \Omega.$$

Suppose that $f : \Omega \to \mathbb{C}$ is holomorphic and smooth. Prove that the real and imaginary components of f are both harmonic. Hint: Prove that

$$\frac{\partial}{\partial \overline{z}}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial \overline{z}}\right) = \frac{1}{4}\Delta.$$

1.3. Power series and analytic functions

A power series is an expansion

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n,$$

in which $a_n \in \mathbb{C}$ and $z_0 \in \mathbb{C}$.

Theorem 1.4 (Radius of convergence). *The radius of convergence of the above power series is given by*[i]

$$\frac{1}{R} = \limsup_{n\to\infty} \sqrt[n]{|a_n|}.$$

That is,

(i). the series converges in $D_R(z_0)$, which is called the disc of convergence,
(ii). the series diverges in $\mathbb{C} \setminus \overline{D}_R(z_0)$.

Remark. A power series on the circle $C_R(z_0)$ is more delicate. There are cases of convergence or divergence.

- $\sum nz^n$ has radius of convergence 1 and diverges everywhere on $C_1(0)$.
- $\sum z^n/n^2$ has radius of convergence 1 and converges everywhere on $C_1(0)$.
- $\sum z^n/n$ has radius of convergence 1 and converges everywhere on $C_1(0)$ except at $z = 1$.

Theorem 1.5. *Let*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

be a power series with radius of convergence R. Then

$$f'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$$

has radius of convergence R. In particular, f is holomorphic in $D_R(z_0)$. In fact,

$$f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1) a_n (z - z_0)^{n-k}$$

has radius of convergence R. In particular, f is infinitely differentiable at z_0.

Example.

(1). The exponential function

$$e^z := \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

has radius of convergence ∞, i.e., e^z is entire.

(2). The trigonometric functions sin and cos

$$\cos z := \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \quad \text{and} \quad \sin z := \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

have radius of convergence ∞.

The exponential function and the trigonometric functions are related by the Euler's formula

$$e^{iz} = \cos z + i \sin z.$$

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[i] Here, we use the convention that $1/0 = \infty$ and $1/\infty = 0$.

Definition (Analytic functions). Let $\Omega \subset \mathbb{C}$ be open and $z_0 \in \Omega$. We say $f : \Omega \to \mathbb{C}$ is analytic at z_0 if f has a power expansion around z_0

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

with positive radius of convergence.

We then conclude that if f is analytic at z_0, then f is holomorphic at z_0. We later on will show that the converse is also true.

Homework Assignment.

1-5. Prove the following statements.
(a). $\sum nz^n$ has radius of convergence 1 and diverges everywhere on $C_1(0)$.
(b). $\sum z^n/n^2$ has radius of convergence 1 and converges everywhere on $C_1(0)$.
(c). $\sum z^n/n$ has radius of convergence 1 and converges everywhere on $C_1(0)$ except at $z = 1$. (Hint: Use summation by parts.)

1.4. Integration along curves

Definition (Curves). We say that a function $z : [a, b] \to \mathbb{C}$ defines a smooth curve γ if $z'(t)$ is continuous on $[a, b]$ and $z'(t) \neq 0$. Here,

$$z'(a) = \lim_{t \to a^+} \frac{z(t) - z(a)}{t - a} \quad \text{and} \quad z'(b) = \lim_{t \to b^-} \frac{z(t) - z(b)}{t - b}.$$

We call the function $z(t)$ a parametrization of γ.

The points $z(a)$ and $z(b)$ are called the endpoints of γ. (More specifically, $z(a)$ is the initial point and $z(b)$ is the terminal point.) We say that γ is closed if $z(a) = z(b)$. We say that γ is simple if $z(t) = z(s)$ iff $t = s$. (In the case when γ is closed, we say that γ is simple if $z(t) = z(s)$ iff $t = s$ or $t = a$ and $s = b$.)

Remark. The requirement of $z'(t) \neq 0$ is to avoid the case when the curve becomes stationary/degenerate at the point $z(t)$. For example, in the extreme case of $z'(t) = 0$ for all $t \in [a, b]$, the curve degenerates into a point.

Definition (Equivalent parametrizations). Two parametrizations $z : [a, b] \to \mathbb{C}$ and $\tilde{z} : [c, d] \to \mathbb{C}$ are said to be equivalent if there exists a continuously differentiable bijection $t : [c, d] \to [a, b]$ such that $t'(s) > 0$ and $\tilde{z}(s) = z(t(s))$.

Example.

(1). Let γ_1 be given by $z(t) = z_0 + t(z_1 - z_0)$, $t \in [0, 1]$, for some $z_0, z_1 \in \mathbb{C}$. Then γ_1 is the line segment from z_0 to z_1. There are equivalent parametrizations, for example, $\tilde{z}(s) = z_0 + s(z_1 - z_0)/2$, $s \in [0, 2]$.
(2). Let γ_2 be given by $z(t) = z_0 + re^{it}$, $t \in [0, 2\pi]$, for some $z_0 \in \mathbb{C}$ and $r > 0$. We call this parametrization the positive orientation (counterclockwise) of the circle $C_r(z_0)$. There are equivalent parametrizations, for example, $\tilde{z}(s) = z_0 + re^{2is}$, $s \in [0, \pi]$.
(3). Let γ_3 be given by $z(t) = z_0 + re^{-it}$, $t \in [0, 2\pi]$, for some $z_0 \in \mathbb{C}$ and $r > 0$. We call this parametrization the negative orientation (clockwise) of the circle $C_r(z_0)$. It is not equivalent to the positive orientation of the circle. Indeed, we say that γ_2 and γ_3 have reverse orientations.

Remark (Reverse orientation). Let γ be a smooth curve that is given by $z : [a, b] \to \mathbb{C}$. We say that the curve γ^- given by the parametrization $\tilde{z}(t) = z(b + a - t)$, $t \in [a, b]$, has the reverse orientation of γ. (Again, the parametrization of the reverse orientation is not unique. One can also set $\tilde{z}(t) = z(-t)$, $t \in [-b, -a]$.)

Definition (Length of curves). The length of a smooth curve γ given by $z : [a, b] \to \mathbb{C}$ is

$$\text{Length}(\gamma) := \int_a^b |z'(t)|\, dt.$$

Example. For the curves γ_1, γ_2, and γ_3 defined above,

$$\text{Length}(\gamma_1) = |z_1 - z_0| \quad \text{and} \quad \text{Length}(\gamma_2) = \text{Length}(\gamma_3) = 2\pi r.$$

The parametrization of a curve is not unique. However, the length of a curve is independent of its parametrization. Furthermore, two curves with reverse orientations have the same length.

Definition (Integration of functions along curves). Let γ be a smooth curve that is given by $z : [a, b] \to \mathbb{C}$ and f be continuous on γ. Then the integral of f along γ is

$$\int_\gamma f(z)\, dz := \int_a^b f(z(t)) z'(t)\, dt.$$

Remark. The integral of a function along a curve is independent of its parametrization. Indeed, let $\tilde{z}(s) = z(t(s))$, $s \in [c, d]$, be an equivalent parametrization of γ. Then

$$\begin{aligned}
\int_\gamma f(z)\, dz &= \int_c^d f(\tilde{z}(s))\tilde{z}'(s)\, ds \\
&= \int_c^d f(z(t(s)))\, (z(t(s)))'\, ds \\
&= \int_c^d f(z(t(s)))z'(t(s))t'(s)\, ds \\
&= \int_a^b f(z(t))z'(t)\, dt.
\end{aligned}$$

Example. Let γ be given by $z(t) = e^{it}$, $t \in [0, 2\pi]$. Then

$$\int_\gamma z\, dz = \int_0^{2\pi} e^{it} \left(e^{it}\right)'\, dt = i \int_0^{2\pi} e^{2it}\, dt = 0,$$

and

$$\int_\gamma \frac{1}{z}\, dz = \int_0^{2\pi} e^{-it} \left(e^{it}\right)'\, dt = i \int_0^{2\pi} dt = 2\pi i.$$

PROPOSITION 1.6. *Let γ be a smooth curve that is given by $z : [a, b] \to \mathbb{C}$ and f, g be continuous on γ.*

(i). If $\alpha, \beta \in \mathbb{C}$, then

$$\int_\gamma (\alpha f(z) + \beta g(z))\, dz = \alpha \int_\gamma f(z)\, dz + \beta \int_\gamma g(z)\, dz.$$

(ii).

$$\int_\gamma f(z)\, dz = -\int_{\gamma^-} f(z)\, dz.$$

(iii).

$$\left| \int_\gamma f(z)\,dz \right| \le \sup_{z\in\gamma} |f(z)| \cdot \mathrm{Length}(\gamma).$$

Remark (Piecewise smooth curves). We say that a function $z : [a,b] \to \mathbb{C}$ defines a piecewise smooth curve γ if there is a sequence of points

$$a = a_0 < a_1 < \cdots < a_{n-1} < a_n = b$$

such that it is smooth on each subinterval $[a_{k-1}, a_k]$, $k = 1, ..., n$. (We henceforth use "curve" for a piecewise smooth curve.)

Then the length of the curve is the sum of the lengths of each piece and the integral of a function along the curve is the sum of the integrals along each piece.

$\boxed{\textbf{Homework Assignment}}$.

1-6. Let γ be a curve that is given by $z : [a,b] \to \mathbb{C}$ and f be continuous on γ. Prove that

$$\int_\gamma f(z)\,dz = -\int_{\gamma^-} f(z)\,dz.$$

1-7. Let γ be the unit circle with positive orientation. Compute the following integrals.

(a). For $n \ge 0$,

$$\int_\gamma z^n\,dz.$$

(b). For $n \ge 1$,

$$\int_\gamma z^{-n}\,dz.$$

CHAPTER 2

Cauchy's theory and its applications

Cauchy's theory of holomorphic functions consists of two parts.

- Part I concerns the characterization of holomorphic functions via integration along curves. In particular, we study the conditions under which $\int_\gamma f = 0$ for closed curves γ and holomorphic functions f.
- Part II concerns the Cauchy integral formula: If f is holomorphic in Ω, then
$$f(z) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{w-z}\, dw$$
for any $\overline{D}_r(z_0) \subset \Omega$ and $z \in D_r(z_0)$.

We then develop the consequences of the above theory, including

- ○ evaluation of integrals by "contour shifting",
- ○ smoothness and analyticity of holomorphic functions,
- ○ Liouville's theorem and the fundamental theorem of algebra.

2.1. Primitives

Definition (Primitives). Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$. We say that F is a primitive of f if F is holomorphic on Ω and $F'(z) = f(z)$ for all $z \in \Omega$.

Remark. The primitive is not unique. Indeed, if F is a primitive of f, then $F + c$ is also a primitive for any $c \in \mathbb{C}$. In fact, two primitives of a function on an open and connected domain differ by a constant.

Theorem 2.1. *Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ be continuous. Suppose that f has a primitive F. Then for any curve γ in Ω,*
$$\int_\gamma f(z)\, dz = F(w_2) - F(w_1),$$
in which w_1 and w_2 are the initial point and terminal point of γ.

Proof. Suppose that γ is smooth and let $z : [a, b] \to \Omega$ be a parametrization. Then
$$\begin{aligned} \int_\gamma f(z)\, dz &= \int_a^b f(z(t))z'(t)\, dt \\ &= \int_a^b F'(z(t))z'(t)\, dt \\ &= \int_a^b (F \circ z)'(t)\, dt \\ &= F(z(b)) - F(z(a)) \\ &= F(w_2) - F(w_1). \end{aligned}$$
If γ is piecewise smooth, then write the above integral in pieces and the theorem follows. □

COROLLARY 2.2. *Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ be continuous. Suppose that f has a primitive F. Then for any closed curve γ in Ω,*

$$\int_\gamma f(z)\,dz = 0.$$

Example.

(1). The function z^n for $n \geq 0$ has a primitive $z^{n+1}/(n+1)$ in $\mathbb{C}$. So $\int_\gamma z^n = 0$ for all closed curves. It then follows that the integral of polynomials along any closed curve is zero.

(2). The function z^{-n} for $n \geq 2$ has a primitive $z^{-n+1}/(-n+1)$ in $\mathbb{C} \setminus \{0\}$. So $\int_\gamma z^{-n} = 0$ for all closed curves that do not pass through the origin.

(3). e^z, $\sin z$, and $\cos z$ have primitives e^z, $-\cos z$, and $\sin z$, respectively, in $\mathbb{C}$. So the integrals of these functions along any closed curve are zero.

COROLLARY 2.3. *Let $\Omega \subset \mathbb{C}$ be open and connected and $f : \Omega \to \mathbb{C}$ such that $f'(z) = 0$ for all $z \in \Omega$. Then f is constant on Ω.*

PROOF. Let $w_0 \in \Omega$ be fixed. Choose any $w \in \Omega$. Since Ω is connected, there is a curve γ that connects w_0 and w, that is, the initial point and terminal point of γ are w_0 and w. Notice that f is a primitive of f'. Then

$$0 = \int_\gamma f'(z)\,dz = f(w) - f(w_0),$$

which implies that f is constant on Ω. □

Homework Assignment.

2-1. Let $\Omega \subset \mathbb{C}$ be open and connected and $f : \Omega \to \mathbb{C}$ be continuous. Prove that any two primitives (if they exist) differ by a constant.

2.2. Goursat's theorem

THEOREM 2.4 (Goursat's theorem). *Let $\Omega \subset \mathbb{C}$ be open and $T \subset \Omega$ be a triangle such that its interior is also contained in Ω. Then*

$$\int_T f(z)\,dz = 0$$

for all holomorphic functions f in Ω.

PROOF. Denote $T_0 = T$ the original triangle (with a fixed orientation). Let d_0 and p_0 be the diameter and perimeter (i.e., length) of T_0. We next divide the triangle into smaller ones successively.

Step 1. Bisect each side of T_0 to create four smaller triangles $T_{1,j}$, $j = 1, 2, 3, 4$, with proper orientations so that

$$\int_{T_0} f(z)\,dz = \sum_{j=1}^{4} \int_{T_{1,j}} f(z)\,dz.$$

Therefore, there is $T_{1,j}$ for some $j = 1, 2, 3, 4$, simply denoted by T_1, such that

$$\left| \int_{T_0} f(z)\,dz \right| \leq 4 \left| \int_{T_1} f(z)\,dz \right|.$$

Note also that the diameter of $T_{1,j}$ is $d_1 = d_0/2$ and the perimeter of $T_{1,j}$ is $p_1 = p_0/2$.

Step n. The triangle T_{n-1} in Step $n-1$ satisfies

$$\left|\int_{T_0} f(z)\,dz\right| \le 4^{n-1}\left|\int_{T_{n-1}} f(z)\,dz\right|.$$

In Step n, T_{n-1} is divided into four smaller triangles $T_{n,j}$, $j=1,...,4$ with proper orientations, each of which has diameter $d_n = 2^{-n}d_0$ and perimeter $p_n = 2^{-n}p_0$. Furthermore, there is $T_{n,j}$ for some $j=1,...,4$, simply denoted by T_n, such that

$$\left|\int_{T_{n-1}} f(z)\,dz\right| \le 4^{n}\left|\int_{T_n} f(z)\,dz\right|.$$

Therefore,

$$\left|\int_{T_0} f(z)\,dz\right| \le 4^{n-1}\left|\int_{T_{n-1}} f(z)\,dz\right| \le 4^{n}\left|\int_{T_n} f(z)\,dz\right|.$$

Let Ω_n be the solid closed triangle of T_n. Then $\Omega_0 \supset \Omega_1 \supset \cdots$ and $\operatorname{diam}(\Omega_n) = \operatorname{diam}(T_n) = d_n \to 0$ as $n \to \infty$. Therefore, $\cap_n \Omega_n = \{z_0\}$ for some z_0.

Since f is holomorphic at z_0,

$$f(z) = f(z_0) + f'(z_0)(z-z_0) + \psi(z)(z-z_0),$$

in which $\psi(z) \to 0$ as $z \to z_0$.

Let N be large enough so the above equation holds in Ω_n for all $n \ge N$. Then

$$\varepsilon_n := \sup_{z\in\Omega_n} |\psi(z)| \to 0 \quad \text{as } n \to \infty.$$

Moreover, compute that

$$\int_{T_n} f(z)\,dz = \int_{T_n} \left(f(z_0) + f'(z_0)(z-z_0) + \psi(z)(z-z_0)\right)\,dz = \int_{T_n} \psi(z)(z-z_0)\,dz,$$

because $f(z_0)$ and $f'(z_0)(z-z_0)$ both have primitives (for example, $f(z_0)z$ and $f'(z_0)(z^2/2 - z_0 z)$) so their integrals are zero. Hence,

$$\begin{aligned}
\left|\int_{T_n} \psi(z)(z-z_0)\,dz\right| &\le \sup_{z\in T_n}|\psi(z)| \cdot \sup_{z\in T_n}|z-z_0| \cdot \operatorname{Length}(T_n) \\
&\le \sup_{z\in\Omega_n}|\psi(z)| \cdot \operatorname{diam}(T_n) \cdot \operatorname{Length}(T_n) \\
&= \varepsilon_n d_n p_n \\
&= 4^{-n}\varepsilon_n d_0 p_0.
\end{aligned}$$

Finally we have that

$$\left|\int_{T_0} f(z)\,dz\right| \le 4^{n}\left|\int_{T_n} f(z)\,dz\right| = 4^{n}\left|\int_{T_n} \psi(z)(z-z_0)\,dz\right| = \varepsilon_n d_0 p_0 \to 0$$

as $n \to \infty$. Therefore,

$$\int_{T_0} f(z)\,dz = 0.$$

□

By writing the integral of a function along a rectangle as the sum of two integrals along two triangles (with proper orientations), we derive that

Corollary 2.5. *Let $\Omega \subset \mathbb{C}$ be open and $R \subset \Omega$ be a rectangle such that its interior is also contained in Ω. Then*

$$\int_R f(z)\, dz = 0$$

for all holomorphic functions f in Ω.

