General Relativity and Gravitation, Vol. 36, No. 5, May 2004 (© 2004)

LETTER

Frame Dragging Anomalies for Rotating Bodies

Peter Collas[1] and David Klein[2]

Received October 7, 2003

Examples of axially symmetric solutions to Einstein's field equations are given that exhibit anomalous "negative frame dragging" in the sense that zero angular momentum test particles acquire angular velocities in the opposite direction of rotation from the source of the metric.

KEY WORDS: Frame dragging; Kerr-Newman; Bonnor; van Stockum; Brill-Cohen spacetimes.

1. INTRODUCTION

The prototype example of frame dragging arises in the Kerr metric. A test particle with zero angular momentum released from a nonrotating frame, far from the source of the Kerr metric, accumulates nonzero angular velocity in the same angular direction as the source of the metric, as the test particle plunges toward the origin (in Boyer-Lindquist coordinates). This "dragging of inertial frames," or frame dragging, is due to the influence of gravity alone, and has no counterpart in Newtonian physics.

Frame dragging is a general relativistic feature, not only of the exterior Kerr solution, but of all solutions to the Einstein field equations associated with rotating sources. In this paper we show that surprising frame dragging anomalies can occur

[1] Department of Physics and Astronomy, California State University, Northridge, Northridge, California 91330-8268; USA; e-mail: peter.collas@csun.edu
[2] Department of Mathematics, California State University, Northridge, Northridge, California 91330-8313; USA; e-mail: david.klein@csun.edu

0001-7701/04/0500-1197/0

in certain situations. We give examples of axially symmetric solutions to the field equations in which zero angular momentum test particles, with respect to nonrotating coordinate systems, acquire angular velocities in the opposite direction of rotation from the sources of the metrics. We refer to this phenomenon as "negative frame dragging."

The mathematical considerations in this paper are straightforward, but from a physical point of view, negative frame dragging is counterintuitive and intriguing. The negative frame dragging in some of the models we consider is associated with closed timelike curves due to singularities, and one might therefore expect to explain the phenomenon entirely in terms of temporal anomalies (Bonnor [1], Kerr-Newman [2]). However, we also show that negative frame dragging occurs relative to nonrotating, inertial observers on the axes of symmetry of metrics that are completely free of causality violations and singularities, such as the low density, slowly rotating van Stockum dust cylinder [3], (see also Tipler [4]), and the slowly rotating spherical shell of Brill and Cohen [5].

In Section 2 we define frame dragging, and introduce notation. In Section 3 we prove the existence of negative frame dragging for a model of a rotating dust cloud obtained by Bonnor [1] and investigated by Steadman [6], and for the Kerr-Newman metric [2]. Section 4 contains a proof of the existence of negative frame dragging for the low mass van Stockum dust cylinder [3], and Brill and Cohen's slowly rotating spherical shell [5]. Our concluding remarks are in Section 5.

2. FRAME DRAGGING

A convenient way of writing the general stationary axisymmetric metric (vacuum or nonvacuum) is

$$ds^2 = -F(dt)^2 + L(d\phi)^2 + 2Mdtd\phi + H_2(dx^2)^2 + H_3(dx^3)^2, \tag{1}$$

where F, L, M, H_2, H_3 are functions of x^2 and x^3 only; consequently the canonical momenta p_t and p_ϕ are conserved along geodesics. From (1) we find that

$$p_t = -F\dot{t} + M\dot{\phi} \equiv -E, \tag{2}$$

$$p_\phi = M\dot{t} + L\dot{\phi}. \tag{3}$$

The overdot stands for $d/d\tau$ for timelike particles and $d/d\lambda$ for lightlike particles, where τ denotes proper time, and λ is an affine parameter. E and p_ϕ are the energy and angular momentum, respectively, of massless or massive particles. We may then write,

$$\dot{t} = \frac{Mp_\phi + LE}{FL + M^2}, \tag{4}$$

$$\dot{\phi} = \frac{Fp_\phi - ME}{FL + M^2}. \tag{5}$$

Thus,

$$\frac{d\phi}{dt} = \frac{\dot{\phi}}{\dot{t}} = \frac{Fp_\phi - ME}{Mp_\phi + LE}. \tag{6}$$

