# Convergence of Hamiltonian systems to billiards

Peter Collas
*Department of Physics and Astronomy, and Mathematical Physics Institute, California State University, Northridge, Northridge, California 91330-8268*

David Klein
*Department of Mathematics, and Mathematical Physics Institute, California State University, Northridge, Northridge, California 91330-8313*

Hans-Peter Schwebler
*Institut für Theoretische Physik, University of Bremen, Postfach 33 04 40, D-28334 Bremen, Germany*

(Received 15 October 1997; accepted for publication 6 February 1998)

We examine in detail a physically natural and general scheme for gradually deforming a Hamiltonian to its corresponding billiard, as a certain parameter $k$ varies from one to infinity. We apply this limiting process to a class of Hamiltonians with homogeneous potential-energy functions and further investigate the extent to which the limiting billiards inherit properties from the corresponding sequences of Hamiltonians. The results are mixed. Using theorems of Yoshida for the case of two degrees of freedom, we prove a general theorem establishing the ''inheritability'' of stability properties of certain orbits. This result follows naturally from the convergence of the traces of appropriate monodromy matrices to the billiard analog. However, in spite of the close analogy between the concepts of integrability for Hamiltonian systems and billiards, integrability properties of Hamiltonians in a sequence are not necessarily inherited by the limiting billiard, as we show by example. In addition to rigorous results, we include numerical examples of certain interesting cases, along with computer simulations. © *1998 American Institute of Physics.*
[S1054-1500(98)00402-9]

**A billiard is a point particle which moves freely between elastic bounces off the walls of its two-dimensional enclosure. By contrast, a particle moving under the influence of conservative forces is a Hamiltonian system. The orbits of billiards and Hamiltonian systems have been extensively studied. Natural lines of inquiry concern the stability of periodic orbits, integrability, i.e., the existence of constants of the motion, ergodicity, and the extent of chaos. In this paper we show how to deform mathematically a Hamiltonian system into a billiard. We do this by gradually weakening the forces acting on the particle near the center of a natural enclosure defined by conservation of energy and concentrating those forces near the boundary. In the limit as forces vanish in the interior of the enclosure, the Hamiltonian becomes a billiard. We prove mathematical results regarding the ''inheritability'' of the stability of certain periodic orbits and integrability, from the Hamiltonian systems to the billiards. Computer simulations and numerical arguments provide additional insights. Properties of billiards may thus be understood by studying Hamiltonian systems with suitable restrictions and vice versa.**

## I. INTRODUCTION

The idea that a billiard system can be thought of as a limiting case of a Hamiltonian system has been known since Birkhoff suggested[1] (see also Ref. 2) that one can obtain the elliptic billiard from the motion of a particle on the surface of a triaxial ellipsoid when one of the ellipsoid's semiaxes goes to zero. Unfortunately Birkhoff's example is unique and cannot be used to connect Hamiltonians to billiards in general. A method which promises to be more generally applicable was used by Dahlqvist and Russberg.[3] They examined the one-parameter family (a sequence) of Hamiltonians

$$H=\tfrac{1}{2}[p_x^2+p_y^2+(x^2y^2)^{1/a}]. \tag{1}$$

where for $a=1$ Eq. (1) becomes the Hamiltonian with the quartic potential $x^2y^2/2$, while for $a=0$, one obtains the hyperbola billiard. In the present paper we generalize the Dahlqvist–Russberg idea, discuss it in detail, and apply it to a class of two degree of freedom Hamiltonians with homogeneous potential-energy functions. In particular we investigate the extent to which the limiting billiards inherit properties from the corresponding sequences of Hamiltonians. We use some results of Yoshida[4–6] and prove a general theorem establishing the ''inheritability'' of the stability properties of straight-line periodic orbits.

In Sec. II we discuss the Hamiltonian to billiard scheme in general. In Sec. III we review certain results from Floquet–Liapunov theory and introduce the monodromy matrix for straight-line periodic orbits in the Hamiltonian, as well as the billiard case. In Sec. IV we review some of Yoshida's results on the stability of straight-line orbits and nonintegrability for homogeneous potentials of even positive integer degree.[4–6] In Sec. V we prove a number of results, and show that the trace of the monodromy matrix for a straight-line periodic orbit of the Hamiltonian sequence, in the limit as $k\to\infty$, is equal to the corresponding trace for the billiard. Finally, in Sec. VI, we investigate two examples and conjecture that Yoshida's trace formula gives correct results also for noninteger $k$ in certain cases.

## II. THE HAMILTONIAN TO BILLIARD SEQUENCE

We define the one-parameter family of Hamiltonians

$$H_k=\frac{1}{2}\,(p_x^2+p_y^2)+E\left[\frac{V(x,y)}{E}\right]^k=E, \tag{2}$$

where $V(x,y)\geqslant 0$ is the potential energy and $E$ is the energy (we let the particle mass $m=1$). We are assuming, throughout the present paper, that the physical region (Hill's region). $Q=\{(x,y)|V(x,y)\leqslant E\}$, is compact, and that $E>0$. However, these assumptions are not all necessary and may be relaxed. When $k=1$, $H_k$ becomes the Hamiltonian

$$H_1=\tfrac{1}{2}(p_x^2+p_y^2)+V=E. \tag{3}$$

When $k\rightarrow\infty$, $H_k$ goes into the billiard with boundary $\partial Q=\{(x,y)|V(x,y)=E\}$, since

$$H_\infty=\tfrac{1}{2}(p_x^2+p_y^2)=E \quad \text{if } V<E, \tag{4}$$

$$H_\infty=\tfrac{1}{2}(p_x^2+p_y^2)+E=E \quad \text{if } V=E, \tag{5}$$

$$H_\infty=\infty \quad \text{if } V>E. \tag{6}$$

A more formal way to view the Hamiltonian $H_k$, given by Eq. (2), is to consider the Hamiltonian

$$H_k(h)=\frac{1}{2}\,(p_x^2+p_y^2)+h\left(\frac{V(x,y)}{h}\right)^k=E, \tag{7}$$

which depends on a parameter $h>0$. $H_k(h=E)$ as $k\rightarrow\infty$ goes into the billiard with boundary $\partial Q=\{(x,y)|V(x,y)=E\}$. We define the unit normal and tangent vectors to $\partial Q$ by

$$n=-\frac{\text{grad } V}{\|\text{grad } V\|}, \quad t=\frac{(-V_y,V_x)}{\|\text{grad } V\|}, \tag{8}$$

where $n$ is the *inward* normal and $t$ is the *counterclockwise* tangent ($V_x=\partial V/\partial x$, etc.). On $\partial Q$, the normal component of the momentum $p_n$ and the tangential component $p_t$ are given by

$$p_n=-\frac{(V_xp_x+V_yp_y)}{\|\text{grad } V\|}, \quad p_t=\frac{(V_xp_y-V_yp_x)}{\|\text{grad } V\|}. \tag{9}$$

Equation (5) implies that $p_x^2+p_y^2=p_n^2+p_t^2=0$ on $\partial Q$. This is the usual situation for a Hamiltonian system, where any trajectory that meets the boundary is normal to $\partial Q$. However, for a billiard this need not be so, and we may have $p_t\neq 0$. We shall show below that, in the billiard limit and on $\partial Q$, $\dot{p}_n=+\infty$, while $\dot{p}_t=0$, thus the particle is reflected properly at the boundary.