$\boxed{\textbf{Homework Assignment}}$.

2-2. Let $\Omega_0 \supset \Omega_1 \supset \cdots$ be a sequence of non-empty compact sets in $\mathbb{C}$ such that $\operatorname{diam}(\Omega_n) \to 0$ as $n \to \infty$. Prove that $\cap_n \Omega_n = \{z_0\}$ for some z_0.

2-3. Let $\Omega \subset \mathbb{C}$ be open and $T \subset \Omega$ be a triangle such that its interior is also contained in Ω. Suppose that f is holomorphic and smooth on Ω. Apply Green's theorem to prove that

$$\int_T f(z)\, dz = 0.$$

Hint: Green's theorem states that if (F, G) is a continuously differentiable vector field, then

$$\int_T F\, dx + G\, dy = \int_\Omega \left(\frac{\partial G}{\partial x} - \frac{\partial F}{\partial y} \right) dx dy,$$

in which Ω is the interior of T. Given any holomorphic function $f = u + iv$ and a curve γ with parametrization $z(t) = x(t) + iy(t)$, compute that

$$\begin{aligned}
\int_\gamma f(z)\, dz &= \int f(z(t)) z'(t)\, dt \\
&= \int_\gamma (u + iv)(x'(t) + iy'(t))\, dt \\
&= \int_\gamma (u\, dx - v\, dy) + i \int_\gamma (u\, dy + v\, dx)\,.
\end{aligned}$$

One can then assign F and G in Green's theorem appropriately and use Cauchy-Riemann equations (1.1) to show that the above integrals are both zero.

2.3. Local existence of primitives

From the example that $\int_C z^{-1} = 2\pi i$ for a circle C centered at the origin, we know that the holomorphic function z^{-1} does not have a primitive on $\mathbb{C} \setminus \{0\}$ (otherwise the integral would be zero according to the previous section). Indeed, $\log z$ is the natural candidate of primitive but it can not be defined as a single-valued function on $\mathbb{C} \setminus \{0\}$. Nevertheless, in this section, we show that a holomorphic function always has a primitive in a local region.

Theorem 2.6. *A holomorphic function in an open disc has a primitive in that disc.*

Proof. Without loss of generality, let $f : D_R(0) \to \mathbb{C}$. (If $f : D_R(z_0) \to \mathbb{C}$, then consider $\tilde{f}(z) := f(z + z_0)$ which is holomorphic and is defined on $D_R(0)$.)

Given a point $z \in D_R(0)$, consider the piecewise-smooth curve γ_z that joints 0 to z first by moving in the horizontal direction from 0 to $\Re(z)$ and then in the vertical direction from $\Re(z)$ to z. Define

$$F(z) = \int_{\gamma_z} f(w)\, dw.$$

Next we show that F is a primitive of f in $D_R(0)$, i.e., F is holomorphic and $F'(z) = f(z)$ at all $z \in D_R(0)$. By repeatedly using Goursat's theorem, we have that

$$F(z+h) - F(z) = \int_\eta f(w)\,dw,$$

in which η is the line segment from z to $z+h$. Since f is holomorphic, f is continuous. Hence, $f(w) \to f(z)$ as $w \to z$. In particular, there is $\psi(w)$ such that

$$f(w) = f(z) + \psi(w),$$

and $\psi(w) \to 0$ as $w \to z$. Compute that

$$F(z+h) - F(z) = \int_\eta f(z)\,dw + \int_\eta \psi(w)\,dw = f(z)\int_\eta dw + \int_\eta \psi(w)\,dw = f(z)h + \int_\eta \psi(w)\,dw,$$

in which we used the fact that 1 has a primitive in $\mathbb{C}$ so $\int_\eta dw = z + h - z = h$.

Thus,

$$\lim_{h\to 0} \frac{F(z+h) - F(z)}{h} = f(z) + \lim_{h\to 0} \frac{1}{h}\int_\eta \psi(w)\,dw = f(z),$$

because as $h \to 0$,

$$\left|\frac{1}{h}\int_\eta \psi(w)\,dw\right| \le \frac{1}{h}\cdot \sup_{w\in\eta} |\psi(w)| \cdot \mathrm{Length}(\eta) = \sup_{w\in\eta} |\psi(w)| \to 0.$$

$\square$

THEOREM 2.7 (Cauchy's theorem for a disc). *If f is holomorphic in a disc, then $\int_\gamma f(z)\,dz = 0$ for any closed curve γ in that disc.*

PROOF. By the above theorem, f has a primitive in the disc. By Corollary 2.5, $\int_\gamma f = 0$ for any closed curve in that disc. $\square$

COROLLARY 2.8. *Suppose that f is holomorphic in an open set that contains a circle C and its interior. Then $\int_C f(z)\,dz = 0$.*

PROOF. Denote the open set Ω. Let D be the disc with boundary C. Then $\overline{D} \subset \Omega$. There is $\tilde{D} \supset D$ such that $\tilde{D} \subset \Omega$. Since f is holomorphic on $\tilde{D}$, $\int_C f = 0$. $\square$

Remark (Zigzag curves). By a zigzag curve, we mean a piecewise smooth curve that consists of a finite number of horizontal or vertical segments. If f is holomorphic in an open disc D and γ_1, γ_2 are two zigzag curves with common end-points in D, then

$$\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz,$$

which is a consequence of the Goursat's theorem for rectangles in Corollary 2.5. Therefore, one can use any zigzag curve to define the primitive of a holomorphic function in a disc. Notice that such construction of the primitive becomes invalid in $D \setminus \{z_0\}$ for some $z_0 \in D$.

2.4. Existence of primitives on simply connected domains

In this section, we extend the result on the local existence of primitives of holomorphic functions (i.e., on discs) to more general classes of domains. The first generalization is rather modest. That is, we call any closed curve where the notion of interior is obvious a toy contour. The examples include circles, triangles, rectangles, etc. We do not attempt to provide the precise definition of a toy contour, but rather be content with its intuition and applications later on.

Example (Keyhole contour). A keyhole contour is comprised of two almost complete circles, one large and one small, connected by a narrow corridor.

Let Γ be a toy contour and denote Ω_Γ its interior. Let f be holomorphic in Ω_Γ. Then similarly as in the previous section, one can show that $\int_{\gamma_1} f = \int_{\gamma_2} f$ for any two zigzag curves $\gamma_1, \gamma_2 \subset \Omega_\Gamma$. (This can usually be verified by Goursat's theorem for rectangles.) Hence, we can use any zigzag curve to define a primitive of f as in Theorem 2.6. It therefore follows that $\int_\gamma f = 0$ for any closed curve $\gamma \subset \Omega_\Gamma$.

In the reverse direction, suppose that γ is a closed curve and f is holomorphic. To establish $\int_\gamma f = 0$, it suffices to find a toy contour Γ with interior Ω_Γ such that $\Omega_\Gamma \supset \gamma$ and f has a primitive in Ω_Γ. For example, take Γ as a keyhole contour in an open set Ω. Then one can find another keyhole contour $\tilde{\Gamma}$ such that its interior $\Omega_{\tilde{\Gamma}}$ satisfies $\Gamma \subset \Omega_{\tilde{\Gamma}} \subset \Omega$. Now for any holomorphic function f in Ω, one can construct a primitive in $\Omega_{\tilde{\Gamma}}$. It then follows that $\int_\Gamma f = 0$.

The second generalization of existence of primitives involves simply connected domains.

Definition (Homotopies and simply connected domains). Let $\Omega \subset \mathbb{C}$. Suppose that $\gamma_0, \gamma_1 : [a,b] \to \Omega$ have common endpoints, i.e., $\gamma_0(a) = \gamma_1(a) = \alpha$ and $\gamma_0(b) = \gamma_1(b) = \beta$ for some $\alpha, \beta \in \Omega$. Then γ_0 and γ_1 are said to be homotopic in Ω if there is a continuous function

$$\Gamma(t,s) : [a,b] \times [0,1] \to \Omega$$

such that

- $\Gamma(a,s) = \alpha$ and $\Gamma(b,s) = \beta$ for all $s \in [0,1]$,
- $\Gamma(t,0) = \gamma_0(t)$ and $\Gamma(t,1) = \gamma_1(t)$ for all $t \in [a,b]$.

That is, two curves are homotopic in Ω if one can be continuously deformed into another within the set, while the two endpoints are fixed. We say that Ω is simply connected if any two distinct points can be connected by curves and any two curves in Ω with common endpoints are homotopic.

Example.

(1). The discs are simply connected.
(2). The interior of the keyhole contour is simply connected.
(3). If a set is open and convex, then it is simply connected.
(4). If a set is star-shaped, then it is simply connected.
(5). A open disc with a point removed is not simply connected.

THEOREM 2.9. *Let $\Omega \subset \mathbb{C}$ be open and simply connected and $f : \Omega \to \mathbb{C}$ be holomorphic on Ω. Then*

$$\int_{\gamma_0} f(z)\,dz = \int_{\gamma_1} f(z)\,dz$$

for any two homotopic curves $\gamma_0, \gamma_1 \in \Omega$.

OUTLINE OF THE PROOF.

Step 1. Suppose that γ_0 and γ_1 are close to each other. In particular, one can find a chain of discs $\{\overline{D}_j\}_{j=1}^n \subset \Omega$ such that $D_j \cap D_{j+1} \neq \emptyset$ for all $j = 1, ..., n-1$ and

$$\gamma_0, \gamma_1 \subset \bigcup_{j=1}^n D_j.$$

Moreover, there are points $z_0, ..., z_n \in \gamma_0$ and $w_0, ..., w_n \in \gamma_1$ such that

- $z_0 = w_0 \in D_1$ is the initial point of γ_0 and γ_1 while $z_n = w_n \in D_n$ is the terminal point of γ_0 and γ_1;

- $z_j, w_j \in D_j \cap D_{j+1}$ for $j = 1, ..., n-1$.

Using the local existence of primitive in Theorem 2.6, there is a primitive F_j of f in D_j for each $j = 1, ..., n$. Since $D_j \cap D_{j+1} \neq \emptyset$, the primitives F_j and F_{j+1} of f on this set differ by a constant $c_j \in \mathbb{C}$:

$$F_j(z) - F_{j+1}(z) = c_j \quad \text{for all } z \in D_j \cap D_{j+1}.$$

In particular, for $j = 1, ..., n-1$,

$$F_j(z_j) - F_{j+1}(z_j) = F_j(w_j) - F_{j+1}(w_j) = c_j,$$

which means that

$$F_{j+1}(z_j) - F_{j+1}(w_j) = F_j(z_j) - F_j(w_j).$$

Denote $\gamma_{z_j z_{j+1}}$ the piece of curve on γ_0 that is from z_j to z_{j+1} and $\gamma_{w_j w_{j+1}}$ the piece of curve on γ_1 that is from w_j to w_{j+1}. Then $\gamma_{z_j z_{j+1}}, \gamma_{w_j w_{j+1}} \in D_j$ for $j = 1, ..., n-1$. Moreover, construct $\gamma_{z_j w_j}$ as a curve from z_j to w_j in $D_j \cap D_{j+1}$.

By Theorem 2.7, we have the following estimates.

- On D_1,

$$\int_{\gamma_{z_0 z_1}} f + \int_{\gamma_{z_1 w_1}} f + \int_{\gamma_{w_1 w_0}} f = 0,$$

  which implies that

$$\int_{\gamma_{z_0 z_1}} f - \int_{\gamma_{w_0 w_1}} f = \int_{\gamma_{w_1 z_1}} f = F_1(z_1) - F_1(w_1).$$

- On D_j for $j = 1, ..., n-2$,

$$\int_{\gamma_{z_j z_{j+1}}} f + \int_{\gamma_{z_{j+1} w_{j+1}}} f + \int_{\gamma_{w_{j+1} w_j}} f + \int_{\gamma_{w_j z_j}} f = 0,$$

  which implies that

$$\begin{aligned}\int_{\gamma_{z_j z_{j+1}}} f - \int_{\gamma_{w_j w_{j+1}}} f &= \int_{\gamma_{w_{j+1} z_{j+1}}} f + \int_{\gamma_{z_j w_j}} f \\ &= F_{j+1}(z_{j+1}) - F_{j+1}(w_{j+1}) + F_j(w_j) - F_j(z_j) \\ &= F_{j+1}(z_{j+1}) - F_{j+1}(w_{j+1}) - \big(F_{j+1}(z_j) - F_{j+1}(w_j)\big).\end{aligned}$$

- On D_n,

$$\int_{\gamma_{z_{n-1} z_n}} f + \int_{\gamma_{w_n w_{n-1}}} f + \int_{\gamma_{w_{n-1} z_{n-1}}} f = 0,$$

  which implies that

$$\int_{\gamma_{z_{n-1} z_n}} f - \int_{\gamma_{w_{n-1} w_n}} f = \int_{\gamma_{z_{n-1} w_{n-1}}} f = F_n(w_{n-1}) - F_n(z_{n-1}).$$

Summing these terms, we have that

$$\begin{aligned}\int_{\gamma_0} f - \int_{\gamma_1} f &= \sum_{j=0}^{n-1} \int_{\gamma_{z_j z_{j+1}}} f - \sum_{j=0}^{n-1} \int_{\gamma_{w_j w_{j+1}}} f \\ &= \sum_{j=0}^{n-1} \left(\int_{\gamma_{z_j z_{j+1}}} f - \int_{\gamma_{w_j w_{j+1}}} f \right) \\ &= F_1(z_1) - F_1(w_1)\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{n-2}\left[F_{j+1}(z_{j+1})-F_{j+1}(w_{j+1})-\big(F_{j+1}(z_j)-F_{j+1}(w_j)\big)\right]\\
&+F_n(w_{n-1})-F_n(z_{n-1})\\
=\;& 0.
\end{aligned}
$$

Step 2. Let $\Gamma(t,s)$ be the homotopy between two curves γ_0 and γ_1, that is,

- $\Gamma(a,s)=\alpha$ and $\Gamma(b,s)=\beta$ for all $s\in[0,1]$,
- $\Gamma(t,0)=\gamma_0(t)$ and $\Gamma(t,1)=\gamma_1(t)$ for all $t\in[a,b]$.

Divide $s\in[0,1]$ into subintervals $[s_k,s_{k+1}]$ short enough such that $\Gamma(t,s_k)$ and $\Gamma(t,s_{k+1})$ are close enough so Step 1 applies. Now

$$\int_{\gamma_0} f-\int_{\gamma_1} f=\sum_k\left(\int_{\Gamma(t,s_k)} f-\int_{\Gamma(t,s_{k+1})} f\right)=0.$$

□

Similarly as in the local existence of primitives, we can use any curve to define the primitive of a holomorphic function on a simply connected set. It then follows that

Theorem 2.10. *A holomorphic function on a simply connected set has a primitive in that set. Therefore, the integral of holomorphic functions along closed curves in a simply connected set is zero.*

2.5. Evaluation of some integrals

In this subsection, we use Cauchy's theorem in Theorem 2.7 to evaluate some integrals. The evaluation usually involves a (toy) contour that contains the original integral domain as a piece. If we can use Cauchy's theorem to conclude the integral of a holomorphic function along the contour is zero, then the original integral is "shifted" to the ones on the other pieces of the contour.

Lemma 2.11.

$$\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx=1.$$

Proof. By Fubini's theorem,

$$\int_{\mathbb{R}^2} e^{-\pi(x^2+y^2)}\,dxdy=\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx\int_{-\infty}^{\infty} e^{-\pi y^2}\,dy=\left(\int_{-\infty}^{\infty} e^{-\pi x^2}\,dx\right)^2.$$

By polar coordinates,

$$\int_{\mathbb{R}^2} e^{-\pi(x^2+y^2)}\,dxdy=\int_0^{\infty}\int_0^{2\pi} e^{-\pi r^2} r\,d\theta dr=-e^{-\pi r^2}\Big|_0^{\infty}=1.$$

The lemma then follows. □

Remark (Gaussian distribution). The Gaussian distribution is given by the probability density function

$$f(x)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{|x-x_0|^2}{2\sigma^2}}.$$

By change of variables, one can check that $\int f=1$ so it is a probability density. The mean (expectation) and the standard deviation of this distribution are x_0 and σ. Most of the values are concentrated within several deviations from the mean value, for example, 95% of the values are within 2σ of x_0.

Definition (Fourier transform). The Fourier transform of a function $f(x)$ is defined as

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x\xi}\, dx.$$

THEOREM 2.12. *Let* $f(x) = e^{-\pi x^2}$. *Then* $\hat{f}(\xi) = e^{-\pi\xi^2}$.

PROOF. The theorem is obvious if $\xi = 0$.

Assume that $\xi > 0$. Let γ_R be the rectangular contour with vertices $R, R + i\xi, -R + i\xi, -R$ and positive orientation. Since $f(z) = e^{-\pi z^2}$ is holomorphic in $\mathbb{C}$, $\int_{\gamma_R} f = 0$ by Cauchy's theorem. On the other hand,

$$\int_{\gamma_R} f(z)\, dz = \int_{-R}^{R} f(x)\, dx + \int_0^{\xi} f(R+iy) i\, dy + \int_R^{-R} f(x+i\xi)\, dx + \int_{\xi}^{0} f(-R+iy) i\, dy.$$

Next we estimate the four integrals.

(1).

$$\int_{-R}^{R} f(x)\, dx = \int_{-R}^{R} e^{-\pi x^2}\, dx \to 1 \quad \text{as } R \to \infty,$$

by the previous lemma.

