Week 7 Lecture

Note that for a total of k effects, there are (1/2) k (k – 1) pairwise-contrast effects of the form $\mu_i - \mu_j$. Each of these contrasts may be set up as a null hypothesis. However, not all these contrasts are independent of one another. There are thus only two *independent* contrasts among the three effects. In general, for k effects, only k – 1 independent contrasts may be set up. Since there are k – 1 degrees of freedom for treatments in the analysis of variance, we say that there is one independent contrast for each degree of freedom.

Within the context of a single experiment, there is no entirely satisfactory resolution to the problem raised above. Operationally, however, the experimenter can feel more confidence in his hypothesis decision if the contrasts tested are preplanned (or a priori) and directional alternatives are set up. For example, if the experimenter predicts a specific rank order among the treatment effects and if the analysis of the pair-wise contrasts conforms to this a priori ordering, the experimenter will have few qualms concerning the total Type I risk. However, in many experimental contexts, such a priori decisions cannot be reasonably established. The major exception occurs in experiments that evolve from a background of theoretical reasoning. A well-defined theory (or theories) will yield specific predictions concerning the outcomes of a suitably designed experiment. Unfortunately, at present, a relatively small proportion of educational research is based on theory. Although the experimenter may have "hunches" (i.e., guesses) concerning the outcomes of an experiment, these can seldom provide a satisfactory foundation for the specification of a priori contrasts.

In order to properly test specific contrasts among sample means, a fundamental distinction must be maintained between preplanned (or a priori) comparisons and post-hoc (or a posteriori) comparisons. Preplanned contrasts arise, presumably, from the nature of the theoretical and empirical background out of and around which the research is being conducted. They exist prior to the collection of data; this is a critical point since once the sample data are known, the experiment represents an entirely new probability situation. Post-hoc comparisons by their very nature capitalize on chance occurrences within the experimental situation. A common post-hoc procedure is to contrast the two most extreme, out of k sample means. Which treatment levels result in the most extreme outcomes is not known until the data are collected; it is critical in the process of assigning a probability to the corresponding contrast to take into account the fact that the extremes were located after collection and inspection of the empirical data. There are statistically sound procedures for carrying out postmortem comparisons, and we dealt with that last week in the form of the Scheffé, Tukey, Newman-Keuls, and Dunnett tests. The point being emphasized here is that the researcher must select the analytical procedures partly in terms of the *origins* of the contrasts that are being tested.

Orthogonal Contrasts

For k treatment levels, it is always possible to design a set of k –1 linear contrasts which are mutually orthogonal (i.e., statistically independent). In this section, we develop procedures for testing hypotheses based on such sets of orthogonal contrasts. Often, a single set of orthogonal contrasts will be sufficient to test all a priori hypotheses that are of interest to the researcher. However, it may occur that overlapping, nonorthogonal sets of contrasts are necessary in specific settings. Consider, again, the case of a CRD with three treatment levels. Further, assume that the first treatment group received a control treatment (e.g., a placebo treatment), while treatment groups 2 and 3 were variations of an experimental treatment dimension. A meaningful and orthogonal set of contrasts would be M1 – (1/2)(M2 + M3) and M2 – M3. The first contrast compares the average of the experimental treatment effects with the control effect. Empirically, the question being answered can be stated: Is, on the average, any treatment better than no treatment at all? Whether or not this contrast proves significant, the second contrast compares the two levels of the experimental treatment to see whether differential effects are produced.

Now let us change the meaning of the treatment levels. Suppose the three treatments are differing amounts of treatment along one dimension in which, at best, only a rank ordering of levels is possible[4] (e.g., different intensities of individualized, in-school vocational counseling). It might then be of interest to contrast successive differences of the form M1 – M2, M2 – M3. These contrasts are not independent, yet in an applied situation we may still wish to test them.

For samples of *equal size*, the following conditions define a set of orthogonal contrasts:

1. The sum of the contrast coefficients for each contrast must be 0.
2. The sum of cross products of coefficients for every pair of contrasts must be 0.