From Eq. (2), when $p_x^2+p_y^2=0$, we see that $\partial Q$ is given by $V(x,y)=E$ and so it is independent of $k$. Furthermore, since we have assumed that $E>0$, we have that $E(V/E)^k\geqslant 0$; therefore, $p_{x\,\max}^2$ and $p_{y\,\max}^2$ are obtained when $V=0$ as in the $k=1$ case. Thus the physical range of the variables $x,y$ [determined by $V(x,y)=E$], and the physical range of the variables $p_x,p_y$ (determined by $V=0$), are independent of $k$.

In the following sections we shall consider potentials $V(x,y)$ which are homogeneous of *even* degree $m$; thus

$$V(\alpha x,\alpha y)=\alpha^m V(x,y). \tag{10}$$

Let us rewrite Eq. (7) as follows:

$$H_k(h)=\tfrac{1}{2}(p_x^2+p_y^2)+h^{1-k}(V(x,y))^k=E. \tag{11}$$

It is a well-known fact[7,4] that if the potential energy is a homogeneous function of the (Cartesian) coordinates, the scaling $x,y\rightarrow\alpha x,\alpha y$, and $t\rightarrow\beta t$ [with a particular choice of $\beta=\beta(\alpha)$], amounts to a rescaling of the energy and vice versa. Therefore the energy $E$ is not an essential parameter of the Hamiltonian and in particular it does not affect the integrability or the stability properties of the system. It is also true that the value of an overall constant factor of the potential energy, like $h^{1-k}$ in Eq. (11), cannot affect the integrability or the stability properties of the system, and in fact can be eliminated from the equations of motion by the reparametrization $\bar{t}=h^{(1-k)/2}t$, in which case $d^2x/d\bar{t}^2=-\partial[V(x(\bar{t}),y(\bar{t}))]^k/\partial x$, and likewise for $d^2y/d\bar{t}^2$. We shall, therefore, let $E=h=1$ without loss of generality. Our Hamiltonian, Eq. (7), then becomes

$$H_k=\tfrac{1}{2}(p_x^2+p_y^2)+V_k(x,y)=1, \tag{12}$$

where

$$V_k(x,y)=[V(x,y)]^k \tag{13}$$

and

$$V_k(\alpha x,\alpha y)=\alpha^{mk}V_k(x,y). \tag{14}$$

The equations of motion for $H_k$ are

$$\begin{aligned}
\dot{x}&=p_x,\\
\dot{y}&=p_y,\\
\dot{p}_x&=-kV^{k-1}V_x,\\
\dot{p}_y&=-kV^{k-1}V_y.
\end{aligned} \tag{15}$$

On $\partial Q$, when $k\rightarrow\infty$, Eqs. (15) give

$$\begin{aligned}
&\dot{x}=p_x, \quad \dot{y}=p_y,\\
&\dot{p}_x\rightarrow\begin{cases}\pm\infty & \text{if } V_x\neq 0\\ 0 & \text{if } V_x=0,\end{cases}\\
&\dot{p}_y\rightarrow\begin{cases}\pm\infty & \text{if } V_y\neq 0\\ 0 & \text{if } V_y=0.\end{cases}
\end{aligned} \tag{16}$$

Furthermore, if we evaluate $\dot{p}_n$ and $\dot{p}_t$, using Eqs. (9) and (15), we obtain

$$\begin{aligned}
\dot{p}_n&=kV^{k-1}\|\text{grad } V\|+[\text{terms independent of } k],\\
\dot{p}_t&=p_n\,\frac{[V_xV_{yy}p_y-V_yV_{xx}p_x+V_{xy}(V_xp_x-V_yp_y)]}{\|\text{grad } V\|^2}.
\end{aligned} \tag{17}$$

It follows from Eqs. (17) that on $\partial Q$, when $V=1$, and $p_n=0$,

$$\lim_{k\rightarrow\infty}\dot{p}_n=+\infty \quad \text{and} \quad \dot{p}_t=0, \tag{18}$$

as mentioned above (recall that our normal is the inward normal). Figure 1 in Sec. VI shows the implications of Eqs. (18) graphically.

## III. THE MONODROMY MATRIX

In order to investigate the stability of an orbit, one examines the behavior of the solutions of the so-called variational equations and the trace of the corresponding monodromy matrices. In general, to obtain the variational equations for a reference orbit $(x(t),y(t))$, of $H_k$, we let $(x(t)+\xi_x(t),y(t)+\xi_y(t))$ be a nearby orbit, where $\xi_x$ and $\xi_y$ measure the separation of the two orbits. Then the variational equations are given by $\ddot{\xi}_i+\Sigma_j V_{k,ij}\xi_j=0$, ($i=x,y$ and $j=x,y$). By a simple rotation of the vector $(\xi_x,\xi_y)$ we obtain the variational equations for $(\xi_n,\xi_t)$, the normal and tangential variations. As we shall show below the normal variational equation for any straight-line periodic orbit of our homogeneous potential may be put in the form of Hill's equation

$$\frac{d^2\xi_n}{dt^2}+\lambda_k\psi_k^{mk-2}\xi_n=0, \tag{19}$$

where $\psi_k(t)$ is defined by $(x(t),y(t))=\psi_k(t)(x_0,y_0)$, with $V(x_0,y_0)=1$, and $\lambda_k$ is called the *integrability coefficient*.[6] We may rewrite Eq. (19) as

$$\frac{d\Xi}{dt}=A(t)\Xi, \tag{20}$$

where

$$\Xi(t)=\begin{pmatrix}\xi_n\\ \dot{\xi}_n\end{pmatrix},\quad A(t)=\begin{pmatrix}0 & 1\\ -\lambda_k\psi_k^{mk-2} & 0\end{pmatrix}, \tag{21}$$

and $A(t)$ is periodic with minimal period $T$. By Floquet–Liapunov theory, any fundamental matrix of solutions $\Phi(t)$ for Eq. (20) can be expressed as

$$\Phi(t)=\begin{pmatrix}\xi_n(t) & \zeta_n(t)\\ \dot{\xi}_n(t) & \dot{\zeta}_n(t)\end{pmatrix}=P(t)e^{Bt}, \tag{22}$$

where $\xi_n$ and $\zeta_n$ are two independent solutions of Eq. (19), $P(t)$ is a nonsingular matrix of periodic functions with the same period $T$ as $A(t)$, and $B$ is a constant matrix whose eigenvalues are called the characteristic exponents of the system (20). Since $\mathrm{tr}\,A(t)=0$, the solutions of Eq. (20) are never asymptotically stable. The matrix, $M(T)=e^{BT}$ is called the *monodromy matrix* for the system (20). The trace of $M$ determines the stability of the system. When $|\mathrm{tr}\,M|>2$, the system (20) is unstable and as a consequence, the periodic solution $(x(t),y(t))$ is also unstable.