(2).

$$\int_0^{\xi} f(R+iy) i\, dy = i \int_0^{\xi} e^{-\pi(R+iy)^2}\, dy = i \int_0^{\xi} e^{-\pi(R^2+2iRy-y^2)}\, dy \to 0 \quad \text{as } R \to \infty,$$

because

$$\left| \int_0^{\xi} e^{-\pi(R^2+2iRy-y^2)}\, dy \right| \le |\xi| e^{\pi|\xi|^2} \cdot e^{-\pi R^2} \to 0 \quad \text{as } R \to \infty.$$

(3).

$$\int_R^{-R} f(x+i\xi)\, dx = -\int_{-R}^{R} e^{-\pi(x+i\xi)^2}\, dx = -e^{\pi\xi^2} \int_{-R}^{R} e^{-\pi x^2} e^{-2\pi i x\xi}\, dx.$$

(4).

$$\int_{\xi}^{0} f(-R+iy) i\, dy = -i \int_0^{\xi} e^{-\pi(-R+iy)^2}\, dy = -i \int_0^{\xi} e^{-\pi(R^2-2iRy-y^2)}\, dy \to 0 \quad \text{as } R \to \infty,$$

because

$$\left| \int_0^{\xi} e^{-\pi(R^2-2iRy-y^2)}\, dy \right| \le |\xi| e^{\pi|\xi|^2} \cdot e^{-\pi R^2} \to 0 \quad \text{as } R \to \infty.$$

Therefore, as $R \to \infty$,

$$0 = 1 - e^{\pi\xi^2} \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-2\pi i x\xi}\, dx,$$

which finishes the proof for the case of $\xi > 0$. The case when $\xi < 0$ can be treated similarly by considering a rectangular contour below the real line. $\square$

Remark (Uncertainty principle). Let

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{|x|^2}{2\sigma^2}}$$

be a Gaussian function with standard deviation σ. Then f is concentrated in the σ neighborhood of the mean value 0. By change of variables, the Fourier transform

$$\hat{f}(\xi) = e^{-\frac{|\xi|^2}{2\tilde{\sigma}^2}} \quad \text{with } \tilde{\sigma} = \frac{1}{2\sqrt{\pi}\sigma},$$

which is also a Gaussian function with standard deviation $\tilde{\sigma}$. We then see that more concentrated f is, less concentrated $\hat{f}$ will be. This phenomenon reflects the uncertainty principle that a function and its Fourier transform can not be simultaneously localized.

THEOREM 2.13 (Fresnel integrals).

$$\int_0^\infty \sin(x^2)\,dx = \int_0^\infty \cos(x^2)\,dx = \frac{\sqrt{2\pi}}{4}.$$

PROOF. Let γ_R be the sector contour with center at the origin, radius R, and opening angle $\pi/4$. Let γ_R be positively orientated. Since $f(z) = e^{-z^2}$ is holomorphic in $\mathbb{C}$, $\int_{\gamma_R} f = 0$ by Cauchy's theorem. On the other hand,

$$\int_{\gamma_R} f(z)\,dz = \int_0^R f(x)\,dx + \int_0^{\frac{\pi}{4}} f(Re^{i\theta})iRe^{i\theta}\,d\theta + \int_{\frac{\sqrt{2}R}{2}}^0 f(t+it)(1+i)\,dt.$$

Next we estimate the three integrals.

(1).

$$\int_0^R f(x)\,dx = \int_0^R e^{-x^2}\,dx \to \int_0^\infty e^{-x^2} = \frac{\sqrt{\pi}}{2} \quad \text{as } R\to\infty.$$

(2).

$$\int_0^{\frac{\pi}{4}} f(Re^{i\theta})iRe^{i\theta}\,d\theta = i\int_0^{\frac{\pi}{4}} e^{-R^2e^{2i\theta}}Re^{i\theta}\,d\theta \to 0 \quad \text{as } R\to\infty,$$

because

$$\begin{aligned}\left|\int_0^{\frac{\pi}{4}} e^{-R^2e^{2i\theta}}Re^{i\theta}\,d\theta\right| &\le R\int_0^{\frac{\pi}{4}} e^{-R^2\cos(2\theta)}\,d\theta \\ &\le R\int_0^{\frac{\pi}{4}} e^{-R^2\left(1-\frac{4}{\pi}\theta\right)}\,d\theta \\ &= \frac{\pi}{4R}\left(1-e^{-R^2}\right) \\ &\to 0 \quad \text{as } R\to\infty.\end{aligned}$$

Here, we use the fact that $\cos(2\theta) \ge 1 - \frac{4}{\pi}\theta$ when $0 \le \theta \le \frac{\pi}{4}$.

(3).

$$\begin{aligned}&\int_{\frac{\sqrt{2}R}{2}}^0 f(t+it)(1+i)\,dt \\ =\ & -\int_0^{\frac{\sqrt{2}R}{2}} e^{-2it^2}(1+i)\,dt \\ =\ & -\int_0^{\frac{\sqrt{2}R}{2}} \left[(\cos(2t^2)+\sin(2t^2)\right]\,dt - i\int_0^{\frac{\sqrt{2}R}{2}} \left[(\cos(2t^2)-\sin(2t^2)\right]\,dt \\ \to\ & -\int_0^\infty \left[(\cos(2t^2)+\sin(2t^2)\right]\,dt - i\int_0^\infty \left[(\cos(2t^2)-\sin(2t^2)\right]\,dt \quad \text{as } R\to\infty.\end{aligned}$$

Therefore, as $R \to \infty$,

$$0 = \frac{\sqrt{\pi}}{2} - \int_0^\infty \left[(\cos(2t^2) + \sin(2t^2)\right] dt - i \int_0^\infty \left[(\cos(2t^2) - \sin(2t^2)\right] dt,$$

which implies that

$$\int_0^\infty \cos(2t^2)\, dt = \int_0^\infty \sin(2t^2)\, dt = \frac{\sqrt{\pi}}{4}.$$

The theorem then follows by changing the variable. □

THEOREM 2.14.

$$\int_0^\infty \frac{1-\cos x}{x^2}\, dx = \frac{\pi}{2}.$$

PROOF. Let $0 < \varepsilon < R$ and $\gamma_{R,\varepsilon}$ be the indented semicircle in the upper half-plane positioned on the x-axis. Let $\gamma_{R,\varepsilon}$ be positively orientated. Since $f(z) = (1 - e^{iz})/z^2$ is holomorphic in $\mathbb{C} \setminus \{0\}$, $\int_{\gamma_{R,\varepsilon}} f = 0$ by Cauchy's theorem. On the other hand,

$$\int_{\gamma_{R,\varepsilon}} f(z)\, dz = \int_\varepsilon^R f(x)\, dx + \int_0^\pi f(Re^{i\theta}) iRe^{i\theta}\, d\theta + \int_{-R}^{-\varepsilon} f(x)\, dx + \int_\pi^0 f(\varepsilon e^{i\theta}) i\varepsilon e^{i\theta}\, d\theta.$$

Next we estimate the above integrals.

(1).

$$\int_\varepsilon^R f(x)\, dx + \int_{-R}^{-\varepsilon} f(x)\, dx = 2\int_\varepsilon^R \frac{1-\cos x}{x^2}\, dx \to 2\int_0^\infty \frac{1-\cos x}{x^2}\, dx$$

as $R \to \infty$ and $\varepsilon \to 0$.

(2).

$$\int_0^\pi f(Re^{i\theta}) iRe^{i\theta}\, d\theta = i\int_0^\pi \frac{1-e^{iRe^{i\theta}}}{R^2 e^{2i\theta}} Re^{i\theta}\, d\theta = i\int_0^\pi \frac{1-e^{iRe^{i\theta}}}{Re^{i\theta}}\, d\theta \to 0 \quad \text{as } R \to \infty,$$

because

$$\left| \int_0^\pi \frac{1-e^{iRe^{i\theta}}}{Re^{i\theta}}\, d\theta \right| \le \frac{\pi}{R} \left|1 - e^{iRe^{i\theta}}\right| \le \frac{2\pi}{R} \to 0 \quad \text{as } R \to \infty.$$

Here, we use the fact that $\left|1 - e^{iRe^{i\theta}}\right| \le 2$ when $0 \le \theta \le \pi$.

(3). Using the power series definition of

$$e^{iz} = \sum_{n=0}^\infty \frac{(iz)^n}{n!} = 1 + iz + \frac{(iz)^2}{2} + \frac{(iz)^3}{6} + \cdots,$$

we have that

$$f(z) = \frac{1-e^{iz}}{z^2} = -\frac{i}{z} + \psi(z),$$

in which $|\psi(z)| \le C$ for some constant $C > 0$ for all $|z| \le \varepsilon$. Then

$$\begin{aligned} & \int_\pi^0 f(\varepsilon e^{i\theta}) i\varepsilon e^{i\theta}\, d\theta \\ = \quad & -i\int_0^\pi f(\varepsilon e^{i\theta}) \varepsilon e^{i\theta}\, d\theta \\ = \quad & i\int_0^\pi \frac{i}{\varepsilon e^{i\theta}} \varepsilon e^{i\theta}\, d\theta - i\int_0^\pi \psi(\varepsilon e^{i\theta}) \varepsilon e^{i\theta}\, d\theta \\ \to \quad & -\pi \quad \text{as } \varepsilon \to 0, \end{aligned}$$

because
$$\left|\int_0^\pi \psi(\varepsilon e^{i\theta})\varepsilon e^{i\theta}\,d\theta\right| \le C\pi\varepsilon \to 0 \quad \text{as } \varepsilon \to 0.$$
Therefore, as $R \to \infty$ and $\varepsilon \to 0$,
$$0 = -\pi + 2\int_0^\infty \frac{1-\cos x}{x^2}\,dx,$$
which implies the theorem. □

$\boxed{\textbf{Homework Assignment}}$.

2-4. [Dirichlet integral] Prove that
$$\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}.$$

2.6. Cauchy integral formulas

In this section, we prove the Cauchy integral formulas. As immediate consequences, we show that holomorphic functions are smooth and analytic.

THEOREM 2.15 (Cauchy integral formula). *Let $\Omega \subset \mathbb{C}$ be open and $\overline{D}_r(z_0) \subset \Omega$. Then*
$$f(z) = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(w)}{w-z}\,dw,$$
for all holomorphic functions f in Ω and $z \in D_r(z_0)$.

PROOF. Let $\delta, \varepsilon > 0$. Consider the keyhole contour $\Gamma_{\delta,\varepsilon}$ that is comprised of an almost complete circle γ_1 of radius r and with center z_0, an almost complete circle γ_3 of radius ε and with center z, and corridor γ_2 and γ_4 of width δ, with positive orientation. Write
$$g(w) = \frac{f(w)}{w-z}.$$
Then g is holomorphic in $\Omega_{\Gamma_{\delta,\varepsilon}}$, the interior of $\Gamma_{\delta,\varepsilon}$. Therefore,
$$0 = \int_{\Gamma_{\delta,\varepsilon}} g(w)\,dw = \sum_{j=1}^4 \int_{\gamma_j} g(w)\,dw.$$
Since g is holomorphic therefore is continuous,
$$\int_{\gamma_2} g(w)\,dw + \int_{\gamma_4} g(w)\,dw \to 0 \quad \text{as } \delta \to 0.$$
Furthermore, since f is holomorphic,
$$g(w) = \frac{f(w)}{w-z} = \frac{f(z)+\psi(w)(w-z)}{w-z} = \frac{f(z)}{w-z} + \psi(w)$$
in a neighborhood of z with $|\psi(w)| \le C$ for some $C > 0$. Then for $\varepsilon \to 0$,
$$\left|\int_{\gamma_3} \psi(w)\,dw\right| \le \sup_{w\in\gamma_3} |\psi(w)| \cdot \text{Length}(\gamma_3) \le 2\pi C\varepsilon \to 0,$$
and
$$\int_{\gamma_3} \frac{f(z)}{w-z}\,dw \to -\int_{C_\varepsilon(z)} \frac{f(z)}{w-z}\,dw = -2\pi i f(z),$$

in which $C_\varepsilon(z)$ has positive orientation. Taking $\varepsilon \to 0$, we have that

$$\int_{\gamma_3} g(w)\,dw = \int_{\gamma_3} \frac{f(z)}{w-z}\,dw + \int_{\gamma_3} \psi(w)\,dw \to -2\pi i f(z).$$

Finally, since

$$\int_{\gamma_1} g(w)\,dw \to \int_{C_r(z_0)} g(w)\,dw$$

as $\delta \to 0$, we have that

$$\int_{C_r(z_0)} g(w)\,dw = 2\pi i f(z).$$

□

The following corollary provides the integral formulas of any n-th order complex derivative of a holomorphic functions. Therefore, we derive a striking property of holomorphic function that they are in fact infinitely complex differentiable.

Corollary 2.16 (Cauchy integral formulas). *Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ be holomorphic. Then f is infinitely complex differentiable in Ω. Moreover, for any $n \in \mathbb{N}$ and $\overline{D}_r(z_0) \subset \Omega$,*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{(w-z)^{n+1}}\,dw$$

for all $z \in D_r(z_0)$.

Proof. We prove by induction. The case when $n = 0$ is proved in the previous theorem. Let $n \geq 1$. Suppose that f is $(n-1)$-th complex differentiable and

$$f^{(n-1)}(z) = \frac{(n-1)!}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{(w-z)^n}\,dw$$

for any $\overline{D}_r(z_0) \subset \Omega$ and $z \in D_r(z_0)$. As $h \to 0$,

$$\begin{aligned}
& \frac{f^{(n-1)}(z+h) - f^{(n-1)}(z)}{h} \\
= \;& \frac{(n-1)!}{2\pi i} \int_{C_r(z_0)} \frac{1}{h}\left(\frac{f(w)}{(w-z-h)^n} - \frac{f(w)}{(w-z)^n}\right) dw \\
= \;& \frac{(n-1)!}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{h} \cdot \frac{(w-z)^n - (w-z-h)^n}{(w-z-h)^n(w-z)^n}\,dw \\
= \;& \frac{(n-1)!}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{h} \cdot \frac{1 - \left(1 - \frac{h}{w-z}\right)^n}{(w-z-h)^n}\,dw \\
\to \;& \frac{(n-1)!}{2\pi i} \int_{C_r(z_0)} f(w) \cdot \frac{n}{(w-z)^{n+1}}\,dw \\
= \;& \frac{n!}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{(w-z)^{n+1}}\,dw.
\end{aligned}$$

□

COROLLARY 2.17 (Cauchy inequalities). *Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ be holomorphic. Then for any $n \in \mathbb{N}$ and $\overline{D}_r(z_0) \subset \Omega$,*

$$\left|f^{(n)}(z_0)\right| \leq \frac{n!}{r^n} \cdot \sup_{w \in C_r(z_0)} |f(w)|.$$

PROOF. By the previous corollary,

$$\begin{aligned} \left|f^{(n)}(z_0)\right| &= \left|\frac{n!}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}\, dw\right| \\ &\leq \frac{n!}{2\pi} \cdot \frac{\sup_{w \in C_r(z_0)} |f(w)|}{r^{n+1}} \cdot \mathrm{Length}(C_r(z_0)) \\ &\leq \frac{n!}{r^n} \cdot \sup_{w \in C_r(z_0)} |f(w)|. \end{aligned}$$

$\square$

Immediately following Cauchy inequalities, we show Liouville's theorem and Morera's theorem.

THEOREM 2.18 (Liouville's theorem). *If f is entire and bounded, then f is constant.*

PROOF. See Problem 2-5. $\square$

THEOREM 2.19 (Morera's theorem). *Suppose that $f : D_r(z_0) \to \mathbb{C}$ is continuous and $\int_T f = 0$ for all triangles $T \subset D_r(z_0)$. Then f is holomorphic in $D_r(z_0)$.*

PROOF. See Problem 2-6. $\square$

We next prove another striking property of holomorphic functions that they are analytic. Since the reverse is obviously true, we use analytic functions and holomorphic functions in complex analysis interchangeably.

THEOREM 2.20 (Holomorphic functions are analytic). *Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ be holomorphic. Then f is analytic in Ω. That is, for any $z_0 \in \Omega$, there is $R > 0$ and $\{a_n\} \subset \mathbb{C}$ such that*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \textit{for all } z \in D_R(z_0).$$

Moreover, we have that

$$a_n = \frac{f^{(n)}(z_0)}{n!}.$$

PROOF. Let $r = \mathrm{dist}(z_0, \partial\Omega)$ and fix $z \in D_R(z_0)$. Then there is $r \in (0, 1)$ such that

$$\left|\frac{z - z_0}{w - z_0}\right| < r$$

for all $w \in C_R(z_0)$. Therefore, the geometric series

$$\sum_{n=0}^{\infty} \left(\frac{z - z_0}{w - z_0}\right)^n = \frac{1}{1 - \frac{z - z_0}{w - z_0}}$$

converges uniformly for $w \in C_R(z_0)$. Now

$$f(z) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{w - z}\, dw$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(w)}{w-z_0-(z-z_0)}\,dw \\
&= \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(w)}{(w-z_0)\left(1-\frac{z-z_0}{w-z_0}\right)}\,dw \\
&= \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(w)}{w-z_0}\cdot\left(\sum_{n=0}^{\infty}\left(\frac{z-z_0}{w-z_0}\right)^n\right)dw \\
&= \sum_{n=0}^{\infty}\left(\frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}\,dw\right)\cdot(z-z_0)^n \\
&= \sum_{n=0}^{\infty} a_n(z-z_0)^n,
\end{aligned}
$$

in which

$$a_n = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f(w)}{(w-z_0)^{n+1}}\,dw = \frac{f^{(n)}(z_0)}{n!}$$

by Cauchy integral formulas. □

Homework Assignment.

2-5. Prove Liouville's theorem: If f is entire and bounded, then f is constant.

2-6. Prove Morera's theorem: Suppose that $f : D_r(z_0) \to \mathbb{C}$ is continuous and $\int_T f = 0$ for all triangles $T \subset \mathbb{D}_r(z_0)$. Then f is holomorphic in $D_r(z_0)$.

2-7. Let $f : \mathbb{D} \to \mathbb{C}$ be holomorphic. Prove that

$$2|f'(0)| \leq \sup_{z,w\in\mathbb{D}} |f(z)-f(w)|,$$

and moreover, verify that the equality holds if f is linear, i.e., $f(z) = a_0 + a_1 z$ for some $a_0, a_1 \in \mathbb{C}$. (In fact, the equality holds iff the function is linear, but the proof of the necessity is much more challenging.)