If we let $p_\phi = 0$ in Eq. (6), we obtain the angular velocity ω of a zero angular momentum particle as measured by an observer for whom t is the proper time. This is the angular velocity of the frame dragging and it is given by,

$$\omega = -\frac{M}{L}. \tag{7}$$

3. SINGULAR METRICS

A solution to the field equations given by Bonnor in [1] describes a cloud of rigidly rotating dust particles moving along circular geodesics in hypersurfaces of $z =$ constant. In contrast to the van Stockum dust cylinder considered in the next section, this metric has a singularity at $r = z = 0$. Bonnor's metric has the form of Eq. (1) where F, L, M, and $H \equiv H_2 = H_3$ are functions of $x^2 = r$ and $x^3 = z$. In comoving (i.e., corotating) coordinates these functions are given by

$$F = 1, \qquad L = r^2 - n^2, \qquad M = n, \qquad H = e^\mu, \tag{8}$$

where

$$n = \frac{2hr^2}{R^3}, \qquad \mu = \frac{h^2 r^2 (r^2 - 8z^2)}{2R^8}, \qquad R^2 = r^2 + z^2, \tag{9}$$

and we have the coordinate condition

$$FL + M^2 = r^2. \tag{10}$$

The rotation parameter h has dimensions of length squared. The energy density ρ is given by

$$8\pi\rho = \frac{4e^{-\mu} h^2 (r^2 + 4z^2)}{R^8}. \tag{11}$$

As $R \to \infty$, ρ approaches zero rapidly and the metric coefficients tend to Minkowski values. Moreover, all the Riemann curvature tensor elements vanish at spatial infinity. Thus an observer at spatial infinity may be regarded as nonrotating, as in the case of the Kerr metric (in Boyer-Lindquist coordinates).

Steadman [6] observed that null geodesics with angular momentum p_ϕ are restricted to the region S_B given by

$$S_B = \{(t, \phi, r, z) | -p_\phi^2 + 2nEp_\phi + (r^2 - n^2)E^2 \geq 0\}. \tag{12}$$

For the case where $p_\phi = 0$ we let $S_B = S_{B0}$. Then $S_{B0} = \{(t, \phi, r, z) | L \geq 0\}$, and $\partial S_{B0} = \{(t, \phi, r, z) | L = 0\}$. The proof of the next proposition follows from direct calculation, using Eq. (7).

Proposition 1: *In Bonnor's metric, $\omega \to 0$ as either r or z go to ∞, $\omega < 0$ everywhere in S_{B0}, and $\omega \to -\infty$ on ∂S_{B0}.*

Since $\omega \to 0$ as either r or z go to ∞, an observer at spatial infinity observes a zero angular momentum test particle to be nonrotating (at infinity). The same observer observes negative frame dragging at all finite r and z coordinate values. This negative frame dragging is associated with temporal anomalies as we explain at the end of this section.

The Kerr-Newman metric [2] is a vacuum metric. It is a generalization of the Kerr metric that accounts for an electrical charge of the source. We write it below in Boyer-Lindquist coordinates. Using the notation of Eq. (1) where F, L, M, H_2, H_3 are now functions of $x^2 = r$ and $x^3 = \theta$, we have

$$F = \frac{\Delta - a^2 \sin^2\theta}{\rho^2}, \qquad L = \frac{[(r^2 + a^2)^2 - a^2\Delta\sin^2\theta]\sin^2\theta}{\rho^2}, \tag{13}$$

$$M = -\frac{a\sin^2\theta(r^2 + a^2 - \Delta)}{\rho^2}, \qquad H_r = \frac{\rho^2}{\Delta}, \quad H_\theta = \rho^2, \tag{14}$$

where

$$\Delta = r^2 + a^2 + e^2 - 2mr, \quad \text{and} \quad \rho^2 = r^2 + a^2\cos^2\theta. \tag{15}$$

In Eqs. (13)–(15) m is associated with the mass of the source of the metric, e is the electric charge, and the parameter a is the angular momentum per unit mass. We note that the Kerr-Newman metric differs from the Kerr metric only in the definition of Δ. For simplicity we consider only the case where $a^2 + e^2 > m^2$. In this case the Kerr-Newman metric has a naked (ring) singularity at $\rho^2 = 0$. There are no event horizons since under the above condition $\Delta > 0$ for all r. We also have that

$$FL + M^2 = \Delta\sin^2\theta, \tag{16}$$

thus $FL + M^2 > 0$ for $\theta \neq 0$.