Clearly, from Eq. (22), $\Phi(T)=P(T)M(T)$, so if we choose $\Phi(0)=I$, where $I$ is the identity matrix, we have that $P(0)=P(T)=I$, and therefore $\Phi(T)=M(T)$. We may then write

$$\begin{pmatrix}\xi_n(T)\\ \dot{\xi}_n(T)\end{pmatrix}=M(T)\begin{pmatrix}\xi_n(0)\\ \dot{\xi}_n(0)\end{pmatrix}. \tag{23}$$

Consider now the billiard whose boundary is determined by $V(x,y)=1$. The billiard map is given by

$$s_{n+1}=f(s_n,p_n),\quad p_{n+1}=g(s_n,p_n), \tag{24}$$

where $s$ and $p$ are the phase space coordinates for the billiard; $s_i$ is the arclength, and $p_i=\cos\theta_i$ is the tangential component of the momentum at the $i$th bounce. The stability of a periodic orbit, with initial point $(s_0,p_0)$, is determined by considering the behavior of a nearby orbit starting at $(s_0+\delta s_0,p_0+\delta p_0)$, where $\delta s_0$ and $\delta p_0$ are small. It is easy to show, by keeping only linear terms in the expansions for $\delta s_1$ and $\delta p_1$, that after one bounce we have

$$\begin{pmatrix}\delta s_1\\ \delta p_1\end{pmatrix}=m_{1,0}\begin{pmatrix}\delta s_0\\ \delta p_0\end{pmatrix}, \tag{25}$$

where

$$m_{1,0}=\begin{pmatrix}\partial s_1/\partial s_0 & \partial s_1/\partial p_0\\ \partial p_1/\partial s_0 & \partial p_1/\partial p_0\end{pmatrix}. \tag{26}$$

Equation (25) is the billiard analog to Eq. (23). An explicit expression for the matrix $m_{1,0}$ was given by Berry.[8] For the case of a period 2 orbit in a billiard [from the point $(s_0,p_0)$ to the point $(s_1,p_1)$ and back] $m_{1,0}$ becomes

$$m_{1,0}=\begin{pmatrix}\rho_{01}K_0-1 & -\rho_{01}\\ K_0+K_1-\rho_{01}K_0K_1 & \rho_{01}K_1-1\end{pmatrix}. \tag{27}$$

In Eq. (27) $\rho_{01}$ is the length of the chord from $(s_0,p_0)$ to $(s_1,p_1)$, while $K_0$ and $K_1$ are the curvatures at $(s_0,p_0)$ and $(s_1,p_1)$, respectively. One should recall here that $m_{1,0}$ is the linearized map for *one bounce*, which corresponds to half the period of our periodic orbit. Likewise the period $T$ appearing in $M(T)$, Eq. (23), is the period of $\psi_k^{mk-2}$ which is half that of $\psi_k$.[4] We see that

$$\mathrm{tr}\ m_{1,0}=\rho_{01}(K_0+K_1)-2. \tag{28}$$

The absolute value of $\mathrm{tr}\ m_{1,0}$ determines the stability of the period 2 orbits for the billiard. If $|\mathrm{tr}\ m_{1,0}|>2$, then the orbit is unstable.[8] A formula analogous to Eq. (28) was derived by Yoshida for the trace of the monodromy matrix in Eq. (23). We discuss it in the next section, as it applies to our sequence of Hamiltonians, and will compare it to Eq. (28).

## IV. YOSHIDA'S THEOREMS, INSTABILITY, AND NONINTEGRABILITY

Yoshida, in a series of papers,[4–6] proved certain theorems relating to the stability of straight-line periodic orbits, and the nonintegrability of Hamiltonian systems with homogeneous potentials. We collect some results of Yoshida in the form of the two theorems below.

We consider a Hamiltonian system with two degrees of freedom, and a homogeneous potential of *even* positive integer degree $mk$. By the assumption of the homogeneity of the potential, Hamilton's equations have, in general, at least two straight line periodic solutions[6] (see also Sec. V). The normal variational equation of such a solution is given by Eq. (19). Yoshida[5] has shown that $\psi_k(t)$ has $k$ independent periods (in the complex $t$ plane) and thus we have $k$ monodromy matrices. Under the above conditions we have the following theorem:

**Theorem I:**[4] The trace of any monodromy matrix $M$ of the normal variational equation (19), is given by[5]

$$\mathrm{tr}\ M=2\,\frac{\cos[(\pi/2mk)\sqrt{(mk-2)^2+8mk\lambda_k}]}{\cos[(mk-2)(\pi/2mk)]}. \tag{29}$$

The periodic solution $(x(t),y(t))=\psi_k(t)(x_0,y_0)$, with $\psi_k(0)=1$ and $d\psi_k/dt(0)=0$, is unstable if $|\mathrm{tr}\ M|>2$.

Yoshida[5] also proved, for the type of system under consideration, that under certain conditions, the existence of an exponentially unstable straight-line periodic solution implies the nonintegrability of the system, i.e., the nonexistence of an additional global analytic first integral. We now summarize some arguments of Yoshida[5] based on results of Ziglin,[9] in the form of Theorem II below.

**Theorem II:**[5] If a Hamiltonian with two degrees of freedom and a homogeneous potential of even (positive) degree has a straight-line periodic solution for which the trace of the monodromy matrix is greater than 2, then the Hamiltonian system is nonintegrable.

## V. THE GENERAL HOMOGENEOUS CASE

We consider again the class of non-negative smooth homogeneous potentials with even degree $m$. We assume that the physical region $Q=\{(x,y)|V(x,y)\leqslant 1\}$ is compact with nonvanishing gradient on the boundary. It follows from homogeneity that the gradient is nonvanishing in $Q$ except at the origin. As before we let

$$H_k=\tfrac{1}{2}(p_x^2+p_y^2)+V_k(x,y)=1, \tag{30}$$

Since $\partial Q=\{(x,y)|V(x,y)=1\}$ is compact and smooth, there is a point, say $(x_0,y_0)$, on $\partial Q$ closest to the origin, and this point has (nonvanishing) gradient proportional to the vector from the origin to $(x_0,y_0)$. From the homogeneity of $V$ the same is true for the point $-(x_0,y_0)$ on $\partial Q$. From this it follows that the straight line segment from $(x_0,y_0)$ to $-(x_0,y_0)$ is the trajectory of a straight-line periodic orbit for the Hamiltonian $H_1$. The same argument shows that the farthest point from the origin is the end point of a straight-line periodic orbit. Thus there are at least two straight-line periodic orbits for this class of Hamiltonians.