In general, the *i*th contrast of sample means (based on equal sample sizes) may be represented as $\Psi_i = \sum_{j=1}^{k} c_{ij} M_{.j}$. The above conditions require that $\sum_{j=1}^{k} c_{ij} = 0$ for all *i*, and that for all i and k, $\sum_{j=1}^{k} c_{ij} c_{kj} = 0$ where the coefficients are for the *i*th and *k*th contrasts. For example, with three treatment levels, the contrasts M1 – (1/2)(M2 + M3) and M2 – M3 *are* orthogonal. To demonstrate this, we rewrite the contrasts with specific coefficients:

$1M_1 - .5M_2 - .5M_3$
$0M_1 + .1M_2 - .1M_3$

The sum of the coefficients for each contrast is 0 (that is, 1 –. 5 – .5 = 0 and 0 + 1 – 1 = 0). Also, the sum of the cross products of the coefficients is (1)(0) + (–. 5)(1) + (–. 5)(–1) = 0 – .5 + .5 = 0. For more than three treatment levels, cross products must be computed for all possible pairs of contrasts [in general, there will be $(k - 1)(k - 2)$ pairs of contrasts].

Kirk's Notation.

Kirk designates each comparison with the symbol: $\hat{\Psi}_i$. If an experiment contains k treatment levels, there can only be k – 1 orthogonal contrasts. That is, k – 1 independent comparisons. Comparisons among means that are orthogonal and a priori can be carried out using a t-test. These tests are used in answering a limited number of questions concerning the data. The researcher should start with the hypothesis that is the most important. Then develop the other contrasts so that they are orthogonal with each other and with the main hypothesis. Kirk uses C_i for the coefficients of the orthogonal contrast. For example, C_1 would be the contrast for the first treatment level or Group1; C_2 would be for the second treatment level or Group 2, an so on. In order for the contrasts to be orthogonal (independent), the definition must be satisfied. The formula for the t-test looks like:

$$t = \frac{C_1 M_1 + C_2 M_2 + \cdots + C_k M_k}{\sqrt{MS_{Error}\left[\frac{C_1^2}{n_1} + \frac{C_2^2}{n_2} + \cdots + \frac{C_k^2}{n_k}\right]}}$$

With $N - k$ degrees of freedom.

When developing orthogonal contrasts, there are a number of different sets of contrasts that can be developed. These contrasts will be a combination of pair-wise comparisons and non pair-wise comparisons.

Given the following descriptive statistics and ANOVA Summary Table:

	Group 1	Group 2	Group 3	Group 4	Group 5
Mean	2.605	2.645	2.890	3.185	3.260

Source	df	SS	MS	F
Between Groups	4	2.1074	0.5426	3.13
Within Groups	25	4.3281	0.1731	
Total	29	6.4985		

With 5 levels of the independent variable, i.e., $k = 5$, we will have at most 4 orthogonal comparisons. Using the notation of Kirk, we can specify the contrasts as:

Comparison	Hypotheses	Coefficients				
		C1	C2	C3	C4	C5
$\hat{\Psi}_1$	$H_0: \mu_1 = \mu_2$ $H_1: \mu_1 \neq \mu_2$	1	–1	0	0	0

$\hat{\Psi}_2$	H_0: $\mu_3 = \mu_4$ H_1: $\mu_3 \neq \mu_4$	0	0	1	–1	0
$\hat{\Psi}_3$	H_0: $(\mu_1 + \mu_2)/2 = (\mu_3 + \mu_4)/2$ H_1: $(\mu_1 + \mu_2)/2 \neq (\mu_3 + \mu_4)/2$	½	½	–½	–½	0
$\hat{\Psi}_4$	H_0: $(\mu_1 + \mu_2 + \mu_3 + \mu_4)/4 = \mu_5$ H_1: $(\mu_1 + \mu_2 + \mu_3 + \mu_4)/4 \neq \mu_5$	¼	¼	¼	¼	–1

Remember that this is only one possible set of orthogonal contrasts. There are others that can be constructed. A t–test can be conducted on each of these contrasts where $n = 6$.

$$For\ \hat{\Psi}_1 \qquad t = \frac{C_1(M_1) + C_2(M_2)}{\sqrt{.1731\left[\frac{1}{6} + \frac{1}{6}\right]}} = \frac{(1)(2.605) + (-1)(2.645)}{\sqrt{.0677}} = \frac{-.04}{.240} = -0.166$$

df = 25, critical value = 2.06 (from t-distribution table). Since |–. 166| < 2.06 do not reject H_0: $\hat{\Psi}_1 = 0$.

$$For\ \hat{\Psi}_2 \qquad t = \frac{C_3(M_3) + C_4(M_4)}{\sqrt{.1731\left[\frac{1}{6} + \frac{1}{6}\right]}} = \frac{(1)(2.89) + (-1)(3.185)}{\sqrt{.0677}} = -1.23$$