The Kerr-Newman metric also has a "forbidden region" like the one found by Steadman [6] for Bonnor's metric. From Eqs. (14) and (15) we see that

$$H_r dr^2 + H_\theta d\theta^2 \geq 0, \tag{17}$$

and since for null paths $ds^2 = 0$, it follows that

$$F\dot{t}^2 - L\dot{\phi}^2 - 2M\dot{t}\dot{\phi} \geq 0. \tag{18}$$

Now substituting Eqs. (4) and (5) into Eq. (18), we obtain the inequality,

$$-Fp_\phi^2 + LE^2 + 2Mp_\phi E \geq 0, \qquad \theta \neq 0. \tag{19}$$

Null geodesics with angular momentum p_ϕ are restricted to a region S_{KN} given by

$$S_{KN} = \{(t, \phi, r, \theta) | - F p_\phi^2 + L E^2 + 2M p_\phi E \geq 0\}. \tag{20}$$

For the case where $p_\phi = 0$ we let $S_{KN} = S_{KN0}$. Then $S_{KN0} = \{(t, \phi, r, \theta) | L \geq 0\}$, and $\partial S_{KN0} = \{(t, \phi, r, \theta) | L = 0\}$. We then have the following proposition:

Proposition 2: *In the Kerr-Newman metric, ω vanishes at ∞ and at $r = e^2/(2m)$. Furthermore $\omega \to -\infty$ on ∂S_{KN0}.*

Proof. From Eqs. (7) and (13)–(15) we have that

$$\omega = -\frac{M}{L} = \frac{a(2mr - e^2)}{B(r)}, \tag{21}$$

where

$$\begin{aligned} B(r) &= (a^2 + r^2)^2 - a^2 \Delta \sin^2\theta \\ &= r^4 + a^2(2 - \sin^2\theta) r^2 + (2a^2 m \sin^2\theta) r \\ &\quad - a^2 \left[(a^2 + e^2)\sin^2\theta - a^2\right]. \end{aligned} \tag{22}$$

It is clear that ω vanishes at $r = e^2/(2m)$ and at infinity. We wish to show that $B(r)$ has one positive root and that this root cannot coincide with the root of the numerator.

By Descartes' theorem $B(r)$ can have at most one positive root if

$$(a^2 + e^2)\sin^2\theta > a^2. \tag{23}$$

Since the negative roots of $B(r)$ are the same as the positive roots of $B(-r)$, it follows that, under condition (23), $B(r)$ can have at most one negative root. Since $B(0) < 0$ if condition (23) holds, we see that indeed $B(r)$ has exactly one positive, one negative, and two complex roots. The positive root is the root of $L(r)$ that causes $\omega \to -\infty$. The proof is completed by observing that the equation $B(r = e^2/(2m)) = 0$ cannot be satisfied for any real θ.

Referring to Eq. (1), consider a curve with fixed (t, x^2, x^3) coordinates, i.e., an integral curve of the ϕ coordinate. When $L > 0$, this curve is a closed spacelike curve of length given by $s^2 = L(2\pi)^2$. However, when $L < 0$, it is a closed timelike curve, while when $L = 0$, it is a closed null curve. The last two cases are examples of causality violating paths. Thus, the forbidden regions for $p_\phi = 0$ for the Bonnor and Kerr-Newman metrics coincide with the region described here where closed azimuthal timelike curves first appear. The sign of the metric coefficient L determines the sign of the frame dragging, ω, as well as a region of causality violations for integral curves of the ϕ coordinate. While negative frame dragging may be correlated in this way with temporal anomalies for the Bonnor and Kerr-Newman metrics, the metrics considered in Section 4 are free of causality violations. Yet, negative frame dragging occurs in nonrotating reference frames for those metrics.