2-8. Weierstrass's theorem states that a continuous function on $[0,1]$ can be uniformly approximated by polynomials. Can every continuous function on $\overline{D}_1(0)$ be approximated uniformly by polynomials? Prove your assertion.

2.7. Applications of Cauchy's theory

In this section, we prove several consequences of Cauchy's theory.

2.7.1. Fundamental theorem of algebra.

THEOREM 2.21. *Every non-constant polynomial (with complex coefficients) has at least one root in* $\mathbb{C}$.

PROOF. Let $P(z) = a_n z^n + \cdots + a_1 z + a_0$ with $a_n, ..., a_1, a_0 \in \mathbb{C}$ and $a_n \neq 0$. Suppose that P has no root in $\mathbb{C}$. Then $Q(z) = 1/P(z)$ is entire.

Notice that as $|z| \to \infty$,

$$\left|\frac{P(z)}{z^n} - a_n\right| = \left|\frac{a_{n-1}}{z} + \cdots + \frac{a_1}{z^{n-1}} + \frac{a_0}{z^n}\right| \to 0.$$

Hence, there is $R > 0$ such that for all $|z| \geq R$, we have that

$$\left|\frac{P(z)}{z^n} - a_n\right| \leq \frac{|a_n|}{2},$$

which implies that

$$\left|\frac{P(z)}{z^n}\right| = \left|\frac{P(z)}{z^n} - a_n + a_n\right| \geq |a_n| - \left|\frac{P(z)}{z^n} - a_n\right| \geq \frac{|a_n|}{2},$$

by triangle inequality. Thus, for all $z \in \mathbb{C} \setminus \overline{D}_R(0)$,

$$|P(z)| \geq \frac{|a_n|}{2}|z|^n > \frac{|a_n|R^n}{2}.$$

Moreover, since $P \neq 0$ in $\overline{D}_R(0)$,

$$|P(z)| \geq \min_{\overline{D}_R(0)} |P(z)| > c,$$

for some $c > 0$. Therefore, for all $z \in \mathbb{C}$,

$$|P(z)| \geq \min\left\{\frac{|a_n|R^n}{2}, c\right\}.$$

It then follows that $Q(z) = 1/P(z)$ is bounded. Now Q is entire and bounded in $\mathbb{C}$. By Liouville's theorem, $Q(z)$ is constant so $P(z) = 1/Q(z)$ is also constant. This contradicts with the fact that P is not constant. □

Corollary 2.22 (Fundamental theorem of algebra). *Let $P(z) = a_n z^n + \cdots + a_1 z + a_0$ with $a_n, ..., a_1, a_0 \in \mathbb{C}$ and $a_n \neq 0$. Then P has n roots in $\mathbb{C}$. Denote the roots by $w_1, ..., w_n$. Then $P(z) = a_n(z - w_1)\cdots(z - w_n)$.*

Proof. By the previous theorem, there is a root, denoted by w_1, of $P(z)$. Writing $z = (z - w_1) + w_1$, we have that

$$\begin{aligned} P(z) &= a_n\big[(z - w_1) + w_1\big]^n + \cdots + a_1\big[(z - w_1) + w_1\big] + a_0 \\ &= b_n(z - w_1)^n + \cdots + b_1(z - w_1) + b_0, \end{aligned}$$

in which $b_n = a_n$. Since $P(w_1) = 0$, $b_0 = 0$. Therefore,

$$P(z) = b_n(z - w_1)^n + \cdots + b_1(z - w_1) = (z - w_1)\big[b_n(z - w_1)^{n-1} + \cdots + b_1\big] = (z - w_1)Q(z),$$

in which $Q(z) = b_n(z - w_1)^{n-1} + \cdots + b_1$ has degree $n - 1$.

Applying the previous theorem again, $Q(z)$ has a root, denoted by w_2, which is also a root of $P(z)$. Inductively for n steps, we conclude that $P(z)$ has n roots, if denoted by $w_1, ..., w_n$,

$$P(z) = a_n(z - w_1)\cdots(z - w_n).$$

□

2.7.2. Analytic continuation.

Theorem 2.23. *Let $\Omega \subset \mathbb{C}$ be open and connected and $f : \Omega \to \mathbb{C}$ be holomorphic. Suppose that there is a sequence of distinct points $\{z_n\} \subset \Omega$ such that $z_n \to z_0 \in \Omega$ and $f(z_n) = 0$ for all $n \in \mathbb{N}$. Then $f = 0$ in Ω.*

In other words, if the zeros of a holomorphic function f in a connected and open set Ω accumulate in Ω, then $f = 0$ in Ω.

PROOF. Since f is holomorphic (and therefore is analytic) at $z_0 \in \mathbb{C}$, there is $r > 0$ such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in D_r(z_0)$.

We first show that $f = 0$ in $D_r(z_0)$. Suppose otherwise. Then

$$m = \min\{n \in \mathbb{N} : a_n \neq 0\} < \infty.$$

Now

$$\begin{aligned} f(z) &= \sum_{n=m}^{\infty} a_n (z - z_0)^n \\ &= a_m (z - z_0)^m \sum_{n=m}^{\infty} \frac{a_n}{a_m} (z - z_0)^{n-m} \\ &= a_m (z - z_0)^m \left(1 + \sum_{j=1}^{\infty} \frac{a_{m+j}}{a_m} (z - z_0)^j\right) \\ &= a_m (z - z_0)^m \left(1 + g(z)\right), \end{aligned}$$

in which

$$g(z) = \sum_{j=1}^{\infty} \frac{a_{m+j}}{a_m} (z - z_0)^j$$

is holomorphic at z_0 and $g(z_0) = 0$. Because $z_n \to z_0$, we know that $g(z_n) \to 0$ as $n \to \infty$. Hence, $1 + g(z_n) \neq 0$ as n is sufficiently large. We also see that $(z_n - z_0)^m \neq 0$ since z_n are distinct. This contradicts with the fact that $f(z_n) = 0$.

Next we show that $f = 0$ in Ω. Let $Z = \{z \in \Omega : f(z) = 0\}$ and $U = \mathrm{Int}(Z)$. Then U is open by definition and is non-empty since $z_0 \in U$. We now show that U is also closed. Indeed, if $w_n \in U$ and $w_n \to w_0$, then $f(w_n) = 0$ for all $n \in \mathbb{N}$. Hence, $f(w_0) = 0$ since f is continuous. Repeating the same argument as above, we know that $f = 0$ on $D_r(w_0)$ for some $r > 0$. Hence, $w_0 \in U$ so U is closed. It therefore follows that $U = \Omega$ since Ω is open and connected. $\square$

COROLLARY 2.24. *Let $\Omega \subset \mathbb{C}$ be open and connected and $f, g : \Omega \to \mathbb{C}$ be holomorphic. Suppose that there is a sequence of distinct points $\{z_n\} \subset \Omega$ such that $z_n \to z_0 \in \Omega$ and $f(z_n) = g(z_n)$ for all $n \in \mathbb{N}$. Then $f = g$ in Ω.*

THEOREM 2.25 (Analytic continuation). *Let $\Omega \subset \mathbb{C}$ be open and connected and $f, F : \Omega \to \mathbb{C}$ be holomorphic (i.e., analytic). Suppose that $F = f$ in Ω_1 for some non-empty and open set $\Omega_1 \subset \Omega$. Then $F = f$ in Ω. We say that F is an analytic continuation of f and is uniquely determined by f.*

2.7.3. Sequences of holomorphic functions.

LEMMA 2.26. *Let $\Omega \subset \mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be holomorphic on Ω. Then*

$$\sup_{z \in \Omega_\delta} |f^{(n)}(z)| \leq \frac{n!}{\delta^n} \cdot \sup_{w \in \Omega} |f(w)|,$$

in which $\Omega_\delta = \{z \in \Omega : \overline{D}_\delta(z) \subset \Omega\}$.

Proof. By the Cauchy integral formula,

$$\begin{aligned} |f^{(n)}(z)| &= \left|\frac{n!}{2\pi i}\int_{C_\delta(z)} \frac{f(w)}{(w-z)^{n+1}}\,dw\right| \\ &\leq \frac{n!}{2\pi}\cdot\frac{\sup_{w\in\Omega}|f(w)|}{\delta^{n+1}}\cdot \mathrm{Length}(C_\delta(z)) \\ &= \frac{n!}{\delta^n}\cdot\sup_{w\in\Omega}|f(w)|. \end{aligned}$$

□

Theorem 2.27. *Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$. Suppose that $\{f_j\}_{j=1}^{\infty}$ is a sequence of holomorphic functions such that $f_j \to f$ uniformly on any compact subset of Ω. Then*

(i). f is holomorphic on Ω.

(ii). $f_j^{(n)} \to f^{(n)}$ uniformly on any compact subset of Ω.

Remark. Let $f_j(z) = z^j$. Then $f_j(z) \to 0$ if $|z| < 1$ and $f_j(1) = 1$. One then sees that the pointwise limit of holomorphic functions is not necessarily holomorphic (or even continuous).

Proof. Let $z_0 \in \Omega$ and $\overline{D}_r(z_0) \subset \Omega$. Since $f_j \to f$ uniformly on $\overline{D}_r(z_0)$, f is continuous. Moreover, for any triangle $T \subset D_r(z_0)$,

$$\int_T f_j \to \int_T f,$$

in which $\int_T f_j = 0$ because f_j is holomorphic. Therefore, $\int_T f = 0$ for any triangle $T \subset D_r(z_0)$. By Morera's theorem 2.19, f is holomorphic in $D_r(z_0)$. It then follows that f is holomorphic on Ω.

To prove (ii), for any compact subset $\Omega_1 \subset \Omega$, there are compact set $\tilde{\Omega} \subset \Omega$ and $\delta > 0$ such that $\Omega_1 \subset \tilde{\Omega}_\delta \subset \tilde{\Omega} \subset \Omega$. Since $f_j - f \to 0$ uniformly on $\tilde{\Omega}$,

$$\sup_{w\in\tilde{\Omega}} |f_j(w) - f(w)| \to 0,$$

as $j \to \infty$. Then by the previous lemma,

$$\sup_{z\in\Omega_1} |f_j^{(n)}(z) - f^{(n)}(z)| \leq \sup_{z\in\tilde{\Omega}_\delta} |f_j^{(n)}(z) - f^{(n)}(z)| \leq \frac{n!}{\delta^n}\cdot\sup_{w\in\tilde{\Omega}}|f_j(w) - f(w)| \to 0,$$

that is, $f_j^{(n)} \to f^{(n)}$ uniformly on Ω_1. □

2.7.4. Holomorphic functions defined by integrals.

Theorem 2.28. *Let $\Omega \subset \mathbb{C}$ be open and*

$$f(z) = \int_0^1 F(z,s)\,ds,$$

in which $F : \Omega \times [0,1] \to \mathbb{C}$ satisfies that

(i). $F(z,s)$ is holomorphic in z for each $s \in [0,1]$,

(ii). $F(z,s)$ is continuous on $\Omega \times [0,1]$.

Then f is holomorphic on Ω.

PROOF. Write the Riemann sum

$$f_k(z) = \frac{1}{k}\sum_{j=1}^{k} F\left(z, \frac{j}{k}\right),$$

Then $f_k(z) \to f(z)$ as $k \to \infty$. Moreover, f_k is holomorphic on Ω since $F(z, j/k)$ is holomorphic for each $j = 1, ..., k$.

Let Ω_1 be a compact subset of Ω. Then F is uniformly continuous on $\Omega_1 \times [0,1]$. That is, for any $\varepsilon > 0$, there is $\delta > 0$ such that $|F(z,s) - F(w,t)| \le \varepsilon$ if $|(z,s) - (w,t)| \le \delta$. In particular,

$$\sup_{z\in\Omega_1} |F(z,s) - F(z,t)| \le \varepsilon, \quad \text{if } |s-t| \le \delta.$$

Now if $k > 1/\delta$, then

$$\begin{aligned}
|f_k(z) - f(z)| &= \left|\frac{1}{k}\sum_{j=1}^{k} F\left(z, \frac{j}{k}\right) - \int_0^1 F(z,s)\,ds\right| \\
&= \left|\sum_{j=1}^{k}\int_{\frac{j-1}{k}}^{\frac{j}{k}} \left(F\left(z, \frac{j}{k}\right) - F(z,s)\right) ds\right| \\
&\le \sum_{j=1}^{k}\int_{\frac{j-1}{k}}^{\frac{j}{k}} \left|F\left(z, \frac{j}{k}\right) - F(z,s)\right| ds \\
&\le \sum_{j=1}^{k}\int_{\frac{j-1}{k}}^{\frac{j}{k}} \varepsilon\, ds \\
&= \varepsilon.
\end{aligned}$$

This means that $f_k \to f$ uniformly on Ω_1. By Theorem 2.27, f is holomorphic on Ω. □

Remark. The interval $[0,1]$ is of course an arbitrary choice and the theorem remains valid for any interval $[a,b]$. In addition, since integrals along curves can be written as integrations as above, we have the following generalization of theorem: Let γ be a curve and

$$f(z) = \int_\gamma F(z,w)\,dw,$$

in which $F : \Omega \times \gamma \to \mathbb{C}$ satisfies that

(i). $F(z,w)$ is holomorphic in z for each $w \in \gamma$,
(ii). $F(z,w)$ is continuous on $\Omega \times \gamma$.

Then f is holomorphic on Ω.

CHAPTER 3

Meromorphic functions

Meromorphic functions are quotients of holomorphic functions (at least locally). Suppose that $f = g/h$ in which g and h are holomorphic. Then the zeros of g produce zeros of f while the zeros of h produce poles of f. In this chapter, we study and classify the zeros/poles of meromorphic functions. Then we provide some applications of such theory.

3.1. Zeros and poles

Definition (Zeros). Let $\Omega \subset \mathbb{C}$ and $f : \Omega \to \mathbb{C}$ be holomorphic. A point $z_0 \in \Omega$ is said to be a zero of f if $f(z_0) = 0$.

Remark. By Theorem 2.23, the set of zeros of a holomorphic function f on an open and connected set Ω does not have a limit point unless $f = 0$ on Ω.

THEOREM 3.1 (Order/multiplicity of zeros). *Let $\Omega \subset \mathbb{C}$ be open and connected and $f : \Omega \to \mathbb{C}$ be holomorphic and not identically zero. Suppose that $z_0 \in \Omega$ is a zero of f. Then there are $r > 0$, a holomorphic function $g : D_r(z_0) \to \mathbb{C}$ that does not have zeros in $D_r(z_0)$, and a unique positive integer n such that*

$$f(z) = (z - z_0)^n g(z) \quad \text{for } z \in D_r(z_0).$$

We say that f has a zero of order/multiplicity n at z_0. If a zero has order 1, we then say that it is simple.

PROOF. Since f is analytic, there is $r_1 > 0$ such that $D_{r_1}(z_0) \subset \Omega$ and

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k \quad \text{for all } z \in D_{r_1}(z_0).$$

Since f is not identically zero, f is not identically zero in $D_{r_1}(z_0)$. Hence, there exists a smallest integer n such that $a_n \neq 0$. Therefore,

$$f(z) = \sum_{k=n}^{\infty} a_k (z - z_0)^k = (z - z_0)^n \sum_{k=0}^{\infty} a_{k+n} (z - z_0)^k = (z - z_0)^n g(z),$$

in which

$$g(z) = \sum_{k=0}^{\infty} a_{k+n} (z - z_0)^k$$

is holomorphic in $D_{r_1}(z_0)$. Since $g(z_0) = a_n \neq 0$, there is $0 < r \leq r_1$ such that $g(z) \neq 0$ for all $z \in D_r(z_0)$.

To prove the uniqueness of the integer n, suppose that $f(z) = (z - z_0)^m h(z)$ for some $m \in \mathbb{N}$ and $h(z) \neq 0$ for all z in a neighborhood of z_0. That is,

$$(z - z_0)^n g(z) = (z - z_0)^m h(z).$$

Without loss of generality, assume that $m > n$. Then

$$g(z) = (z - z_0)^{m-n} h(z)$$

so $g(z_0) = 0$, a contradiction. We therefore conclude that $m = n$ so $h(z) = g(z)$. □

Definition (Poles). A point $z_0 \in \mathbb{C}$ is said to be a pole of a function f if there is $r > 0$ such that $f : D_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ and $1/f : D_r(z_0) \to \mathbb{C}$ is holomorphic and vanishes at z_0.

THEOREM 3.2 (Order/multiplicity of poles). *Suppose that $z_0 \in \mathbb{C}$ is a pole of f. Then there are $r > 0$, a holomorphic function $g : D_r(z_0) \to \mathbb{C}$ that does not have zeros in $D_r(z_0)$, and a unique positive integer n such that*

$$f(z) = (z - z_0)^{-n} g(z) \quad \textit{for } z \in D_r(z_0) \setminus \{z_0\}.$$

We say that f has a pole of order/multiplicity n at z_0. If a pole has order 1, we then say that it is simple.

PROOF. By the definition of poles, there is $r_1 > 0$ such that $f : D_{r_1}(z_0) \setminus \{z_0\} \to \mathbb{C}$ and $1/f : D_{r_1}(z_0) \to \mathbb{C}$ is holomorphic and vanishes at z_0. Since $1/f$ is not identically zero (otherwise f can not be defined in $D_{r_1}(z_0) \setminus \{z_0\}$), there are $0 < r \leq r_1$, a holomorphic function $h : D_r(z_0) \to \mathbb{C}$ that does not have zeros in $D_r(z_0)$, and a unique positive integer n such that

$$\frac{1}{f(z)} = (z - z_0)^n h(z) \quad \text{for } z \in D_r(z_0).$$

Letting $g(z) = 1/h(z)$ so $g(z) \neq 0$ for all $z \in D_r(z_0)$. Therefore,

$$f(z) = (z - z_0)^{-n} g(z) \quad \text{for } z \in D_r(z_0) \setminus \{z_0\}.$$

□

Remark. In practice, if one can write $f(z) = (z - z_0)^{-n} g(z)$ for some holomorphic function g that does not vanish anywhere in a neighborhood of z_0, then f has a pole of order n at z_0.