*Lemma I:* If $H_1$ has a straight-line periodic orbit

$$(x,y)=\psi_1(t)(x_0,y_0), \tag{31}$$

with $V(x_0,y_0)=1$, then for every $k$, $H_k$ has a periodic orbit with the same trajectory from $(x_0,y_0)$ to $-(x_0,y_0)$.

*Proof:* From the equations of motion

$$\ddot{x}=-V_x,\quad \ddot{y}=-V_y, \tag{32}$$

we get from the homogeneity of $V$.

$$\begin{aligned} x_0\ddot{\psi}_1(t)&=-\psi_1^{m-1}(t)V_x(x_0,y_0),\\ y_0\ddot{\psi}_1(t)&=-\psi_1^{m-1}(t)V_y(x_0,y_0). \end{aligned} \tag{33}$$

We have that $\psi_1$ is a solution of

$$\ddot{\psi}(t)+\frac{V_x(x_0,y_0)}{x_0}\,\psi^{m-1}(t)=0 \qquad \text{if } x_0\neq 0,$$

$$\ddot{\psi}(t)+\frac{V_y(x_0,y_0)}{y_0}\,\psi^{m-1}(t)=0 \qquad \text{if } y_0\neq 0. \tag{34}$$

If $x_0,y_0\neq 0$, it follows that

$$\frac{V_x(x_0,y_0)}{x_0}=\frac{V_y(x_0,y_0)}{y_0}. \tag{35}$$

If $x_0=0$, $y_0\neq 0$, Eq. (33) implies that $V_x(x_0,y_0)=0$ and therefore $V_y(x_0,y_0)\neq 0$. Similarly if $x_0\neq 0$, $y_0=0$. Thus in all cases grad $V(x_0,y_0)=a(x_0,y_0)$ for some nonzero constant $a$. Now suppose $\psi_k$ is a solution of

$$\ddot{\psi}(t)+ak\psi^{mk-1}(t)=0, \tag{36}$$

satisfying $\psi(0)=1$, $\dot{\psi}(0)=0$. Using the homogeneity of $V_k=[V]^k$ and its derivatives, it follows that

$$\begin{aligned} x_0\ddot{\psi}_k&=-V_{k,x}(\psi_k x_0,\psi_k y_0),\\ y_0\ddot{\psi}_k&=-V_{k,y}(\psi_k x_0,\psi_k y_0), \end{aligned} \tag{37}$$

and $(x,y)=\psi_k(t)(x_0,y_0)$ is a straight-line periodic solution for $H_k$. ■

From the proof of Lemma I it follows that

$$\begin{aligned} \mathrm{grad}\ V_k(x_0,y_0)&=ak(x_0,y_0)\\ &\equiv\frac{V_x(x_0,y_0)}{x_0}\,k(x_0,y_0)\quad \text{if } x_0\neq 0. \end{aligned} \tag{38}$$

$$\begin{aligned} \mathrm{grad}\ V_k(x_0,y_0)&=ak(x_0,y_0)\\ &\equiv\frac{V_y(x_0,y_0)}{y_0}\,k(x_0,y_0)\quad \text{if } y_0\neq 0, \end{aligned}$$

where $V(x_0,y_0)=1$ and $(x_0,y_0)$ is on the trajectory of a straight-line periodic orbit. For $H_k$ the straight-line periodic solution has the form

$$(x,y)=\psi_k(t)(x_0,y_0), \tag{39}$$

where $\psi_k$ satisfies Eq. (36).

*Lemma II:* The Hessian of $V_k$, denoted Hess$[V_k]$, satisfies the eigenvalue equations

$$\mathrm{Hess}[V_k]\begin{pmatrix}x_0\\ y_0\end{pmatrix}=ak(mk-1)\begin{pmatrix}x_0\\ y_0\end{pmatrix} \tag{40}$$

and

$$\mathrm{Hess}[V_k]\begin{pmatrix}-y_0\\ x_0\end{pmatrix}=[\Delta V_k-ak(mk-1)]\begin{pmatrix}-y_0\\ x_0\end{pmatrix}, \tag{41}$$

where $\Delta V_k$ is the Laplacian of $V_k$.

*Proof:* Equation (40) follows from differentiating Eqs. (39) and (37) with respect to $t$, using the chain rule, and comparing terms. Equation (41) follows from Eq. (40), the fact that the sum of the eigenvalues of Hess$[V_k]$ equals its trace, and the symmetric matrix Hess$[V_k]$ has an orthogonal basis of eigenvectors. ■

Let $(x(t),y(t))$ and $(x(t)+\xi_x(t),y(t)+\xi_y(t))$ be neighboring solutions for the position coordinates of the equations of motion

$$\ddot{x}=-V_{k,x},\quad \ddot{y}=-V_{k,y}, \tag{42}$$

for $H_k$. Note that this may rewritten as Eq. (36). The variational equations for $\xi_x$, $\xi_y$ are

$$\begin{pmatrix} \ddot{\xi}_x \\ \ddot{\xi}_y \end{pmatrix} = -\mathrm{Hess}[V_k(x,y)] \begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix}. \tag{43}$$

which may be written as

$$\begin{pmatrix} \ddot{\xi}_x \\ \ddot{\xi}_y \end{pmatrix} = -\psi_k^{mk-2}(t)\mathrm{Hess}[V_k(x_0,y_0)] \begin{pmatrix} \xi_x \\ \xi_y \end{pmatrix}. \tag{44}$$

Define $\xi_n = -y_0\xi_x + x_0\xi_y$ the normal variation, and $\xi_t = x_0\xi_x + y_0\xi_y$ the tangential variation. In terms of $\xi_n$ and $\xi_t$, Eq. (44) becomes

$$\begin{pmatrix} \ddot{\xi}_n \\ \ddot{\xi}_t \end{pmatrix} = -\psi_k^{mk-2}(t) \begin{pmatrix} -y_0 & x_0 \\ x_0 & y_0 \end{pmatrix} \mathrm{Hess}[V_k(x_0,y_0)] \times \begin{pmatrix} -y_0 & x_0 \\ x_0 & y_0 \end{pmatrix}^{-1} \begin{pmatrix} \xi_n \\ \xi_t \end{pmatrix}. \tag{45}$$