4. NONSINGULAR METRICS

The van Stockum solution [3] represents a rotating dust cylinder of infinite extent along the axis of symmetry (z-axis) but of finite radius. There are three vacuum exterior solutions that can be matched to the interior solution, depending on the mass per unit length of the interior. Bonnor [7] labeled these exterior solutions: (I) the low mass case, (II) the null case, (III) the ultrarelativistic case. Tipler [4] (see also Steadman [8]) showed that in case III, there exist causality violating paths in the spacetime. We focus on the low mass case, as it is the most physically realistic of the three, and the most significant from the point of view of frame dragging. We comment briefly on the frame dragging properties of the other two cases.

For the van Stockum metric, $H_2 = H_3 = H$ in Eq. (1), $x^3 = z$, and the functions F, L, M, H depend only on $x^2 = r$. The metric coefficients for the interior of the cylinder in comoving coordinates, i.e., coordinates corotating with the dust particles, are given by

$$F = 1, \qquad L = r^2(1 - a^2r^2), \qquad M = ar^2, \qquad H = e^{-a^2r^2} \tag{24}$$

In Eq. (24), $0 \leq r \leq R$ for a constant R that determines the radius of the cylinder, a is the angular velocity of the dust particles, and the density ρ is given by $8\pi\rho = 4a^2e^{a^2r^2}$. The coordinate condition

$$FL + M^2 = r^2, \tag{25}$$

holds for the interior as well as the exterior solutions below. Furthermore since $g = \det(g_{\mu\nu}) = -(FL + M^2)H^2 = -r^2H^2 < 0$, the metric signature is $(-, +, +, +)$ for all $r > 0$, provided $H \neq 0$; this is true in particular even if L changes sign.

The low mass vacuum exterior solution (Case I) is valid for $0 < aR < 1/2$ and $r \geq R$. The metric coefficients are

$$F = \frac{r \sinh(\epsilon - \theta)}{R \sinh \epsilon}, \qquad L = \frac{Rr \sinh(3\epsilon + \theta)}{2 \sinh 2\epsilon \cosh \epsilon}, \tag{26}$$

$$M = \frac{r \sinh(\epsilon + \theta)}{\sinh 2\epsilon}, \qquad H = e^{-a^2R^2} \left(\frac{R}{r}\right)^{2a^2R^2}, \tag{27}$$

with

$$\epsilon = \tanh^{-1}(1 - 4a^2R^2)^{1/2}, \qquad \theta = (\tanh \epsilon) \log\left(\frac{r}{R}\right). \tag{28}$$

The metric is globally regular, and the algebraic invariants of the Riemann tensor vanish as $r \to \infty$ (this is true also in Cases II and III provided $aR < 1$).

We consider noncomoving coordinates given by the transformation

$$t = \bar{t}, \qquad \phi = \bar{\phi} - \Omega\bar{t}, \qquad r = \bar{r}, \qquad z = \bar{z}, \tag{29}$$

where the barred coordinates are noncomoving. In the barred coordinates, the metric coefficients are:

$$\bar{F} = F + 2\Omega M - \Omega^2 L, \qquad \bar{L} = L, \qquad \bar{M} = M - \Omega L, \qquad \bar{H} = H. \tag{30}$$

Among these barred coordinate systems, two values of Ω may be used to compute physically meaningful values of the angular velocity ω for frame dragging given by Eq. (7): $\Omega = a$ for an observer in a nonrotating inertial reference frame on the axis of symmetry, and $\Omega = \Omega_c$ (determined below) for an observer nonrotating relative to "the fixed stars." The choice $\Omega = a$ is determined by the Fermi-Walker equations. A coordinate system satisfying the Fermi-Walker equations is rotation free (Walker [9], also Misner, Thorne, and Wheeler [10]), and it is therefore natural to study frame dragging in such a coordinate system.