COROLLARY 3.3. *Suppose that f has a pole of order n at z_0. Then there are $r > 0$, $a_{-n} \neq 0, a_{-(n-1)} ..., a_{-1} \in \mathbb{C}$, and a holomorphic function $G : D_r(z_0) \to \mathbb{C}$ such that*

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + G(z) \quad \textit{for } z \in D_r(z_0) \setminus \{z_0\}.$$

PROOF. By the previous theorem, there are $r_1 > 0$ and a holomorphic function $g : D_{r_1}(z_0) \to \mathbb{C}$ that does not have zeros in $D_{r_1}(z_0)$ such that

$$f(z) = (z - z_0)^{-n} g(z) \quad \text{for } z \in D_{r_1}(z_0) \setminus \{z_0\}.$$

Since g is analytic, there is $0 < r \leq r_1$ such that

$$g(z) = \sum_{k=0}^{\infty} b_k (z - z_0)^k \quad \text{for all } z \in D_r(z_0).$$

Notice that $b_0 \neq 0$ since $g(z_0) \neq 0$. Hence, for all $z \in D_r(z_0) \setminus \{z_0\}$,

$$\begin{aligned} f(z) &= (z - z_0)^{-n} g(z) \\ &= (z - z_0)^{-n} \left(\sum_{k=0}^{\infty} b_k (z - z_0)^k \right) \\ &= \frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + G(z), \end{aligned}$$

in which $a_{-n} = b_0 \neq 0, a_{-(n-1)} = b_1, ..., a_{-1} = b_{n-1}$, and

$$G(z) = \sum_{k=n}^{\infty} b_k (z - z_0)^{k-n}$$

is holomorphic in $D_r(z_0)$. $\square$

Definition (Principle part and residue). Given the above representation of f at a neighborhood of the pole z_0,

- $$\frac{a_{-n}}{(z-z_0)^n} + \cdots + \frac{a_{-2}}{(z-z_0)^2} + \frac{a_{-1}}{z-z_0}$$
  is called the principle part at the pole z_0,
- a_{-1} is called the residue of f and is denoted by $\mathrm{res}_{z_0} f$.

Remark (Computing the coefficients in the principle part). There are the following ways to deduce the coefficients $a_{-n}, ..., a_{-1}$ in the principle part of f at a pole z_0.

- It is obvious that
  $$a_{-n} = \lim_{z\to z_0} (z-z_0)^n f(z).$$
  In particular, if z_0 is simple, then
  $$\mathrm{res}_{z_0} f = a_{-1} = \lim_{z\to z_0} (z-z_0) f(z).$$
- Suppose that f is holomorphic on $D_R(z_0)$ except at the pole z_0. Then
  $$a_{-n} = \frac{1}{2\pi i}\int_{C_r(z_0)} (z-z_0)^{n-1} f(z)\,dz \quad \text{for all } r < R.$$
  This follows directly from the fact that for $n \in \mathbb{Z}$,
  $$\frac{1}{2\pi i}\int_{C_r(z_0)} (z-z_0)^n\,dz = \begin{cases} 1 & \text{if } n = -1, \\ 0 & \text{otherwise.} \end{cases}$$
  In particular, if z_0 is simple, then
  $$\mathrm{res}_{z_0} f = a_{-1} = \frac{1}{2\pi i}\int_{C_r(z_0)} f(z)\,dz \quad \text{for all } r < R,$$
  which is called the “residue formula”. See the next section for a detailed proof.

Homework Assignment.

3-1. Prove that $\sin(\pi z)$ has simple zeros at the integers.

3-2. Prove that $1/\sin(\pi z)$ has simple poles at the integers and find the residue at these poles.

3.2. The residue formula

Recall that the Cauchy's theorem in Theorem 2.7 asserts that $\int_\gamma f = 0$ for all $\gamma \in D$ if f is holomorphic in a disc D. (In fact, the theorem holds for more general toy contours.) Cauchy's theorem immediately applies to the evaluation of some integrals. The residue formula in this section consider the case when f has poles. For example, the function $(z-z_0)^{-k}$ $(k \geq 1)$ has a pole of order k at $z_0 = 0$ and

$$\int_{C_r(z_0)} \frac{1}{(z-z_0)^k}\,dz = \begin{cases} 2\pi i & \text{if } k = 1, \\ 0 & \text{if } k \geq 2, \end{cases}$$

for all circles $C_r(z_0)$ with positive orientation.

THEOREM 3.4. *Let $\Omega \subset \mathbb{C}$ be open and $z_0 \in \Omega$. Suppose that $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ is holomorphic and has a pole at z_0. Then*

$$\int_{C_r(z_0)} f(z)\,dz = 2\pi i \cdot \mathrm{res}_{z_0} f$$

for all $\overline{D}_r(z_0) \subset \Omega$.

PROOF. Similar as in the proof of Cauchy integral formula in Theorem 2.15, we use a keyhole contour that avoids the pole z_0. Letting the width of the corridor go to zero,

$$\int_{C_r(z_0)} f(z)\,dz = \int_{C_\varepsilon(z_0)} f(z)\,dz,$$

in which both circles have positive orientation. By Corollary 3.3, there are $r > 0$, $a_{-n}, ..., a_{-1} \in \mathbb{C}$, and $G : D_r(z_0) \to \mathbb{C}$ holomorphic such that

$$f(z) = \frac{a_{-n}}{(z - z_0)^n} + \cdots + \frac{a_{-2}}{(z - z_0)^2} + \frac{a_{-1}}{z - z_0} + G(z) \quad \text{for } z \in D_r(z_0) \setminus \{z_0\}.$$

Now if $\varepsilon < r$, then

$$\begin{aligned} &\int_{C_\varepsilon(z_0)} f(z)\,dz \\ = \ & \int_{C_\varepsilon(z_0)} \frac{a_{-n}}{(z - z_0)^n}\,dz + \cdots + \int_{C_\varepsilon(z_0)} \frac{a_{-2}}{(z - z_0)^2}\,dz + \int_{C_\varepsilon(z_0)} \frac{a_{-1}}{z - z_0}\,dz + \int_{C_\varepsilon(z_0)} G(z)\,dz \\ = \ & 2\pi i \cdot a_{-1} \\ = \ & 2\pi i \cdot \mathrm{res}_{z_0} f, \end{aligned}$$

by Cauchy's theorem. We then complete the proof because $\int_{C_r(z_0)} f(z) = \int_{C_\varepsilon(z_0)} f(z)$. ☐

THEOREM 3.5 (The residue formula). *Suppose that f is holomorphic in an open set containing a toy contour γ and its interior, except for poles at points $z_1, ..., z_n$ inside γ. Then*

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{n} \mathrm{res}_{z_j} f$$

One can design a multiple keyhole which has a loop avoiding each one of the poles. Then the proof is similar as above.

3.2.1. Evaluation of some integrals. In this subsection, we use the residue formula in Theorem 3.5 to evaluate some integrals. The approach is similar but more general as the one in Section 2.3. That is, the evaluation involves a (toy) contour that contains the original integral domain as a piece. If we can use the residue formula to conclude the integral of a holomorphic function along the contour is determined by the poles inside the curve, then the original integral is "shifted" to the ones on the other pieces of the contour.

THEOREM 3.6.

$$\int_{-\infty}^{\infty} \frac{1}{1 + x^2}\,dx = \pi.$$

PROOF. Let γ_R be the semicircle contour with center at the origin and with radius R. Let γ_R be positively orientated. The function $f(z) = 1/(1 + z^2)$ is holomorphic in $\mathbb{C}$ except for simple poles $\pm i$. In addition,

$$f(z) = \frac{1}{1 + z^2} = \frac{1}{(z - i)(z + i)}.$$

So the residue of f at i is

$$\mathrm{res}_i f = \lim_{z\to i}(z-i)f(z) = \frac{1}{2i}.$$

Hence, $\int_{\gamma_R} f = 2\pi i \cdot \mathrm{res}_i f = \pi$ by the residue formula. On the other hand,

$$\int_{\gamma_R} f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_0^\pi f(Re^{i\theta})iRe^{i\theta}\,d\theta.$$

Next we estimate the two integrals.

(1).

$$\int_{-R}^{R} f(x)\,dx = \int_{-R}^{R} \frac{1}{1+x^2}\,dx \to \int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx \quad \text{as } R\to\infty.$$

(2).

$$\int_0^\pi f(Re^{i\theta})iRe^{i\theta}\,d\theta = i\int_0^\pi \frac{Re^{i\theta}}{1+R^2e^{2i\theta}}\,d\theta \to 0 \quad \text{as } R\to\infty,$$

because

$$\frac{Re^{i\theta}}{1+R^2e^{2i\theta}} = \frac{1}{R^{-1}e^{i\theta}+Re^{i\theta}} \to 0 \quad \text{as } R\to\infty$$

uniformly for $\theta\in[0,\pi]$.

Therefore, as $R\to\infty$,

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \pi.$$

□

Theorem 3.7. *Let $0<a<1$. Then*

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = \frac{\pi}{\sin(\pi a)}.$$

Proof. Let γ_R be the rectangular contour with vertices $R, R+2\pi i, -R+2\pi i, -R$ and positive orientation. The function $f(z)=e^{az}/(1+e^z)$ is holomorphic in an open set that contains γ_R and its interior except for a pole πi. In addition,

$$\lim_{z\to\pi i}(z-\pi i)f(z) = e^{\pi ai}\lim_{z\to\pi i}\frac{z-\pi i}{1+e^z} = e^{\pi ai}\lim_{z\to\pi i}\frac{z-\pi i}{e^z-e^{\pi i}} = -e^{\pi ai},$$

in which we use the fact that

$$\lim_{z\to\pi i}\frac{e^z-e^{\pi i}}{z-\pi i} = (e^z)'(\pi i) = e^{\pi i} = -1.$$

Hence,

$$\lim_{z\to\pi i}(z-\pi i)^k f(z) = 0 \quad \text{for all } k\geq 2.$$

So f has a simple pole at πi with residue $\mathrm{res}_{\pi i}f = -e^{\pi ai}$.

Hence, $\int_{\gamma_R} f = 2\pi i\cdot \mathrm{res}_{\pi i} f = -2\pi i e^{\pi ai}$ by the residue formula. On the other hand,

$$\int_{\gamma_R} f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_0^{2\pi} f(R+iy)i\,dy + \int_R^{-R} f(x+2\pi i)\,dx + \int_{2\pi}^0 f(-R+iy)i\,dy.$$

Next we estimate the four integrals.

(1).

$$\int_{-R}^{R} f(x)\,dx \to \int_{-\infty}^{\infty}\frac{e^{ax}}{1+e^x}\,dx \quad \text{as } R\to\infty.$$

(2).
$$\int_0^{2\pi} f(R+iy)i\,dy = i\int_0^{2\pi} \frac{e^{a(R+iy)}}{1+e^{R+iy}}\,dy \to 0 \quad \text{as } R\to\infty,$$
because
$$\left|\int_0^{2\pi} \frac{e^{a(R+iy)}}{1+e^{R+iy}}\,dy\right| \le 2\pi C\cdot e^{(a-1)R} \to 0 \quad \text{as } R\to\infty.$$
(3).
$$\int_R^{-R} f(x+2\pi i)\,dx = -\int_{-R}^{R} \frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}}\,dx \to -e^{2\pi ai}\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx \quad \text{as } R\to\infty.$$
(4).
$$\int_{2\pi}^{0} f(-R+iy)i\,dy \to 0 \quad \text{as } R\to\infty,$$
similar as in (2).

Therefore, as $R\to\infty$,
$$-2\pi i e^{\pi ai} = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx - e^{2\pi ai}\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx,$$
which implies that
$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = \frac{2\pi i e^{\pi ai}}{e^{2\pi ai}-1} = \frac{2\pi i}{e^{\pi ai}-e^{-\pi ai}} = \frac{\pi}{\sin(\pi a)}.$$
□

Homework Assignment.

3-3. Let $n\ge 2$. Prove that
$$\int_0^{\infty} \frac{1}{1+x^n}\,dx = \frac{\pi}{n\sin(\pi/n)}.$$
3-4. Prove that
$$\int_{-\infty}^{\infty} \frac{e^{-2\pi i x\xi}}{\cosh(\pi x)}\,dx = \frac{1}{\cosh(\pi\xi)},$$
that is, the Fourier transform of $1/\cosh(\pi x)$ is itself.

3-5. Prove that
$$\int_0^{\infty} \frac{\log x}{x^2+a^2}\,dx = \frac{\pi}{2a}\log a, \quad \text{in which } a>0.$$

3.3. Classification of isolated singularities

Definition (Isolated singularities). We say that a function f has an isolated singularity at a point $z_0\in\mathbb{C}$ if f is defined in a neighborhood of z_0 but not at z_0.

According to Theorem 3.2, the poles are a class of isolated singularities of holomorphic functions. In this section, we study the full classification of the isolated singularities of holomorphic functions:

(1). removable singularities, e.g., $f(z)=z^2/z$ has a removable singularity at 0,
(2). poles, e.g., $f(z)=1/z$ has a pole at 0,
(3). essential singularities, e.g., $f(z)=e^{1/z}$ has an essential singularity at 0.

Definition (Removable singularities). Suppose that a holomorphic function f has an isolated singularity at a point $z_0 \in \mathbb{C}$, that is, f is holomorphic in a neighborhood of z_0 but not at z_0. We say that z_0 is a removable singularity of f if we can define f at z_0 such that f is holomorphic in that neighborhood of z_0.

THEOREM 3.8 (Riemann). *Let $\Omega \subset \mathbb{C}$ be open and $z_0 \in \Omega$. Suppose that $f : \Omega \setminus \{z_0\} \to \mathbb{C}$ is holomorphic and bounded. Then z_0 is a removable singularity of f.*

PROOF. Let $r > 0$ such that $\overline{D}_r(z_0) \subset \Omega$. Define

$$f_1(z) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{w - z}\, dw \quad \text{for } z \in D_r(z_0) \setminus \{z_0\}.$$

Notice that $F(z, w) := f(w)/(w - z)$ is holomorphic with respect to $z \in D_r(z_0)$ for each $w \in C_r(z_0)$ and is continuous in $D_r(z_0) \times C_r(z_0)$. Therefore, by Theorem 2.28, f_1 is holomorphic on $D_r(z_0)$.

We next show that $f_1(z) = f(z)$ for all $z \in D_r(z_0) \setminus \{z_0\}$. Fix $z \in D_r(z_0)$ and $z \neq z_0$. Construct a multiple keyhole contour with larger circle $C_r(z_0)$ which avoids z_0 and z. Then similarly as in the residue formula of Theorem 3.5, we have that

$$\int_{C_r(z_0)} \frac{f(w)}{w - z}\, dw = \int_{C_\varepsilon(z_0)} \frac{f(w)}{w - z}\, dw + \int_{C_\varepsilon(z)} \frac{f(w)}{w - z}\, dw,$$

in which all three circles have positive orientation.

Since f is bounded, $|f(w)| \leq C$ for all $w \in \Omega$ with some constant $C > 0$. Then estimate that

$$\left| \int_{C_\varepsilon(z_0)} \frac{f(w)}{w - z}\, dw \right| \leq 2\pi\varepsilon \cdot \frac{C}{|z - z_0| - \varepsilon} \to 0 \quad \text{as } \varepsilon \to 0.$$

Here, we use the fact that $|w - z| \geq |z - z_0| - \varepsilon$ for all $w \in C_\varepsilon(z_0)$.

Since f is holomorphic near z, by Cauchy integral formula in Theorem 2.15,

$$\int_{C_\varepsilon(z)} \frac{f(w)}{w - z}\, dw = 2\pi i f(z).$$

Combining these two estimates, we have that

$$f_1(z) = \frac{1}{2\pi i} \int_{C_r(z_0)} \frac{f(w)}{w - z}\, dw = f(z)$$

all $z \in D_r(z_0) \setminus \{z_0\}$.

Finally, we set $f(z_0) = f_1(z_0)$ so f becomes holomorphic in $D_r(z_0)$. □

COROLLARY 3.9. *Suppose that $f : D_r(z_0) \setminus \{z_0\}$ is holomorphic and has an isolated singularity at z_0. Then z_0 is a pole of f iff $|f(z)| \to \infty$ as $z \to z_0$.*

PROOF. Necessity. Suppose that f has a pole at z_0. Then $1/f(z) \to 0$ as $z \to z_0$. It then follows that $|f(z)| \to \infty$ as $z \to z_0$.

Sufficiency. Suppose that $|f(z)| \to \infty$ as $z \to z_0$. Then $1/f(z) \to 0$ as $z \to z_0$. In particular, $1/f$ is bounded near z_0. By the previous theorem, z_0 is removable for $1/f$ and is also removable for $1/f$. We can then (re)define f at z_0 such that $1/f(z_0) = 0$. This means that f has a pole at z_0. □

Definition (Essential singularities). Suppose that a holomorphic function f has an isolated singularity at a point $z_0 \in \mathbb{C}$, that is, f is holomorphic in a neighborhood of z_0 but not at z_0. We say that z_0 is an essential singularity of f if it is not a pole or a removable singularity.

By the definition, we classify the isolated singularities of holomorphic functions into classes of poles, removable singularities, and essential singularities.

Example (Essential singularities). Let $f(z) = e^{1/z}$. We show that 0 is an essential singularity of f.

(1). 0 is not a removable singularity. Suppose that $f(0)$ can be defined so f becomes holomorphic in a neighborhood of 0. Hence, $f(z) \to f(0)$ as $z \to 0$. However, let $z = 1/(k\pi i)$. Then $f(z) = e^{k\pi i} = 1$ if k is even and -1 if k is odd.