Since the columns of

$$\begin{pmatrix} -y_0 & x_0 \\ x_0 & y_0 \end{pmatrix}^{-1}$$

are eigenvectors for $\mathrm{Hess}[V_k(x_0,y_0)]$, we may apply Lemma II to rewrite Eq. (45) as

$$\begin{pmatrix} \ddot{\xi}_n \\ \ddot{\xi}_t \end{pmatrix} = -\psi_k^{mk-2}(t) \times \begin{pmatrix} \Delta V_k(x_0,y_0) - ak(mk-1) & 0 \\ 0 & ak(mk-1) \end{pmatrix} \times \begin{pmatrix} \xi_n \\ \xi_t \end{pmatrix}. \tag{46}$$

This yields the normal and tangential variational equations

$$\ddot{\xi}_n + [\Delta V_k(x_0,y_0) - ak(mk-1)]\psi_k^{mk-2}\xi_n = 0, \tag{47}$$

$$\ddot{\xi}_t + ak(mk-1)\psi_k^{mk-2}\xi_t = 0. \tag{48}$$

Equations (47) and (48) are not suitable in their present form for Yoshida's trace formula. The time variable in Eqs. (36), (47), and (48) must first be rescaled by a factor of $\sqrt{ak}$. This gives us

$$\ddot{\psi}_k + \psi_k^{mk-1} = 0, \tag{49}$$

$$\ddot{\xi}_n + \lambda_k \psi_k^{mk-2}\xi_n = 0, \tag{50}$$

$$\ddot{\xi}_t + (mk-1)\psi_k^{mk-2}\xi_t = 0, \tag{51}$$

respectively, where the *integrability coefficient* $\lambda_k$ in Eq. (50) is given by

$$\lambda_k = \frac{\Delta V_k(x_0,y_0)}{ak} - (mk-1). \tag{52}$$

Evaluating the Laplacian in Eq. (52) we obtain

$$\lambda_k = [a(x_0^2+y_0^2) - m]k - a(x_0^2+y_0^2) + \frac{\Delta V(x_0,y_0)}{a} + 1, \tag{53}$$

where $a$ is determined by the equation,

$$\mathrm{grad}\ V(x_0,y_0) = a(x_0,y_0). \tag{54}$$

Note that by Lemma 1, $\lambda_k$ depends only on $k$ since the trajectory determined by $(x_0,y_0)$ is the same for each of the Hamiltonians $H_k$.

**Theorem III:** With the above assumptions on the potential $V$.

(a) the integrability coefficient $\lambda_k$ is independent of $k$ and the same for all of the Hamiltonians $\{H_k\}$ if and only if $a(x_0^2+y_0^2) - m = 0$.

(b) If $a(x_0^2+y_0^2) - m = 0$, the traces of the monodromy matrices, for the normal variational equations corresponding to the straight-line periodic orbit whose trajectory is the line segment from $(x_0,y_0)$ to $-(x_0,y_0)$, converge, as $k\to\infty$, to the trace of the matrix $m_{1,0}$ for the limiting billiard.

(c) If $a(x_0^2+y_0^2) - m = 0$ and $\Delta V(x_0,y_0) > a^2(x_0^2+y_0^2)$, then the straight-line periodic orbit whose trajectory is the line segment from $(x_0,y_0)$ to $-(x_0,y_0)$ is an unstable orbit for $H_k$, and $H_k$ is nonintegrable for all (finite) sufficiently large $k$. This orbit is also unstable for the limiting billiard.

(d) If $a(x_0^2+y_0^2) - m > 0$, then the straight-line periodic orbit whose trajectory is the line segment from $(x_0,y_0)$ to $-(x_0,y_0)$ is an unstable orbit for $H_k$ for all (finite) sufficiently large $k$.

*Proof:* Part (a) follows immediately from Eq. (53). For part (b), observe that by Eq. (53), if $a(x_0^2+y_0^2) - m = 0$, then $\lambda_k = \lambda$ is independent of $k$. By direct calculation of the limit of Eq. (29) we obtain

$$\lim_{k\to\infty} |\mathrm{tr}\ M| = 2(2\lambda - 1). \tag{55}$$

It now remains to prove that the trace of the matrix $m_{1,0}$ equals $2(2\lambda-1)$.

From the expression for the curvature, $K$, of $V(x,y) = 1$

$$K = \frac{V_{xx}V_y^2 - 2V_{xy}V_xV_y + V_{yy}V_x^2}{(V_x^2+V_y^2)^{3/2}}, \tag{56}$$

and the fact that $V$ is homogeneous of even degree, it follows that the curvatures $K_0(x_0,y_0)$ and $K_1(-x_0,-y_0)$ appearing in Eq. (28) for tr $m_{1,0}$ are equal. Thus we have

$$\mathrm{tr}\ m_{1,0} = 2(\rho_{01}K - 1), \tag{57}$$

where $K = K(x_0,y_0)$, and $\rho_{01} = 2\sqrt{x_0^2+y_0^2}$. Assuming for the moment that both $x_0$, $y_0 \neq 0$, we have from Eq. (40) with $k=1$, that

$$\begin{aligned} V_{xx}x_0^2 + V_{xy}x_0y_0 &= a(m-1)x_0^2, \\ V_{yy}y_0^2 + V_{xy}x_0y_0 &= a(m-1)y_0^2. \end{aligned} \tag{58}$$

Using Eqs. (54) and (58) in Eq. (56) and then substituting the result in Eq. (57) we obtain

$$\mathrm{tr}\ m_{1,0} = 2(2\lambda - 1). \tag{59}$$

If, say, $x_0 \neq 0$ but $y_0 = 0$, then we must have $V_x(x_0,y_0) \neq 0$ and $V_y(x_0,y_0) = 0$, and we obtain again Eq. (59); likewise if $y_0 \neq 0$ and $x_0 = 0$. Part (c) follows from the fact that $\lambda > 1$ and that for $k$ sufficiently large $|\mathrm{tr}\ M| > 2$ by Eq. (55). We then use Theorem II and the comments following Eq. (28).

Part (d) follows from Eq. (55), the arguments for part (c), and the fact that $\lambda_k$ diverges to infinity. ■

*Remark:* It is easy to show by direct calculation that for straight-line periodic orbits along the $x$ and $y$ axes of a polynomial potential, $V=A_0x^m+A_1x^{m-1}y+\cdots+A_my^m$, $m$ even integer, $a(x_0^2+y_0^2)-m=0$, and thus $\lambda_k$ is independent of $k$. The same is true for *all* straight-line periodic solutions of potentials of the form $x^m+y^m$, $m$ even integer.