By changing from polar to Cartesian coordinates, it is easy to see that the spacetime may be extended to include the axis of symmetry ($r = 0$), and the metric is Minkowskian there for any value of Ω. Furthermore, the reference frame of a nonmoving observer with four-velocity, $\vec{u} = (F^{-1/2}, 0, 0, 0)$ and orthonormal spatial frame vectors in the x, y, z directions satisfies the Fermi-Walker equations if and only if $\Omega = a$. Indeed, the fixed observer with this four-velocity lies on a geodesic, and the orthonormal frame satisfies the parallel transport equations when $\Omega = a$. The calculations are straightforward. We note that in [3], van Stockum already argued that an observer on the axis with the above four-velocity is nonrotating if and only if $\Omega = a$, through a calculation that involved taking limits as $r \to 0$ in cylindrical coordinates (our transformation, Eq. (29), differs by a sign from the one that van Stockum used in ref. [3], p. 145).

Proposition 3. *In the nonrotating, inertial reference frame of the low mass van Stockum cylinder corresponding to $\Omega = a$ described above, a zero angular momentum test particle experiences negative frame dragging at all points in the exterior and interior of the cylinder off of the axis of symmetry, that is, a zero angular momentum test particle with positive r coordinate will accumulate an angular velocity in the direction opposite to the rotation of the cylinder. Furthermore, the angular velocity, $\bar{\omega}(r)$, given by* Eq. (7) *for this coordinate system, decreases monotonically to the negative constant $a - (2/R)\,e^{-2\epsilon}\cosh\epsilon$, as $r \to \infty$.*

Proof. From Eqs. (7), (29), and (30), $\bar{\omega}(r) \equiv -\bar{M}(r)/\bar{L}(r) = -M(r)/L(r) + a$. A simple calculation using Eq. (24) shows that $\bar{\omega}(r)$ is negative whenever $0 < r \leq R$. A second calculation using Eqs. (26) and (27) shows that $d\bar{\omega}/dr < 0$ for all $r \geq R$.

It follows that $\bar{\omega}(r) < 0$ for all $r > 0$, and that $\bar{\omega}(r)$ is a decreasing function of r. The limiting value $\bar{\omega}(r \to \infty) = a - (2/R)\, e^{-2\epsilon} \cosh\epsilon$ follows directly from Eqs. (26) through (30).

Instead of $\Omega = a$, we may choose another value, $\Omega = \Omega_c$, in Eqs. (29) and (30) where $\Omega_c \equiv \Omega_c(a, R)$ is the "critical value" of Ω for which $\bar{\omega}(r \to \infty) = 0$. Such an Ω_c exists for Case I (as well as Case II but not for Case III). A short calculation shows that

$$\Omega_c = \left(\frac{2}{R}\right) e^{-2\epsilon} \cosh\epsilon, \tag{31}$$

When $\Omega = \Omega_c$ it follows that $\bar{\omega}(r) > 0$ for all r. In this coordinate system $\bar{t}$ is the proper time of an observer at $\bar{r} = r = 0$ whose frame is nonrotating relative to the distant stars, i.e., relative to $r = \infty$. This observer does not observe negative frame dragging; but rather the usual (positive) frame dragging in the angular direction of rotation of the dust cylinder.

We note that $g_{tt} = -F$ changes sign in the exterior cases in the corotating coordinate systems. However when we rotate the comoving coordinates by Ω_c, we have $g_{\bar{t}\bar{t}} < 0$ for all $\bar{r}$ in Case I. In the exterior Case II, the analogous critical value of Ω results in $g_{\bar{t}\bar{t}} = 0$ for all $\bar{r}$. Therefore in Case II $\partial/\partial\bar{t}$ is a null vector. Finally in Case III ($1/2 < aR < 1$) we have causality violating paths and negative frame dragging that cannot be made positive.

Two coordinate systems are determined by $\Omega = \Omega_c$ and $\Omega = a$, through Eqs. (29) and (30). Nonrotating observers in frames defined in terms of these coordinate systems observe completely different frame dragging properties. In the first case, negative frame dragging occurs, while in the second case it does not. Yet, both observers can claim to be nonrotating in physically reasonable ways. In the first case, the observer is nonrotating in the sense that his reference frame is nonrotating and is locally inertial, while the second observer has the feature that the distant stars are fixed (i.e., nonrotating) in his frame. Van Stockum in [3, 11] already noted that these coordinate systems rotate relative to each other, but it is a peculiar feature that zero angular momentum test particles in one of these frames is dragged in the opposite angular direction from the motion of the cylinder.