(2). 0 is not a pole. Suppose that $f(z) = z^{-n}g(z)$ in $D_r(0)$ for $n \in \mathbb{N}$ and $g(z) \neq 0$ for all $z \in D_r(0)$. Write $z = se^{i\theta}$. Then

$$g(z) = z^n e^{\frac{1}{z}} = s^n e^{in\theta} e^{s^{-1}e^{-i\theta}} = s^n e^{in\theta} e^{s^{-1}\cos\theta} e^{-is^{-1}\sin\theta} \to 0,$$

by choosing $s \to 0$ and $\theta = \pi$. This contradicts with the fact that $g(z) = z^n f(z) \neq 0$.

We next provide a characterization of essential singularities.

THEOREM 3.10 (Casorati-Weierstrass). *Suppose that $f : D_r(z_0) \setminus \{z_0\} \to \mathbb{C}$ is holomorphic and has an essential singularity at z_0. Then the image of f is dense in $\mathbb{C}$, that is, for each $w \in \mathbb{C}$ and $\delta > 0$, there is $z \in D_r(z_0) \setminus \{z_0\}$ such that $|f(z) - w| \leq \delta$.*

PROOF. We prove by contradiction. Suppose that there are $w \in \mathbb{C}$ and $\delta > 0$ such that $|f(z) - w| > \delta$ for all $z \in D_r(z_0) \setminus \{z_0\}$. Write the function

$$g(z) = \frac{1}{f(z) - w} \quad \text{for } z \in D_r(z_0) \setminus \{z_0\}.$$

Then g is holomorphic and is bounded by $1/\delta$. It then follows that g has a removable singularity at z_0, which means that g can be (re)defined as a holomorphic function in $D_r(z_0)$.

Now if $g(z_0) \neq 0$, then $f(z) = w + 1/g(z)$ is holomorphic at z_0. It is not possible.

While if $g(z_0) = 0$, then $f(z) - w$ has a pole at z_0, which means that $f(z)$ also has a pole at z_0 (because $|f(z)| \to \infty$ as $z \to z_0$). It is not possible either. □

Remark (Picard). Captured by the above theorem, we see that the behavior near an essential singularity is quite wild. In fact, Picard proved that a holomorphic function near an essential singularity takes on every complex value infinitely many times with at most one exception. While not proved in this note, we use the example $f(z) = e^{1/z}$ to verify such phenomenon. Let $w = se^{i\varphi} \in \mathbb{C}$ and $e^{1/z} = w$. Write $z = re^{i\theta}$. Then

$$e^{\frac{1}{z}} = e^{r^{-1}\cos\theta} e^{ir^{-1}\sin\theta} = se^{i\varphi},$$

which has infinitely many solutions

$$\begin{cases} e^{r^{-1}\cos\theta} = s, \\ r^{-1}\sin\theta = \varphi. \end{cases}$$

3.4. Meromorphic functions

Definition (Meromorphic functions). Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$. We say that f is meromorphic if

- there is a subset $\Omega_1 = \{z_1, z_2, ...\} \subset \Omega$ with no limit points,
- f is holomorphic on $\Omega \setminus \Omega_1$,
- f has poles at the points in Ω_1.

That is, a function is meromorphic means that it is holomorphic except at possibly some poles that do not accumulate anywhere. In particular, a meromorphic function is holomorphic iff it does not have any poles.

We show that any rational function (i.e., quotient of two holomorphic functions) is meromorphic. Indeed, suppose that $f = g/h$, in which g and h are holomorphic on Ω. Let $z_0 \in \Omega$.

- Case I. If z_0 is not a zero of h, i.e., $h(z_0) \neq 0$, then f is holomorphic at z_0.
- Case II. If z_0 is a zero (of order m) of h, then by Theorem 3.1, $h(z) = (z - z_0)^m h_1(z)$ with $m \in \mathbb{N}$ and $h_1(z) \neq 0$ for all z in a neighborhood of z_0.
  - Case II.A. If z_0 is not a zero of g, i.e., $g(z_0) \neq 0$, then $g(z) \neq 0$ in a neighborhood of z_0. So near z_0,
  $$f(z) = \frac{g(z)}{(z - z_0)^m h_1(z)} = (z - z_0)^{-m} \frac{g(z)}{h_1(z)},$$
  which means that f has a pole (of order m) of z_0.
  - Case II.B. If z_0 is a zero (of order n) of g, then $g(z) = (z - z_0)^n g_1(z)$ with $n \in \mathbb{N}$ and $g_1(z) \neq 0$ for all z in a neighborhood of z_0. So near z_0,
  $$f(z) = \frac{(z - z_0)^n g_1(z)}{(z - z_0)^m h_1(z)} = (z - z_0)^{n-m} \frac{g_1(z)}{h_1(z)},$$
  in which g_1/h_1 is holomorphic and does not vanish anywhere in a neighborhood of z_0, in addition,
    - Case II.B.1. if $n > m$, then f has a zero (of order $n - m$) of z_0,
    - Case II.B.2. if $n = m$, then f is holomorphic at z_0,
    - Case II.B.3. if $n < m$, then f has a pole (of order $m - n$) of z_0.

In summary, $f = g/h$ has a pole at z_0 iff h has a zero of order m at z_0 and $g(z_0) \neq 0$ or g has a zero of order $n < m$ at z_0. Denote the set of such points by Ω_1. Then $\Omega_1 \subset \Omega_h$, in which Ω_h is the set of zeros of h. By Theorem 3.1, Ω_h consists of isolated points so does Ω_1. Therefore, f is meromorphic.

The reverse of the above statement is most convenient to establish on the extended complex plane $\mathbb{C} \cup \{\infty\}$.

Definition (Singularity at infinity). Let $f : \mathbb{C} \setminus \overline{D}_r(0) \to \mathbb{C}$ be holomorphic for some $r \geq 0$. Write $F(z) = f(1/z)$. We say that

- f has a removable singularity at infinity if F has a removable singularity at 0,
- f has a pole (of order n) at infinity if F has a pole (of order n) at 0,
- f has an essential singularity at infinity if F has an essential singularity at 0.

Remark.

- If f has a removable singularity at infinity, then $f(1/z) = F(z) \to F(0)$ as $z \to 0$. This implies that $f(z) \to F(0)$ as $|z| \to \infty$, in particular, $f(z)$ is bounded when $|z| > r$ for some $r > 0$.
- If f has a pole of order n at infinity, then $f(1/z) = F(z) = F_0(z) + G_0(z)$ in a neighborhood of 0. Here, $F_0(z)$ is the principle part (which is a polynomial in $1/z$ of order n) and G_0 is holomorphic near 0. This implies that $f(z) = f_0(z) + g_0(z)$ as $|z| \to \infty$, in which $f_0(z) = F_0(1/z)$ is a polynomial in z of order n and $g_0(z) = F_0(1/z)$ is bounded when $|z| > r$ for some $r > 0$.
- If f has an essential singularity at infinity, then its behavior at infinity can be similarly characterized by Theorem 3.10. That is, $f(D_r(0))$ is dense in $\mathbb{C}$ for some $r > 0$.

Definition (Meromorphic functions on the extended complex plane). We say that a function is meromorphic on the extended complex plane $\mathbb{C} \cup \{\infty\}$ if it is meromorphic on $\mathbb{C}$ and has a pole or removable singularity (i.e., holomorphic) at infinity.

THEOREM 3.11. *A function is meromorphic on the extended complex plane iff f is a rational function of two polynomials.*

PROOF. Sufficiency has been shown above. We only prove the necessity. Let f be meromorphic on $\mathbb{C} \cup \{\infty\}$. In particular, f is holomorphic in $\mathbb{C} \setminus \overline{D}_R(0)$ for some $R > 0$. Therefore, the set of poles $\Omega_1 \subset \overline{D}_R(0)$ is finite since it does not have any limit points. Denote $\Omega_1 = \{z_1, ..., z_n\}$. For each $k = 1, ..., n$, by Corollary 3.3,

$$f(z) = f_k(z) + g_k(z),$$

in which f_k is the principle part and g_k is holomorphic in a neighborhood of z_k. Here, f_k is a polynomial of $1/(z - z_k)$ so defines a meromorphic function on $\mathbb{C}$. In particular, f_k is bounded in $\mathbb{C} \setminus D_{2R}(0)$ since $z_k \in \overline{D}_R(0)$.

Because f has a pole or is holomorphic at infinity, $F(z) = f(1/z)$ has a pole or is holomorphic at 0. Thus, we have that

$$F(z) = F_0(z) + G_0(z),$$

in which F_0 is the principle part and G_0 is holomorphic in $\overline{D}_r(0)$ for some $r > 0$. (In the case when F is holomorphic at 0, we can set $F_0 = 0$.) Here, F_0 is a polynomial of $1/z$ and G_0 is holomorphic on $\overline{D}_r(0)$ so is bounded. Hence,

$$f(z) = F\left(\frac{1}{z}\right) = F_0\left(\frac{1}{z}\right) + G_0\left(\frac{1}{z}\right) = f_\infty(z) + g_\infty(z),$$

in which $f_\infty(z) = F_0(1/z)$ is a polynomial of z and $g_\infty(z) = G_0(1/z)$ is bounded in $\mathbb{C} \setminus \overline{D}_{1/r}(0)$.

Now set

$$H(z) = f(z) - f_\infty(z) - \sum_{k=1}^{n} f_k(z).$$

Then H has removable singularities at z_k for all $k = 1, ..., n$ since the principle parts are removed. Hence, H defines a homomorphic function in $\overline{D}_{1/r}(0)$ so is bounded there. (Notice that f_∞ is a polynomial of z.)

We also know that f_k is holomorphic and bounded in $\mathbb{C} \setminus \overline{D}_{1/r}(0)$ for all $k = 1, ..., n$ since it is a polynomial of $1/z$. In addition, $f - f_\infty = g_\infty$ is holomorphic and bounded in $\mathbb{C} \setminus \overline{D}_{1/r}(0)$.

Putting together, H is entire and bounded. By Liouville's theorem, $H = c$ for some constant $c \in \mathbb{C}$. It then follows that

$$f(z) = f_\infty(z) - \sum_{k=1}^{n} f_k(z) + c$$

is rational. ∎

$\boxed{\textbf{Homework Assignment}}$.

3-6. Construct examples of removable singularity, pole of order n, and essential singularity, at infinity.

3.5. The argument principle and applications

We have seen that $\log f(z)$ in general does not define a single-valued function. Indeed,

$$\log f(z) = \log|f(z)| + i\arg f(z),$$

in which $\arg f(z)$ is unique only up to multiples of 2π. However, it also means that

$$\frac{d}{dz}\log f(z) = \frac{f'(z)}{f(z)}$$

is unique so defines a single-valued function. (This is sometimes called logarithmic differentiation.)

Suppose that f is meromorphic. Then f'/f is holomorphic except at the zeros and poles of f. We investigate the behavior of f'/f around its zeros and poles.

- Around a zero z_0 of order n, $f(z) = (z-z_0)^n g(z)$, in which $g(z)$ is holomorphic does not vanish in $D_r(z_0)$ for some $r > 0$. Therefore,
$$\frac{f'(z)}{f(z)} = \frac{n(z-z_0)^{n-1}g(z) + (z-z_0)^n g'(z)}{(z-z_0)^n g(z)} = \frac{n}{z-z_0} + \frac{g'(z)}{g(z)},$$
which means that f'/f has a simple pole with residue n at z_0.
- Around a pole z_0 of order n, $f(z) = (z-z_0)^{-n} h(z)$, in which $h(z)$ is holomorphic does not vanish in $D_r(z_0)$ for some $r > 0$. Therefore,
$$\frac{f'(z)}{f(z)} = \frac{-n(z-z_0)^{-n-1}h(z) + (z-z_0)^{-n} h'(z)}{(z-z_0)^{-n} h(z)} = \frac{-n}{z-z_0} + \frac{h'(z)}{h(z)},$$
which means that f'/f has a simple pole with residue $-n$ at z_0.

Applying the residue formula in Theorem 3.5 to f'/f, we have the following theorem.

THEOREM 3.12 (Argument principle). *Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ be meromorphic. Denote $N_z(\Omega_1)$ and $N_p(\Omega_1)$ the number of zeros and poles (counting multiplicity) in Ω_1, respectively, for $\Omega_1 \subset \Omega$. Then*

$$\frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f'(z)}{f(z)}\,dz = N_z(D_r(z_0)) - N_p(D_r(z_0)),$$

for all $\overline{D}_r(z_0) \subset \Omega$ such that f has no poles or zeros on $C_r(z_0)$.

In the following, we discuss several applications of the argument principle.

THEOREM 3.13 (Rouché). *Let $\Omega \subset \mathbb{C}$ be open and $f, g : \Omega \to \mathbb{C}$ be holomorphic. Suppose that $\overline{D}_r(z_0) \subset \Omega$ and $|f(z)| > |g(z)|$ for all $z \in C_r(z_0)$. Then f and $f+g$ have the same number of zeros (counting multiplicity) in $D_r(z_0)$.*

PROOF. Set $f_t(z) = f(z) + tg(z)$ for $t \in [0,1]$. So $f(z) = f_0(z)$ and $f_1(z) = f(z) + g(z)$. Denote the number of zeros (counting multiplicity) of f_t in $D_r(z_0)$ by n_t. Then n_t is an integer for all $t \in [0,1]$. It suffices to show that $n_0 = n_1$.

Since $|f(z)| > |g(z)|$ on $C_r(z_0)$,

$$|f_t(z)| = |f(z) + tg(z)| \geq |f(z)| - t|g(z)| \geq |f(z)| - |g(z)| > 0,$$

by the triangle inequality. That is, f_t never vanishes on $C_r(z_0)$. By the argument principle in Theorem 3.12,

$$n_t = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f_t'(z)}{f_t(z)}\,dz = \frac{1}{2\pi i}\int_{C_r(z_0)} \frac{f'(z) + tg'(z)}{f(z) + tg(z)}\,dz.$$

Notice that the integrating function in the above integral is continuous in $(t, z) \in [0, 1] \times C_r(z_0)$ and hence uniformly continuous. This shows that n_t is a continuous function for $t \in [0, 1]$. It then follows that n_t is constant on $[0, 1]$ since it is an integer-valued function. Indeed, if n_t is not constant, then it takes on at least two distinct integer values. By the intermediate value theorem, it must take non-integer values between these two integers, which is not possible. $\square$

THEOREM 3.14 (Open mapping theorem). *Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ be holomorphic and non-constant. Then the image $f(\Omega)$ is open.*

Notice that f is holomorphic so is continuous. Then the pre-image of any open set is open. The open mapping theorem asserts that the image of open sets are also open.

PROOF. The proof is an application of the above Rouché theorem. To show that $f(\Omega)$ is open, pick $w_0 \in f(\Omega)$ so there is $z_0 \in \Omega$ such that $w_0 = f(z_0)$. We need to prove that there is $r > 0$ such that $D_r(w_0) \subset f(\Omega)$, that is, each point $w \in D_r(w_0)$ belongs to the image of f.

Set $g(z) = f(z) - w$ (then $g(z) = 0$ iff $f(z) = w$.) Write

$$g(z) = (f(z) - w_0) + (w_0 - w) = F(z) + G(z),$$

in which $F(z) = f(z) - w_0$ and $G(z) = w_0 - w$.

Since F is holomorphic, the zeros are isolated. Notice that F has a zero at z_0 since $F(z_0) = f(z_0) - w_0 = 0$. There is $r > 0$ such that $F(z) \neq 0$ for all $z \in \overline{D}_r(z_0) \setminus \{z_0\}$. Set $\varepsilon > 0$ such that $|f(z) - w_0| \geq \varepsilon$ for all $z \in C_r(z_0)$ (which is compact).

Now if $|w - w_0| < \varepsilon$, then $|F(z)| = |f(z) - w_0| > \varepsilon > |w - w_0| = |G(z)|$. By Rouché theorem, f and $g = F + G$ have the same number of zeros in $D_r(z_0)$, that is, there is $z \in D_r(z_0)$ such that $g(z) = f(z) - w = 0$, i.e., w belongs to the image of f. $\square$

An immediate consequence of the open mapping theorem is that a non-constant holomorphic function can not attain its maximum.

THEOREM 3.15 (Maximum principle). *Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ be holomorphic and non-constant. Then $|f|$ can not attain a maximum in Ω.*

PROOF. We argue by contradiction. Suppose that there is $z_0 \in \Omega$ such that $|f(z)| \leq |f(z_0)|$ for all $z \in \Omega$. Since Ω is open, there is $r > 0$ such that $D_r(z_0) \subset \Omega$. Because f is holomorphic, $f(D_r(z_0)) \subset \Omega$ is open and contains $f(z_0)$. Then there is $z \in D_r(z_0)$ such that $|w| > |f(z_0)|$, which is not possible. $\square$

Now suppose that Ω is open and bounded and $f : \Omega \to \mathbb{C}$ is holomorphic and non-constant. Then $\overline{\Omega}$ is compact so $|f|$ attains its maximum in $\overline{\Omega}$ but not in Ω. Therefore,

COROLLARY 3.16. *Let $\Omega \subset \mathbb{C}$ be open and bounded and $f : \Omega \to \mathbb{C}$ be holomorphic and continuous on $\overline{\Omega}$. Then*

$$\sup_{z \in \Omega} |f(z)| \leq \sup_{\partial\Omega} |f(z)| \quad \textit{and} \quad \sup_{z \in \overline{\Omega}} |f(z)| = \sup_{\partial\Omega} |f(z)|.$$

Homework Assignment.

3-7. Suppose that $\Omega \subset \mathbb{C}$ and $\overline{D}_1(0) \subset \Omega$. Let $f, g : \Omega \to \mathbb{C}$ be holomorphic such that f has a simple zero at 0 and does not vanish anywhere else in $\overline{D}_1(0)$. Write $f_\varepsilon(z) = f(z) + \varepsilon g(z)$. Prove that if ε is sufficiently small, then

(a). $f_\varepsilon(z)$ has a unique zero in $\overline{D}_1(0)$,

(b). if z_ε is this zero, then the mapping $\varepsilon \to z_\varepsilon$ is continuous.

3-8. Let $\Omega \subset \mathbb{R}^2$ be open and $f : \Omega \to \mathbb{R}$ be continuous and non-constant. Then is the image $f(\Omega)$ open? Prove your assertion.