## VI. THE ELLIPTIC CASE

As an example we now consider the potential energy

$$V=\frac{x^2+\lambda y^2}{2}, \quad \lambda>1, \tag{60}$$

which is homogeneous of degree $m=2$ and thus satisfies the conditions in Theorem III. The Hamiltonians in our sequence are

$$H_k=\tfrac{1}{2}(p_x^2+p_y^2)+V_k(x,y)=1, \tag{61}$$

with

$$V_k=\left(\frac{x^2+\lambda y_2}{2}\right)^k, \tag{62}$$

where $V_k$ satisfies $V_k(\alpha x,\alpha y)=\alpha^{2k}V_k(x,y)$. The Hamiltonians in the above sequence have two straight-line periodic solutions,

$$y=p_y=0 \tag{63}$$

and

$$x=p_x=0. \tag{64}$$

We shall refer to the solutions of Eqs. (63) and (64) as the *long* and *short* straight-line periodic solutions, respectively. For the long straight-line periodic solution $x_0=\sqrt{2}$, $y_0=0$, and so from Eqs. (54) and (53) we find that $a=1$ and $\lambda_k=\lambda$.

By Theorem III (b) we have that $\lim_{k\to\infty}|\mathrm{tr}\ M|=2(2\lambda-1)=\mathrm{tr}\ m_{1,0}$. Since $\lambda>1$, it follows by Theorem III (c) that for all $k$ sufficiently large the long straight-line periodic orbit is unstable, and $H_k$, Eq. (61), is not integrable.

For the *short* straight-line periodic orbit we have that $x_0=0$, $y_0=\sqrt{2/\lambda}$, $a=\lambda$, thus $\lambda_k=1/\lambda$, and

$$|\mathrm{tr}\ M|=2\,\frac{|\cos[(\pi/2k)\sqrt{(k-1)^2+(4k/\lambda)}]|}{\cos[(k-1)(\pi/2k)]}. \tag{65}$$

In this case we find that $\lim_{k\to\infty}|\mathrm{tr}\ M|=2|2/\lambda-1|=|\mathrm{tr}\ m_{1,0}|$. Clearly, $2|2/\lambda-1|<2$, for $\lambda>1$, and one can easily show that in this case $|\mathrm{tr}\ M|<2$ for all $k\geqslant 1$, which suggests that the short straight-line periodic orbit is always stable. Hence one would expect that none of the Hamiltonians in our sequence is ergodic, which is consistent with the nonergodicity of the limiting billiard.

From the above considerations we see that the instability of the long straight-line periodic orbit and the stability of the short straight-line periodic orbit are properties that are inherited by the elliptic billiard from the approximating sequence of Hamiltonian systems. The long straight-line periodic orbit of the elliptic billiard is indeed unstable.[1,8] although the elliptic billiard is integrable.

Liouville integrability in an $n$ degree of freedom Hamiltonian requires the existence of $n$ single-valued, independent integrals of the motion in involution, which completely determine all the invariant tori (Lagrangian submanifolds) in the $2n$-dimensional phase space. Likewise an integrable billiard possesses a conserved quantity which determines all the invariant curves in the billiard global section.[8] In the case of the elliptic billiard a discussion of integrability in terms of a Hamiltonian formulation and action-angle variables was given by Kozlov and Treshchëv;[2] this approach works only for the special cases where the billiard can be written as a Hamiltonian with separable coordinates which is very rare. In our case, the Hamiltonian sequence is nonintegrable for

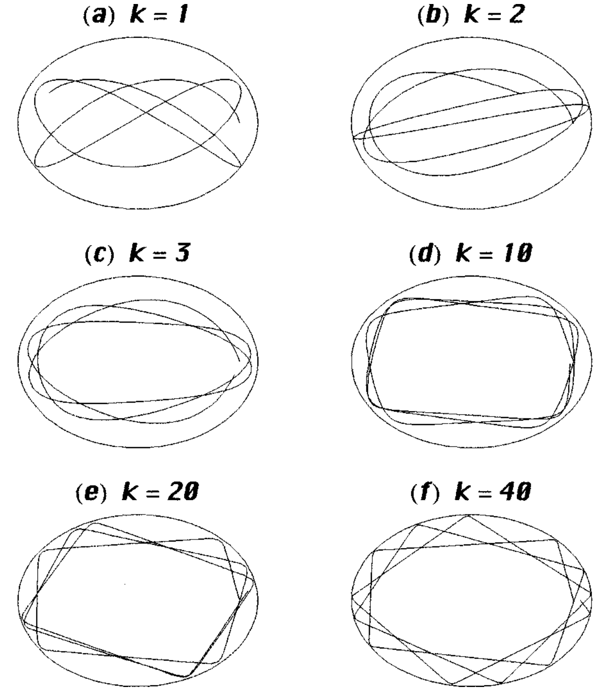


FIG. 1. The orbit with $k=1$ initial conditions $x=1.2$, $y=0$, $p_x=-0.2$, $p_y=0.721\ldots$, of $H_k$, Eq. (66), with $\lambda=2$.

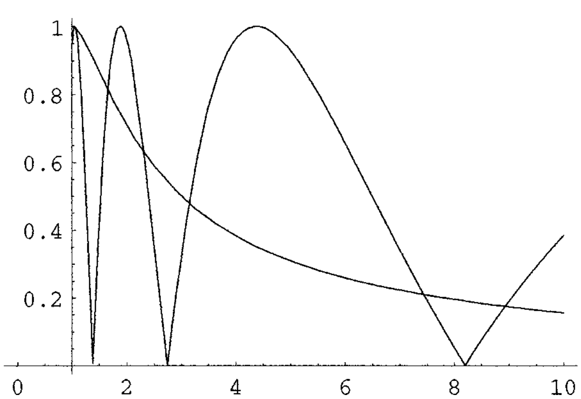


FIG. 2. Plot of the two cosine functions of Eq. (67) (long straight-line orbit) versus $k$ for $\lambda=53/\pi$.

(a)

(b)

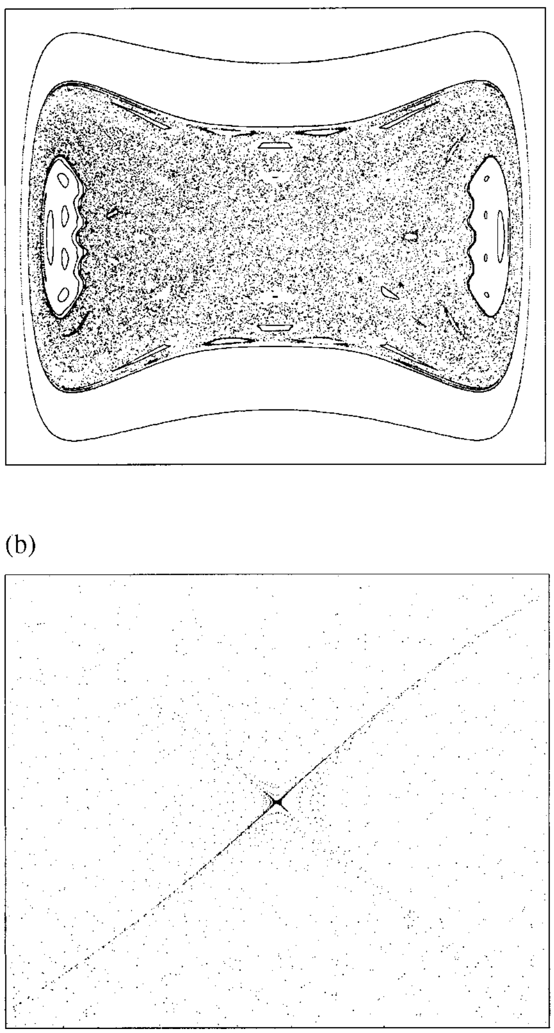

FIG. 3. (a) $x=0$ Poincaré section ($y$, $p_y$ axes) for the elliptic potential, Eq. (62), with $\lambda=53/\pi$ and $k=7.43$. (b) Zoom of the central region where the $y=p_y=0$ long straight-line orbit is unstable.