Brill and Cohen [5] considered frame dragging and Machian effects associated with a slowly rotating thin spherical shell of radius r_0 and mass m. They calculated the metric solution to the Einstein field equations to first order in the angular velocity a of the spherical shell. Their metric may be written in the form of Eq. (1) with $x^2 = r\ x^3 = \theta$, and for $r > r_0$ the metric coefficients are given by

$$H_r = \left(1 + \frac{m}{2r}\right)^4, \qquad H_\theta = r^2 H_r, \qquad L = H_\theta \sin^2\theta, \tag{32}$$

$$M = -L\omega(r), \qquad F = \left(\frac{2r - m}{2r + m}\right)^2 - Lw^2(r), \tag{33}$$

where

$$\omega(r) = \frac{4amr^3(m+2r_0)^5(m-4r_0)}{r_0^3(m+2r)^6(m-6r_0)} \quad \text{for } r > r_0. \tag{34}$$

We note that the function $\omega^2(r)$ in F is not rquired in the lowest order approximation. When $r < r_0$, H_r, H_θ, M, L, F, and ω each take constant values determined by their respective formulas in (32), (33), and (34) evaluated at $r = r_0$, so that, for example,

$$\omega(r_0) = \frac{4am(m-4r_0)}{(m+2r_0)(m-6r_0)} \quad \text{for} \quad r < r_0. \tag{35}$$

We see that ω vanishes for $m = 4r_0$ and diverges for $m = 6r_0$. This unphysical behavior is undoubtedly due to the approximations. Outside of an interval containing $4r_0$ and $6r_0$, ω is monotonically increasing in m. We restrict ourselves to these values of m.

In the interior of the spherical shell, ω is constant and the metric can be made diagonal by simple rotation, Eq. (29) with $\Omega = -\omega$. Therefore an observer on the axis with a coordinate frame associated with this change of coordinates satisfies the Fermi-Walker equations and is nonrotating. With the notation of Eq. (30), we calculate the frame dragging $\bar{\omega}(r)$ for this observer as follows: $\bar{\omega}(r) \equiv -\bar{M}(r)/\bar{L}(r) = -M(r)/L(r) - \omega(r_0) = \omega(r) - \omega(r_0)$. It is easy to see from Eqs. (34) and (35) that $\omega(r_0) > 0$ and $\omega(r) \to 0$ as $r \to \infty$. Brill and Cohen give plots of ω/a versus r/r_0 for various masses, and they show that ω is maximum in the interior of the shell and decreases monotonically for $r > r_0$. It follows that the nonrotating inertial observer on the axis will observe negative frame dragging for all $r > r_0$.

5. CONCLUDING REMARKS

The existence of negative frame dragging depends on the qualitative behavior of the metric coefficients in Eq. (1). The examples considered in this note suggest that the phenomenon is fairly widespread among axially symmetric solutions to the field equations.

ACKNOWLEDGMENT

The authors with to thank Professor John Lawrence for helpful comments.

REFERENCES

[1] Bonnor, W. B. (1977). *J. Phys. A: Math. Gen.* **10**, 1673.
[2] Newman, E. T., Couch, E., Chinnapared, K., Exton, A., Prakash, A., and Torrence, R. (1965). *J. Math. Phys.* **6**, 918.

[3] van Stockum, W. J. (1937). *Proc. R. Soc. Edin.* **57**, 135.
[4] Tipler, F. J. (1974). *Phys. Rev. D* **9**, 2203.
[5] Brill, D. R., and Cohen, J. M. (1966). *Phys. Rev.* **143**, 1011.
[6] Steadman, B. R. (1999). *Class. Quant. Grav.* **16**, 3685.
[7] Bonnor, W. B. (1980). *J. Phys. A: Math. Gen.* **13**, 2121.
[8] Steadman, B. R. (2003). *Gen. Rel. Grav.* **35**, 1721.
[9] Walker, A. G. (1935). *Proc. Edin. Math. Soc.* **4**, 170.
[10] Misner, C. W., Thorne, K. S., and Wheeler, J. A. (1973). *Gravitation*, W. H. Freeman, San Francisco, California.
[11] van Stockum, W. J. (1938). *Proc. R. Irish Acad.* **44**, 109.