3-9. Suppose that $\Omega \subset \mathbb{C}$ and $\overline{D}_1(0) \subset \Omega$. Let $f : \Omega \to \mathbb{C}$ be holomorphic and non-constant.
(a). Prove that if $|f(z)| = 1$ when $|z| = 1$, then $f(\Omega) \supset D_1(0)$.
(b). Prove that if $|f(z)| \geq 1$ when $|z| = 1$ and $|f(z_0)| < 1$ for some $z_0 \in D_1(0)$, then $f(\Omega) \supset D_1(0)$.

3-10. Let $\Omega \subset \mathbb{C}$ be open and $f : \Omega \to \mathbb{C}$ be holomorphic on Ω and continuous on $\overline{\Omega}$. Then is
$$\sup_{z\in\Omega} |f(z)| \leq \sup_{\partial\Omega} |f(z)|?$$
Prove your assertion.

3.6. The complex logarithm

In this section, we define the logarithm function of complex numbers. Since the inverse of $e^z = w$ does not define a single-valued function on $\mathbb{C} \setminus \{0\}$, we instead restrict the domain so that it becomes single-valued. For example, let
$$\Omega = \{z = re^{i\theta} : r > 0 \text{ and } \phi \in (-\pi, \pi)\}.$$
That is, Ω is the complex plane with the non-positive real axis removed and the argument of the complex numbers in Ω is set to be in $(-\pi, \pi)$. Then

Theorem 3.17 (Complex logarithm in the principal branch). *The function*
$$F_\Omega(z) = \log r + i\theta \quad \textit{for } z = re^{i\theta} \in \Omega$$
satisfies that
(i). $F_\Omega(x) = \log x$ *for all* $x \in (0, \infty)$,
(ii). $e^{F_\Omega(z)} = z$ *for all* $z \in \Omega$,
(iii). $F'_\Omega(z)$ *is holomorphic in* Ω, *in particular,* $f'_\Omega(z) = 1/z$ *for all* $z \in \Omega$.
Therefore, F_Ω *is the extension of the standard logarithm function of the positive real numbers. We denote* $F_\Omega = \log z$ *the principle branch of the logarithm function of complex numbers.*

Remark. One can define other branches of the logarithm function of complex numbers. In fact, one can similarly define such function on the complex plane with any ray (and the origin) removed. However, to be able to complex the newly defined function with the standard logarithm function of the positive real numbers, we need to keep the positive real axis in the domain (or at least some part of it).

Proof. For notational simplicity, we drop the subscription in F_Ω. (i) and (ii) are obvious. For (iii), we construct a primitive of $1/z$ in Ω and show that it is $F(z)$. For $z = re^{i\theta} \in \Omega$, let γ_z be the curve that consists of the line segment from 1 to r and the arg from r to z. Then $\int_{\gamma_z} 1/z$ defines a primitive of $1/z$ on Ω (which is simply connected). However,
$$\int_{\gamma_z} \frac{1}{z}\, dz = \int_1^r \frac{1}{x}\, dx + \int_0^\theta \frac{ire^{it}}{re^{it}}\, dt = \log r + i\theta = F(z).$$
Similarly as in the proof of Theorem 2.10, F is the primitive of $1/z$ in Ω and $F'(z) = 1/z$. $\square$

CHAPTER 4

The gamma and zeta functions

Beethoven's Großs Fuge is an absolutely contemporary piece of music
that will be contemporary forever.
– Igor Stravinsky

In this chapter, we study the two most important non-elementary functions,

- the gamma function[i]
$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt,$$
- the zeta function
$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

We are mainly concerned with the following questions.

(1). Where can the function be defined? Since the functions are defined by an infinite integral or summation, it can be defined for $s \in \mathbb{R}$ when the integral or summation is convergent.

(2). Can the function also be defined for certain complex numbers $s \in \mathbb{C}$? This can be done by observing convergence.

(3). Can the function be extended to the complex plane? This follows from some functional identities, which allow the function to be extended to the complex plane.

(4). Is the function meromorphic on the complex plane? If so, where are the zeros and poles?

(5). What do the zeros and poles tell us about other problems? In the case of the zeta function, the location of the zeros are related to the distribution of primes. In particular, the Riemann Hypothesis claims that all (non-trivial) zeros lie on the line $\{s \in \mathbb{C} : \Re(s) = 1/2\}$.

4.1. The gamma function

Let $s > 0$. Then the integrand in the gamma function
$$e^{-t} t^{s-1} \le \begin{cases} t^{s-1} & \text{if } 0 < t \le 1, \\ e^{-\frac{t}{2}} & \text{if } t \gg 1. \end{cases}$$

Furthermore, t^{s-1} is integrable on $[0, 1]$ while $e^{-t/2}$ is integrable on $[1, \infty)$. Therefore, the gamma function $\Gamma(s)$ is well defined for $s > 1$. In fact, a similar argument implies that it is also defined for all complex numbers s such that $\Re(s) > 0$.

Proposition 4.1. *The gamma function*
$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1}\, dt$$
defines a holomorphic function on the half plane $\{s \in \mathbb{C} : \Re(s) > 0\}$.

[i]A history note: Gauss and Riemann originally used the notation $\Pi(s)$ as $\Gamma(s+1)$. Legendre latter introduced the notation Γ and since then widely adopted. Both notations have their superficial advantages: $\Pi(n) = n!$ if $n \in \mathbb{N}$ while $\Gamma(n+1) = n!$; the first pole of $\Gamma(s)$ is at 0 while the one of Π is at -1.

PROOF. For $\delta, M > 0$, denote $S_{\delta,M}$ the strip $\{s \in \mathbb{C} : \delta < \Re(s) < M\}$. The proposition is proved if we show that Γ defines a holomorphic function on $S_{\delta,M}$ for all $0 < \delta < 1 < M < \infty$. Write

$$f_\varepsilon(s) = \int_\varepsilon^{1/\varepsilon} e^{-t} t^{s-1}\, dt.$$

Then f_ε is holomorphic on $S_{\delta,M}$. (The integrand is holomorphic in s for each $t \in [\varepsilon, 1/\varepsilon]$ and is continuous in $(s,t) \in S_{\delta,M} \times [\varepsilon, 1/\varepsilon]$.)

Next we show that $f_\varepsilon(s) \to \Gamma(s)$ uniformly on $S_{\delta,M}$. Let $0 < \varepsilon < 1$. Notice that

$$|f_\varepsilon(s) - \Gamma(s)| = \left| \int_0^\varepsilon e^{-t} t^{s-1}\, dt + \int_{1/\varepsilon}^\infty e^{-t} t^{s-1}\, dt \right| \le \int_0^\varepsilon e^{-t} t^{\sigma-1}\, dt + \int_{1/\varepsilon}^\infty e^{-t} t^{\sigma-1}\, dt,$$

in which $\sigma = \Re(s) \in (\delta, M)$. Now

$$\int_0^\varepsilon e^{-t} t^{\sigma-1}\, dt \le \int_0^\varepsilon t^{\sigma-1}\, dt = \frac{\varepsilon^\sigma}{\sigma} \le \frac{\varepsilon^\delta}{\delta} \to 0$$

uniformly as $\varepsilon \to 0$. Moreover,

$$\int_{1/\varepsilon}^\infty e^{-t} t^{\sigma-1}\, dt \le \int_{1/\varepsilon}^\infty e^{-t} t^{M-1}\, dt \le C \int_{1/\varepsilon}^\infty e^{-t/2}\, dt = 2Ce^{-\frac{1}{2\varepsilon}} \to 0$$

uniformly as $\varepsilon \to 0$. The proposition is therefore complete. $\square$

LEMMA 4.2. *Let $\Re(s) > 0$. Then*

$$\Gamma(s+1) = s\Gamma(s).$$

PROOF. According to integration by parts,

$$\begin{aligned} \Gamma(s+1) &= \int_0^\infty e^{-t} t^s\, dt \\ &= -\int_0^\infty t^s\, de^{-t} \\ &= -t^s e^{-t}|_0^\infty + \int_0^\infty e^{-t}\, d(t^s) \\ &= s \int_0^\infty e^{-t} t^{s-1}\, dt \\ &= s\Gamma(s). \end{aligned}$$

$\square$

Remark (Gamma function and factorial). We compute that

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = -e^{-t}|_0^\infty = 1.$$

Then the functional identity in the above lemma asserts that

$$\Gamma(n+1) = n\Gamma(n-1) = \cdots = n(n-1)\cdots 1 = n!.$$

Furthermore, the functional identity allows one to define $\Gamma(s) = \Gamma(s+1)/s$ when $\Re(s) > -1$ (so $\Re(s+1) > 0$ and the right-hand side is holomorphic.) We can then recursively define the (unique) analytic extension of Γ to the whole complex plane.

THEOREM 4.3. *The gamma function* Γ *on* $\{s \in \mathbb{C} : \Re(s) > 0\}$ *extends analytically to a meromorphic function, still denoted by* Γ, *on* $\mathbb{C}$. *Moreover,* Γ *has simple poles at non-positive integers* $s = -n$, $n \geq 0$, *with residue* $\text{res}_{-n}\Gamma = (-1)^n/n!$.

PROOF. We first extend Γ to a meromorphic function $F_1(s) = \Gamma(s+1)/s$ on $\{s \in \mathbb{C} : \Re(s) > -1\}$. Since Γ is holomorphic, F_1 is meromorphic with a simple pole at $s = 0$ and $\text{res}_0 F_1 = \Gamma(1) = 1$.

We recursively extend F_{n-1} to a meromorphic function

$$F_n(s) = \frac{\Gamma(s+n)}{(s+n-1)\cdots(s+1)s}$$

on $\{s \in \mathbb{C} : \Re(s) > -n\}$. Then F_n has simple poles at $k = 0, -1, ..., -(n-1)$ with residue

$$\begin{aligned} \text{res}_{-k}F_n &= \frac{\Gamma(-k+n)}{(-k+n-1)\cdots(-k+k+1)(-k+k-1)\cdots(-k+1)(-k)} \\ &= \frac{(-k+n-1)!}{(-k+n-1)!\cdot(-1)^k k!} \\ &= \frac{(-1)^k}{k!}. \end{aligned}$$

Notice that $F_n(s) = \Gamma(s)$ on $\{s \in \mathbb{C} : \Re(s) > 0\}$. Hence, $F_n(s) = F_m(s)$ for $\Re(s) > -\min\{n, m\}$ by the uniqueness of analytic continuation in Theorem 2.25. Now for any $s \in \mathbb{C}$, we extend Γ by defining $\Gamma(s) = F_n(s)$ for any n such that $\Re(s) > -n$. $\square$

We next prove another important functional identity, which establishes the certain "symmetry" of $\Gamma(s)$ with respect to the vertical line $\{s \in \mathbb{C} : \Re(s) = 1/2\}$.

THEOREM 4.4. *For all* $s \in \mathbb{C}$,

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}.$$

Remark.

- Notice that $\Gamma(s)$ has simple poles at $s = 0, -1, -2, ...$ while $\Gamma(1-s)$ has simple poles at $s = 1, 2, ...$ They agree with the simples poles of $1/\sin(\pi s)$ at all integers.
- Letting $s = 1/2$ gives $\Gamma(1/2) = \sqrt{\pi}$.
- Note that

$$\Gamma(s) = \frac{\pi}{\sin(\pi s)\Gamma(1-s)},$$

in which $\Gamma(1-s)$ has simple poles at positive integers and $\sin(\pi s)$ has simple zeros at all integers. Therefore, $\Gamma(s)$ has no zeros so $1/\Gamma(s)$ is an entire function.

On the other hand,

$$\frac{1}{\Gamma(s)} = \frac{\sin(\pi s)\Gamma(1-s)}{\pi}$$

has simple zeros at non-positive integers $0, -1, -2, ...$

PROOF. It suffices to prove the identity for $0 < s < 1$ and the other cases follow from the uniqueness of analytic continuation. First note that

$$\Gamma(1-s) = \int_0^\infty e^{-u}u^{-s}\,du = t\int_0^\infty e^{-tv}(tv)^{-s}\,dv,$$

for any $t > 0$. Then

$$\begin{aligned}
\Gamma(s)\Gamma(1-s) &= \int_0^\infty e^{-t}t^{s-1}\left(t\int_0^\infty e^{-tv}(tv)^{-s}\,dv\right)dt \\
&= \int_0^\infty \left(\int_0^\infty e^{-t(v+1)}\,dt\right)v^{-s}\,dv \\
&= \int_0^\infty \frac{v^{-s}}{1+v}\,dv \\
&= \int_{-\infty}^\infty \frac{e^{(1-s)t}}{1+e^t}\,dt \\
&= \frac{\pi}{\sin(\pi(1-s))} \\
&= \frac{\pi}{\sin(\pi s)},
\end{aligned}$$

where we use Theorem 3.7 that

$$\int_{-\infty}^\infty \frac{e^{ax}}{1+e^x}\,dx = \frac{\pi}{\sin(\pi a)}.$$

□

$\boxed{\textbf{Homework Assignment}}$.

4-1. Prove that for $u \in \mathbb{R}$,

$$\left|\Gamma\left(\frac{1}{2}+iu\right)\right| = \sqrt{\frac{\pi}{\cosh(\pi u)}}.$$

4-2. Let $\alpha, \beta \in \mathbb{C}$ such that $\Re(\alpha), \Re(\beta) > 0$. Define the beta function

$$B(\alpha,\beta) = \int_0^1 (1-t)^{\alpha-1}t^{\beta-1}\,dt.$$

(a). Prove that

$$B(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}.$$

(b). Prove that

$$B(\alpha,\beta) = \int_0^\infty \frac{u^{\alpha-1}}{(1+u)^{\alpha+\beta}}\,du.$$

4-3. Define the Mellin transform of $f(t)$ as

$$\mathcal{M}(f)(z) = \int_0^\infty f(t)t^{z-1}\,dt.$$

(a). Observe that

$$\mathcal{M}\left(e^{-t}\right)(z) = \Gamma(z).$$

(b). Prove that if $0 < \Re(z) < 1$, then

$$\mathcal{M}(\cos)(z) = \Gamma(z)\cos\left(\frac{\pi}{2}z\right).$$

4.2. The zeta function

PROPOSITION 4.5. *The zeta function*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

defines a holomorphic function on $\{s \in \mathbb{C} : \Re(s) > 1\}$.

PROOF. It suffices to prove that ζ is holomorphic on $\{s \in \mathbb{C} : \Re(s) \geq \delta\}$ for each fixed $\delta > 1$. Notice that for all $s \in \mathbb{C}$ such that $\Re(s) \geq \delta$,

$$\left|\frac{1}{n^s}\right| = \frac{1}{n^{\Re(s)}} \leq \frac{1}{n^\delta}.$$

Since $\sum \frac{1}{n^\delta}$ is convergent, $\sum \frac{1}{n^s}$ converges uniformly on $\{s \in \mathbb{C} : \Re(s) \geq \delta\}$. Hence, $\zeta(s)$ is holomorphic on $\{s \in \mathbb{C} : \Re(s) \geq \delta\}$. $\square$

Next we derive the functional identity which is used for the analytic continuation of the zeta function. We need some preparation.

LEMMA 4.6 (Possion summation formula for the Gaussian function). *Let* $a > 0$. *Then*

$$\sum_{n=-\infty}^{\infty} e^{-a\pi n^2} = \frac{1}{\sqrt{a}} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi n^2}{a}}.$$

Remark (Schwartz functions and Possion summation formula). Notice that both sides of the summation above are convergent. This is because the Gaussian functions decay fast enough at infinity, in particular, $e^{-a\pi x^2} = O(|x|^{-N})$ as $|x| \to \infty$ for all $N \in \mathbb{N}$. Choosing $N > 1$ implies that the summation converges.

We say that a function $f \in C^\infty(\mathbb{R})$ is Schwartz if $f(x) = O(|x|^{-N})$ as $|x| \to \infty$ for all $N \in \mathbb{N}$, i.e., f decays at infinity faster than any polynomial rate. For example, smooth functions with compact support and Gaussian functions are Schwartz.

Poisson summation formula can be generalized to all Schwartz functions. To this end, recall the Fourier transform

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi}\, dx.$$

Then Poisson summation formula yields that

$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{n=-\infty}^{\infty} \hat{f}(n).$$

If f is Schwartz, then $\hat{f}$ is also Schwartz so both sides of the summation converge. In particular, if $f(x) = e^{-a\pi n^2}$, then $\hat{f}(\xi) = a^{-\frac{1}{2}} e^{-\frac{\pi n^2}{a}}$ is also Schwartz and the above lemma reads

$$\sum_{n=-\infty}^{\infty} e^{-a\pi n^2} = \frac{1}{\sqrt{a}} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi n^2}{a}}.$$

Poisson summation formula can be proved via the distribution theory as follows.

$$\sum_{k=-\infty}^{\infty} \hat{f}(k) \quad = \quad \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) e^{-2\pi i k x}\, dx$$

$$\begin{aligned}
&= \int_{-\infty}^{\infty} f(x)\left(\sum_{k=-\infty}^{\infty} e^{-2\pi ikx}\right) dx \\
&= \int_{-\infty}^{\infty} f(x)\left(\sum_{n=-\infty}^{\infty} \delta(x-n)\right) dx \\
&= \sum_{n=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x)\delta(x-n)\, dx \\
&= \sum_{n=-\infty}^{\infty} f(n),
\end{aligned}$$

in which

$$\sum_{k=-\infty}^{\infty} e^{-2\pi ikx} = \sum_{n=-\infty}^{\infty} \delta(x-n),$$

can be understood intuitively that the left-hand-side is infinity if x is integer and is zero otherwise.

We next prove Poisson summation formula for the Gaussian functions $G_a(x) = e^{-a\pi x^2}$, using contour shifting argument. In fact, moderate modification of the proof then applies to all Schwartz functions.