(a)

(b)

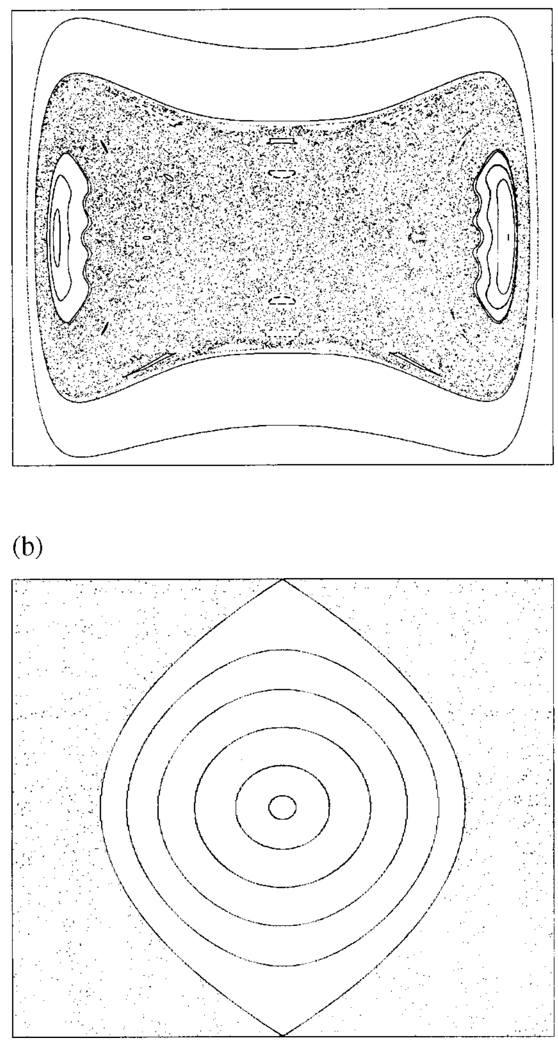

FIG. 4. (a) $x=0$ Poincaré section ($y$, $p_y$ axes) for the elliptic potential, Eq. (62), with $\lambda=53/\pi$ and $k=7.44$. (b) Zoom of the central region where the $y=p_y=0$ long straight-line orbit is now stable.

$k>k_0$, while at $k=\infty$ the elliptic billiard is integrable. Thus in general, nonintegrability is not an inherited property in the Hamiltonian sequence.

We show in Fig. 1 how a typical trajectory changes as we increase the parameter $k$ in the elliptic Hamiltonian sequence

$$H_k=\tfrac{1}{2}(p_x^2+p_y^2)+\left(\frac{x^2+\lambda y^2}{2}\right)^k=1. \tag{66}$$

We define the following mapping for identifying orbits for different values of $k$: An orbit whose initial conditions, for $k=1$, are $(x_1,y_1,p_{x1},p_{y1})$, becomes, for $k>1$, the orbit $(x_1,y_1,p_{xk},p_{yk})$, where $p_{xk}=b_kp_{x1}$ and $p_{yk}=b_kp_{y1}$, and where $b_k$ is chosen so that $H_1(x_1,y_1,p_{x1},p_{y1})=H_k(x_1,y_1,p_{xk},p_{yk})=1$. Thus we only scale the momenta, by the same factor, in order to conserve the energy. In Fig. 1 we show part of the orbit which, for $k=1$, has initial conditions $x=1.2$, $y=0$, $p_x=-0.2$, and $p_y=0.721\,110\,255\,093$, for six different values of the parameter $k$, with $\lambda=2$.

The expression for the trace of the monodromy matrix Eq. (29) was derived by Yoshida for the case where the homogeneity parameter $k$ is an integer, however, our numerical analysis by means of Poincaré sections strongly suggests that Eq. (29) is valid for all $k$ at least for certain potentials $V(x,y)$ which are homogeneous of even positive integer degree. In the case of the long straight-line periodic orbit for the elliptic Hamiltonian, $\lambda_k=\lambda$, and Eq. (29) becomes

$$|\mathrm{tr}\ M|=2\,\frac{|\cos[(\pi/2k)\sqrt{(k-1)^2+4k\lambda}]|}{\cos[(k-1)(\pi/2k)]}. \tag{67}$$

We present a typical set of Poincaré sections. We let $\lambda=53/\pi=16.87...$. Figure 2 is a plot of the functions $|\cos[(\pi/2k)\sqrt{(k-1)^2+4k\lambda}]|$ and $\cos[(k-1)(\pi/2k)]$ appearing in Eq. (67).

As can be seen from Fig. 2, the long straight-line periodic orbit does the stable→unstable, unstable→stable transition several times at every point that the numerator cosine intersects the monotonic denominator cosine of Eq. (67). The values of $k$ at which these intersections occur can be easily