Since |– 1.23| < 2.06, do not reject H_0: $\hat{\Psi}_2 = 0$.

$$For\ \hat{\Psi}_3 \qquad t = \frac{C_1(M_1) + C_2(M_2) + C_3(M_3) + C_4(M_4)}{\sqrt{.1731\left[\frac{1/4}{6} + \frac{1/4}{6} + \frac{1/4}{6} + \frac{1/4}{6}\right]}}$$

$$= \frac{(1/2)(2.605) + (1/2)(2.645) + (-1/2)(2.89) + (-1/2)(3.185)}{\sqrt{.1731(1/6)}} = \frac{-.4125}{.1699} = -2.428$$

Since |– 2.428| > 2.06, reject H_0: $\hat{\Psi}_3 = 0$.

$$For\ \hat{\Psi}_4 \qquad t = \frac{C_1(M_1) + C_2(M_2) + C_3(M_3) + C_4(M_4) + C_5(M_5)}{\sqrt{.1731\left[\frac{1/16}{6} + \frac{1/16}{6} + \frac{1/16}{6} + \frac{1/16}{6} + \frac{1}{6}\right]}}$$

$$= \frac{(1/4)(2.605) + (1/4)(2.645) + (1/4)(2.89) + (1/4)(3.185) + (-1)(3.26)}{.\sqrt{.1731(.20833)}} = \frac{-.4287}{.1899} = -2.2575$$

Since |– 2.2575| > 2.06, reject H_0: $\hat{\Psi}_4 = 0$.

Comparable to the Dunnett test the contrast is –1 ¼ ¼ ¼ ¼

For Assignment 2, students are asked to create a different set than the one given here. Before we do this, let us revisit the notation used in our sample problem.

C1 = Letter Grade No Feedback
C2 = Letter Grade, Minimum Feedback

C3 = Letter Grade, Maximum Feedback
C4 = Mastery Grade No Feedback
C5 = Mastery Grade, Minimum Feedback
C6 = Mastery Grade, Maximum Feedback

Comparison	Hypotheses	Coefficients					
		C1	C2	C3	C4	C5	C6
$\hat{\Psi}_1$	$H_0: \mu_1 = \mu_6 \quad H_1: \mu_1 \neq \mu_6$	1	0	0	0	0	–1
$\hat{\Psi}_2$	$H_0: \mu_2 = \mu_5 \quad H_1: \mu_2 \neq \mu_5$	0	1	0	0	–1	0
$\hat{\Psi}_3$	$H_0: \mu_3 = \mu_4 \quad H_1: \mu_3 \neq \mu_4$	0	0	1	–1	0	0
$\hat{\Psi}_4$	$H_0: (\mu_2 + \mu_5)/2 = (\mu_3 + \mu_4)/2$ $H_1: (\mu_2 + \mu_5)/2 \neq (\mu_3 + \mu_4)/2$	0	½	–½	–½	½	0
$\hat{\Psi}_5$	$H_0: (\mu_1 + \mu_6)/2 = (\mu_2 + \mu_3 + \mu_4 + \mu_5)/4$ $H_1: (\mu_1 + \mu_6)/2 \neq (\mu_2 + \mu_3 + \mu_4 + \mu_5)/4$	½	–¼	–¼	–¼	–¼	½

Partitioning of the Treatment Sum of Squares.

A unique, and fundamentally important, feature of sets of orthogonal contrasts is that they form the basis for a complete *partitioning* of the treatment sum of squares from the analysis of variance. As will be demonstrated in later chapters, more complex analysis-of-variance designs may be constructed by beginning with a CRD and then building orthogonal contrasts on the cells. For example, a 2×2 factorial design (covered later) has four treatment groups or cells. These four cells may be considered to be the levels of a one-way analysis of variance, or CRD. The sum of squares for rows, the sum of squares for columns, and the interaction term may be defined in terms of orthogonal contrasts on the four cells. Thus, a fundamental unity underlies all analysis-of-variance techniques.

For a set of k sample means, sets of $k - 1$ orthogonal contrasts can be constructed; thus, the orthogonal contrasts "consume" the degrees of freedom available in the set of sample means. There are, in general, many different sets of orthogonal contrasts that may be built on the same set of sample means.