PROOF. Let γ_R be the rectangular contour with vertices $R-i, R+i, -R+i, -R-i$ and positive orientation. The function $f(z) = G_a(z)/(e^{2\pi iz}-1) = e^{-a\pi z^2}/(e^{2\pi iz}-1)$ is holomorphic in an open set that contains γ_R and its interior except at simple poles $n \in \mathbb{Z}$ and $|n| < R$. (We suppose that $R > 0$ is not an integer. Later on we actually set $R = N + 1/2$ for $N \in \mathbb{N}$.) In addition,

$$\mathrm{res}_n f = \lim_{z\to\pi i}(z-n)f(z) = G_a(n)\cdot \lim_{z\to n}\frac{z-n}{e^{2\pi iz}-1} = \frac{G_a(n)}{2\pi i}.$$

Hence, by the residue formula,

$$\int_{\gamma_R} f = 2\pi i \cdot \sum_{|n|<R} \mathrm{res}_n f = \sum_{|n|<R} G_a(n).$$

On the other hand,

$$\int_{\gamma_R} f(z)\, dz = \int_{-R}^{R} f(x-i)\, dx + \int_{-1}^{1} f(R+iy)i\, dy + \int_{R}^{-R} f(x+i)\, dx + \int_{1}^{-1} f(-R+iy)i\, dy.$$

Next we estimate the four integrals.

(1). Notice that $|e^{2\pi i(x-i)}| = e^{2\pi} > 1$ for $x \in \mathbb{R}$. Then

$$\frac{1}{e^{2\pi i(x-i)}-1} = e^{2\pi i(x-i)}\cdot\frac{1}{1-e^{-2\pi i(x-i)}} = e^{2\pi i(x-i)}\cdot\sum_{n=0}^{\infty} e^{-2\pi i(x-i)n} = \sum_{n=1}^{\infty} e^{-2\pi i(x-i)n}.$$

Hence,

$$\begin{aligned}
\int_{-R}^{R} f(x-i)\, dx &= \int_{-R}^{R} \frac{e^{-a\pi(x-i)^2}}{e^{2\pi i(x-i)}-1}\, dx \\
&= \int_{-R}^{R} e^{-a\pi(x-i)^2}\cdot\sum_{n=1}^{\infty} e^{-2\pi i(x-i)n}\, dx
\end{aligned}$$

$$= \sum_{n=1}^{\infty} \int_{-R}^{R} e^{-a\pi(x-i)^2} e^{-2\pi i(x-i)n}\, dx.$$

Now we use contour shifting again since the integrating function $g_n(z) = e^{-a\pi z^2} e^{-2\pi i z n}$ is holomorphic.

$$\begin{aligned} & \int_{-R}^{R} e^{-\pi u(x-i)^2} e^{-2\pi i(x-i)n}\, dx \\ = & \int_{-R}^{R} g_n(x-i)\, dx \\ = & \int_{-1}^{0} g_n(-R+iy) i\, dy + \int_{-R}^{R} g_n(x)\, dx + \int_{0}^{-1} g_n(R+iy) i\, dy, \end{aligned}$$

in which the norm of the first and third integrals are bounded by

$$\max_{y\in[-1,0]} |g_n(\pm R + iy)| \leq C e^{-a\pi(R^2+n)}.$$

Therefore,

$$\begin{aligned} \int_{-R}^{R} f(x-i)\, dx &= \sum_{n=1}^{\infty} \int_{-R}^{R} e^{-a\pi(x-i)^2} e^{-2\pi i(x-i)n}\, dx \\ &= \sum_{n=1}^{\infty} \int_{-1}^{0} g_n(-R+iy) i\, dy + \sum_{n=1}^{\infty} \int_{-R}^{R} g_n(x)\, dx + \sum_{n=1}^{\infty} \int_{0}^{-1} g_n(R+iy) i\, dy \\ &\to \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} e^{-a\pi x^2} e^{-2\pi i x n}\, dx \\ &= \sum_{n=1}^{\infty} \hat{G}_a(n). \end{aligned}$$

Here, we use the fact that $\sum_n e^{-a\pi(R^2+n)} \to 0$ as $R \to \infty$.

(2). Set $R = N + 1/2$ for $N \in \mathbb{N}$. Then

$$\int_{-1}^{1} f(R+iy) i\, dy = i \int_{-1}^{1} \frac{e^{-a\pi\left(N+\frac{1}{2}+iy\right)^2}}{e^{2\pi i\left(N+\frac{1}{2}+iy\right)} - 1}\, dy = i \int_{-1}^{1} \frac{e^{-a\pi\left(N+\frac{1}{2}+iy\right)^2}}{-e^{-2\pi y} - 1}\, dy \to 0 \quad \text{as } N \to \infty.$$

(3). Notice that $|e^{2\pi i(x+i)}| = e^{-2\pi} < 1$ for $x \in \mathbb{R}$. Then

$$\frac{1}{e^{2\pi i(x+i)} - 1} = -\sum_{n=0}^{\infty} e^{2\pi i(x+i)n}.$$

Hence, similarly as in (1),

$$\begin{aligned} \int_{R}^{-R} f(x+i)\, dx &= -\int_{-R}^{R} \frac{e^{-a\pi(x+i)^2}}{e^{2\pi i(x+i)} - 1}\, dx \\ &= \int_{-R}^{R} e^{-a\pi(x+i)^2} \cdot \sum_{n=0}^{\infty} e^{2\pi i(x+i)n}\, dx \\ &\to \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} e^{-a\pi x^2} e^{2\pi i x n}\, dx \end{aligned}$$

$$= \sum_{n=0}^{\infty} \hat{G}_a(-n).$$

(4). Similarly as in (2),

$$\int_1^{-1} f(-R+iy)i\,dy \to 0 \quad \text{as } R = N + \frac{1}{2} \to \infty.$$

Therefore, as $R = N + \frac{1}{2} \to \infty$,

$$\sum_{n=-\infty}^{\infty} G_a(n) = \sum_{n=-\infty}^{\infty} \hat{G}_a(n).$$

$\square$

For notational simplicity, we denote

$$\psi(t) = \sum_{n=1}^{\infty} e^{-\pi n^2 t} = \frac{1}{2}\left(\sum_{n=-\infty}^{\infty} e^{-\pi n^2 t} - 1\right)$$

for $t > 0$. (Notice that $\psi(t) = O(t^{-N})$ for any $N \in \mathbb{N}$ as $t \to \infty$.)

Then Poisson summation formula reads

$$\psi(t) = \frac{1}{2}\left(\frac{1}{\sqrt{t}} \sum_{n=-\infty}^{\infty} e^{-\frac{\pi n^2}{t}} - 1\right) = \frac{1}{2}\left[\frac{1}{\sqrt{t}}\left(2\psi\left(t^{-1}\right)+1\right) - 1\right],$$

which is

$$\psi(t) = \frac{1}{\sqrt{t}}\psi\left(t^{-1}\right) + \frac{1}{2\sqrt{t}} - \frac{1}{2}. \tag{4.1}$$

We furthermore need a lemma.

LEMMA 4.7. *Let* $\Re(s) > 1$. *Then*

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \int_0^{\infty} t^{\frac{s}{2}-1}\psi(t)\,dt.$$

PROOF. By a change of the variable $t \to \pi n^2 t$ for $n \neq 0$ in the gamma function, we have that

$$\begin{aligned}
\Gamma\left(\frac{s}{2}\right) &= \int_0^{\infty} e^{-t} t^{\frac{s}{2}-1}\,dt \\
&= \pi n^2 \int_0^{\infty} e^{-\pi n^2 t}\left(\pi n^2 t\right)^{\frac{s}{2}-1}\,dt \\
&= \pi^{\frac{s}{2}} n^s \int_0^{\infty} e^{-\pi n^2 t} t^{\frac{s}{2}-1}\,dt.
\end{aligned}$$

Then

$$\frac{1}{n^s} = \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \int_0^{\infty} e^{-\pi n^2 t} t^{\frac{s}{2}-1}\,dt.$$

Summing over n,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{n=1}^{\infty} \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \int_0^{\infty} e^{-\pi n^2 t} t^{\frac{s}{2}-1}\,dt$$

$$
\begin{aligned}
&= \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \int_0^\infty \left(\sum_{n=1}^\infty e^{-\pi n^2 t} \right) t^{\frac{s}{2}-1}\, dt \\
&= \frac{\pi^{\frac{s}{2}}}{\Gamma\left(\frac{s}{2}\right)} \int_0^\infty t^{\frac{s}{2}-1} \psi(t)\, dt.
\end{aligned}
$$

□

We are now in the position to establish the functional identity of the zeta function. Again for notational simplicity, write

$$\xi(s) = \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s).$$

Then ξ is holomorphic on $\{s \in \mathbb{C} : \Re(s) > 0\}$ and the above lemma reads

$$\xi(s) = \int_0^\infty t^{\frac{s}{2}-1} \psi(t)\, dt.$$

In the next theorem, we prove the analytic extension and the function identity for the xi function ξ, which in turn yield the corresponding results for the zeta function ζ.

Theorem 4.8. *The function ξ is holomorphic on $\{s \in \mathbb{C} : \Re(s) > 1\}$ and has an analytic extension to $\mathbb{C}$ as a meromorphic function with simple poles at $s = 0, 1$. Moreover,*

$$\xi(s) = \xi(1-s) \quad \textit{for all } s \in \mathbb{C}.$$

Proof. Using (4.1) and change of variables, we compute that for $\Re(s) > 1$,

$$
\begin{aligned}
\xi(s) &= \int_0^\infty t^{\frac{s}{2}-1} \psi(t)\, dt \\
&= \int_0^1 t^{\frac{s}{2}-1} \psi(t)\, dt + \int_1^\infty t^{\frac{s}{2}-1} \psi(t)\, dt \\
&= \int_0^1 t^{\frac{s}{2}-1} \left(\frac{1}{\sqrt{t}} \psi\left(t^{-1}\right) + \frac{1}{2\sqrt{t}} - \frac{1}{2} \right) dt + \int_1^\infty t^{\frac{s}{2}-1} \psi(t)\, dt \\
&= \int_1^\infty \left(u^{-\frac{s}{2}-\frac{1}{2}} \psi(u) + \frac{1}{2} u^{-\frac{s}{2}-\frac{1}{2}} - \frac{1}{2} u^{-\frac{s}{2}-1} \right) du + \int_1^\infty t^{\frac{s}{2}-1} \psi(t)\, dt \\
&= \frac{1}{s-1} - \frac{1}{s} + \int_1^\infty \left(t^{-\frac{s}{2}-\frac{1}{2}} + t^{\frac{s}{2}-1} \right) \psi(t)\, dt.
\end{aligned}
$$

Since $\psi(t)$ decays in an exponential rate as $t \to \infty$, the integral above defines an entire function. We then conclude that ξ has an analytic extension to $\mathbb{C}$ with simple poles at $s = 0, 1$. The residues $\mathrm{res}_0 \xi = -1$ and $\mathrm{res}_1 \xi = 1$.

The functional identity $\xi(s) = \xi(1-s)$ is verified directly. It remains valid for all $s \in \mathbb{C}$ by the uniqueness of analytic continuation. □

From the functional identity of ξ, we have the one for the zeta function:

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1-s}{2}} \Gamma\left(\frac{1-s}{2}\right) \zeta(1-s).$$

Now we finally arrive at the analytic extension of the zeta function.

Theorem 4.9. *The zeta function has an analytic extension to $\mathbb{C}$ as a meromorphic function with only singularity of a simple pole at $s = 1$.*

Proof. We have that

$$\zeta(s) = \pi^{\frac{s}{2}} \cdot \frac{\xi(s)}{\Gamma\left(\frac{s}{2}\right)}$$

is meromorphic on $\mathbb{C}$. Notice that ξ has simple poles at $s = 0, 1$ while $\Gamma(s/2)$ has simple poles at $0, -2, -4, ...$ and no zeros. Therefore, the poles at $s = 0$ cancel and ζ has only one singularity of a simple pole at $s = 1$. $\square$

Remark (Riemann Hypothesis). The above proof also shows that $\zeta(s)$ has zeros at negative even integers $-2, -4, -6, ...$ These zeros are usually referred as the "trivial" zeros. The Riemann Hypothesis claims that all other zeros of the zeta function lie on the line $\{s \in \mathbb{C} : \Re(s) = 1/2\}$.

$\boxed{\textbf{Homework Assignment}}$.

4-4. Compute the residue of the zeta function $\zeta(s)$ at $s = 1$.

4.3. Prime number theorem

One of the most important applications (and original motivation) of the Riemann zeta function is to study the distribution of primes among integers. Let

$$\pi(x) = \#\{p \text{ is prime} : p \leq x\}$$

be the prime number counting function. Then

Theorem 4.10 (Prime number theorem). *We have that $\pi(x) \sim x/\log x$ as $x \to \infty$, that is,*

$$\pi(x) = \frac{x}{\log x} + o\left(\frac{x}{\log x}\right) \quad \textit{as } x \to \infty.$$

The prime number theorem is a consequence of the fact that the Riemann zeta function $\zeta(s)$ has no zeros on the line $\{s \in \mathbb{C} : \Re(s) = 1\}$. In this section, we explain the basic idea behind the proof of the theorem. Assuming Riemann Hypothesis, the same proof immediately yields that

$$\pi(x) = \mathrm{Li}(x) + O\left(x^{\frac{1}{2}} \log x\right) \quad \text{as } x \to \infty,$$

in which Li is the offset logarithmic integral or Eulerian logarithmic integral

$$\mathrm{Li}(x) = \int_2^x \frac{1}{\log u}\, du.$$

We now present the steps to prove the prime number theorem via the properties of the Riemann zeta function.

Step 0. An observation of the connection between the primes and the Riemann zeta function.

Lemma 4.11 (Euler product). *Let $\Re(s) > 1$. Then*

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}.$$

Proof. Note that

$$\left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right)\left(1 - \frac{1}{2^s}\right) = \sum_{n=1}^{\infty} \frac{1}{n^s} - \sum_{n=1}^{\infty} \frac{1}{(2n)^s} = \sum_{n=1, 2\nmid n}^{\infty} \frac{1}{n^s}.$$

So

$$\left(\sum_{n=1}^{\infty} \frac{1}{n^s}\right) \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right) = \sum_{n=1, p\nmid n}^{\infty} \frac{1}{n^s} = 1.$$

$\square$

Step 1. Tchebychev's ψ function.

Definition. Define
$$\psi(x) = \sum_{p^m \leq x} \log p.$$
in which the sum is taken over all integers of the form p^m such that $p^m \leq x$.

It is obvious that
$$\psi(x) = \sum_{1 \leq n \leq x} \Lambda(n),$$
in which
$$\Lambda(n) = \begin{cases} \log p & \text{if } n = p^m \text{ for some } m \in \mathbb{N}, \\ 0 & \text{otherwise.} \end{cases}$$
We further write
$$\psi_1(x) = \int_1^x \psi(u)\, du.$$
Then

PROPOSITION 4.12. *The following statements are equivalent as $x \to \infty$.*

(i). $\pi(x) \sim x/\log x$.
(ii). $\psi(x) \sim x$.
(iii). $\psi_1(x) \sim x^2/2$.

We thus reduce the prime number theorem to the estimate of the Tchebychev ψ, ψ_1 functions.
Step 2. Connection between the Tchebychev ψ, ψ_1 functions and the Riemann zeta function.

PROPOSITION 4.13. *Let $c > 1$. Then*
$$\psi_1(x) = \int_{c-i\infty}^{c+i\infty} \frac{x^{s+1}}{s(s+1)} \left(-\frac{\zeta'(s)}{\zeta(s)} \right) ds.$$
Here, the integral is taken on the vertical line $c + iy$, $y \in (-\infty, \infty)$.

Step 3. The zeros of $\zeta(s)$ determine the asymptotic of $\psi_1(x)$.
In the above integral, there is the logarithm derivative $\zeta'(s)/\zeta(s)$. We know that it has a simple pole with residue m if ζ has a zero of order m at s, while it has a simple pole of order $-m$ at s if ζ has a pole of order m at s. Since ζ has only one singularity of a simple pole at $s = 1$, most of the poles of $\zeta'(s)/\zeta(s)$ are from the zeros of ζ.

Furthermore, notice the "dominating term" x^{s+1} in the above integral. To get better estimate of $\psi_1(x)$ (and thus of $\pi(x)$), one would want to integrate on a vertical line with smaller $\Re(s)$ values. To this end, we shift the integral contour from the vertical line $\{s \in \mathbb{C} : \Re(s) = c\}$ with $c > 1$ to another vertical line $\{s \in \mathbb{C} : \Re(s) = c_0\}$ with $c_0 \leq 1$. But according to the residue formula, we have to take consideration of the poles of $\zeta'(s)/\zeta(s)$, which in turn requires to know the location of the zeros of ζ.

Once one establishes that there are no zeros on $\{s \in \mathbb{C} : \Re(s) \geq 1\}$, there is only one simple pole of $\zeta'(s)/\zeta(s)$ at $s = 1$ in this region. It then means that one can shift the contour to $\{s \in \mathbb{C} : \Re(s) = 1\}$. In practice, the new integrating contour needs to avoid $s = 1$. The asymptotic of $\psi_1(x) \sim x^2/2$ follows and then $\pi(x)$.
Step 4. Better asymptotic of $\pi(x)$ follows from the Riemann Hypothesis.

The "proof" in Step 3 indicates that better estimate of $\psi_1(x)$ follows from finding larger "zero-free" region of $\zeta(s)$. That is, suppose that one knows that ζ has no zeros in $\{s \in \mathbb{C} : \Re(s) \geq c_0\}$ for some $c_0 < 1$. Then $\zeta'(s)/\zeta(s)$ has no poles in this region other than the one at $s = 1$. One can shift the integral curve to $\{s \in \mathbb{C} : \Re(s) = c_0\}$ and get better asymptotic of $\psi_1(x)$ and then $\pi(x)$. The Riemann Hypothesis claims that the optimal choice of c_0 is $1/2$, which implies the optimal estimate of the prime number counting function $\pi(x)$.