Ordinarily, the research design and the researcher's interests will lead to the choice of one or, at least, a small number of orthogonal sets. Each set will absorb the $k - 1$ degrees of freedom and will completely partition the treatment sum of squares. If all samples are of the same size n, then a sum of squares for the ith contrast is defined as

$$SS_{\Psi_i} = \frac{n\left(\sum_{j=1}^{k} c_{ij} M_i\right)^2}{\sum_{j=1}^{k} c_{ij}^2}$$

For $k - 1$ orthogonal contrasts, $\sum_{i=1}^{k-1} SS_{\Psi_i} = SS_{Treatment}$, where $SS_{Treatment}$ is the treatment sum of squares based on the k cells of the one-way analysis of variance. The data and the analysis-of-variance summary from above may be used to illustrate the additive property of sums of squares based on sets of orthogonal contrasts.

Consider, first, the contrasts $M_1 - M_2$, $M_3 - M_4$, ½ $(M_1 + M_2) -$ ½ $(M_3 + M_4)$, ¼ $(M_1 + M_2 + M_3 + M_4) - M_5$. As shown previously, these contrasts are orthogonal. The sum of squares for the first contrast is

$$SS_{\Psi_1} = \frac{6[2.605 - 2.645]^2}{1^2 + 1^2} = \frac{.0096}{2} = .0048$$

For the second contrast,

$$SS_{\Psi_2} = \frac{6[2.89 - 3.185]^2}{1^2 + 1^2} = \frac{.52215}{2} = .2611$$

For the third and fourth contrasts:

$$SS_{\Psi_3} = \frac{6[.5(2.605 + 2.645) - .5(2.89 + 3.185)]^2}{.5^2 + .5^2 + (-.5)^2 + (-.5)^2} = \frac{1.0209}{1} = 1.0209$$

$$SS_{\Psi_4} = \frac{6[.25(2.605 + 2.645 + 2.89 + 3.185) - 3.26]^2}{.25^2 + .25^2 + .25^2 + .25^2 - 1} = \frac{1.10295}{1.25} = .88237$$

The sum of squares for treatment from the ANOVA Summary Table given above is 2.1704, and $SS_{\Psi_1} + SS_{\Psi_2} + SS_{\Psi_3} + SS_{\Psi_4} = .0048 + .2611 + 1.021 + .8824 = 2.1693 \approx 2.17$. Each of the sums of squares based on a contrast may be converted to a mean square and tested against the within-cells-error term. Since each contrast has 1 degree of freedom, the mean squares are identical in value with the sums of squares. When *k* is larger than 3, it is sometimes desirable or necessary to pool several of the sums of squares. The number of contrasts that are pooled together determines the degrees of freedom of the resulting sum of squares, and it may be tested against within-cells error. This procedure will be seen later in the course when we deal with more complex designs. A revised ANOVA Summary Table using the partition we just completed would look like:

Source	df	SS	MS	F
Between Groups	4	2.1074	0.5426	3.13
$SS\psi_1$	1	0.0048	0.0048	0.03
$SS\psi_2$	1	0.2611	0.2611	1.51
$SS\psi_3$	1	1.0210	1.0210	5.90
$SS\psi_4$	1	0.8824	0.8824	5.10
Within Groups	25	4.3281	0.1731	
Total	29	6.4985		

Both $SS\psi_3$ and $SS\psi_4$ reached significance at the 0.05 level. Also, eta-squared can be computed for each contrast in order to determine how much variance the contrast shares with the dependent variable.
Any other set of orthogonal contrasts defined on the five sample means would also consume the treatment sum of squares.

The discussion of orthogonal contrasts has, until this point, been restricted to the case in which all samples are the same size. In practice this is often possible, but many times, due to experimental mortality, loss of data, etc., unequal size groups may result. Thus, it is necessary to become familiar with the procedures for use with samples of any sizes. If we return to our original conditions for defining a contrast and for orthogonality, certain changes are necessary. A contrast for samples of varying sizes is defined as

$$\Psi_i = \sum_{j=1}^{k} n_j c_{ij} M_{.j} \quad \text{where} \quad \sum_{j=1}^{k} n_j c_{ij} = 0$$

Note that the sample sizes n_j enter into the expressions as weights. The sum of squares for a contrast becomes

$$SS_{\Psi_i} = \frac{\left(\sum_{j=1}^{k} n_j c_{ij} M_i \right)^2}{\sum_{j=1}^{k} n_j c_{ij}^2}$$

For two contrasts to be orthogonal, the weighted sum of cross products $\sum_{j=1}^{k} n_j c_{ij} c_{kj}$ must be 0. It is apparent that the construction of orthogonal sets is much more tedious when the sample sizes are not the same.