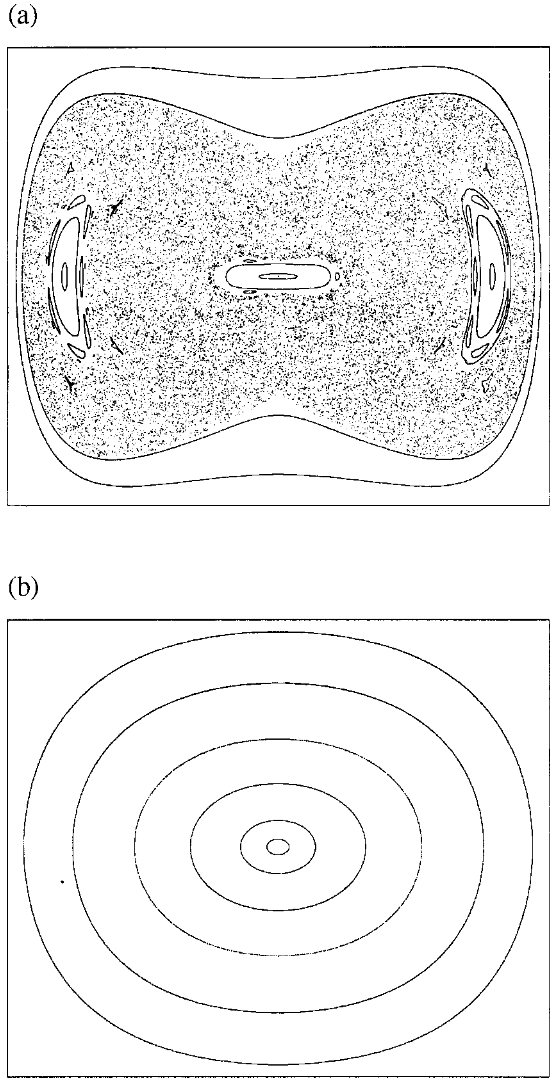


FIG. 5. (a) $x=0$ Poincaré section ($y$, $p_y$ axes) for the potential of Eq. (69) with $k=1.71$. (b) Zoom of the central region where the $y=p_y=0$ long straight-line orbit is stable.

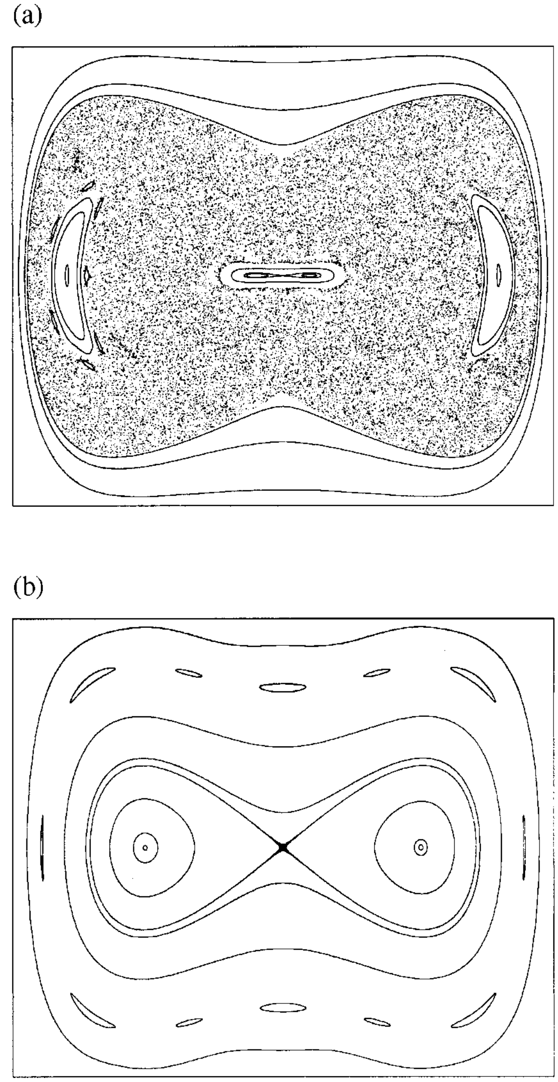


FIG. 6. (a) $x=0$ Poincaré section ($y$, $p_y$ axes) for the potential of Eq. (69) with $k=1.79$. (b) Zoom of the central region where the $y=p_y=0$ long straight-line orbit is now unstable.

obtained from Eq. (67). They are $k=\{1.043\,52,\ 1.072\,54,\ 1.655\,87,\ 2.311\,74,\ 3.145\,07,\ 7.435\,21,\ 8.935\,21\}$, the first one being stable→unstable. The Poincaré sections and related zooms in Figs. 3 and 4 show in detail the situation on either side of the $k=7.435\,21$ transition. The behavior shown in Figs. 3 and 4 is typical and occurs at the other values of $k$.

As an additional example we consider the homogeneous potential of degree $m=4$

$$V=16y^4+12x^2y^2+x^4, \tag{68}$$

thus

$$V_k=(16y^4+12x^2y^2+x^4)^k. \tag{69}$$

The potential of Eq. (68) was shown to be integrable,[10] and like the elliptic one, it has two straight-line periodic orbits along the $x$ and $y$ axes. In this case $m=4$, $\lambda_k=6$, and the trace of the monodromy matrix, Eq. (29), for the periodic orbit along the $x$ axis becomes

$$|\mathrm{tr}\ M|=2\,\frac{|\cos[(\pi/4k)\sqrt{(2k-1)^2+48k}]|}{\cos[(2k-1)(\pi/4k)]}. \tag{70}$$

Using Eq. (70) we find that the periodic orbit along the $x$ axis undergoes a stable→unstable bifurcation at $k=1.75$. This is verified numerically in the Poincaré sections shown in Figs. 5 and 6.

## VII. CONCLUDING REMARKS

Our results provide comparisons of properties of billiards to ''nearby'' Hamiltonian systems and may therefore be viewed in loose analogy to Kolmogorov–Arnold–Moser theory, which considers the inheritability of properties of a given Hamiltonian to nearby Hamiltonians. In our case, the space of Hamiltonians is restricted to those with homogeneous potentials of the type we considered, but the space also includes billiards which may be viewed as highly singular Hamiltonians. There are several avenues open to further research. One could study the behavior of other (non-straight-line) periodic orbits from a sequence of Hamiltonians to a limiting billiard. Hamiltonians in the ''tail'' of our sequences behave like billiards with ''soft'' walls and they have interest in their own right. A different direction would be to investigate alternative deformations of Hamiltonians to billiards. In this way one might be able to establish theorems linking the properties of nearby Hamiltonians to each other when they are in the ''tails'' of different sequences converging to the same billiard. The generalization of our results to Hamiltonians with more than two degrees of freedom along with the limiting higher dimensional billiards is another avenue for further research.

## ACKNOWLEDGMENTS

The authors would like to thank Holger Dullin, Douglas Heggie Martin Lo, and Andreas Wittek for helpful conversations. They are especially grateful to Andreas Wittek for making available to them a custom version of his excellent software for the Power Macintosh.

[1] G. D. Birkhoff, ''On the periodic motions of dynamical systems,'' Acta Math. **50**, 359–379 (1927).

[2] V. V. Kozlov and D. V. Treshchëv, *Billiards. A Genetic Introduction to the Dynamics of Systems with Impacts* (American Mathematical Society translations of Mathematical Monographs Vol. 89, Providence, Rhode Island, 1991).

[3] P. Dahlqvist and G. Russberg, ''Periodic orbit quantization of bound chaotic systems,'' J. Phys. A **24**, 4763–4778 (1991); for example, Ref. 28, p. 687 in Pikovsky *et al*. Chaos **7**, 680 (1997).

[4] H. Yoshida, ''A type of second order linear ordinary differential equations with periodic coefficients for which the characteristic exponents have exact expressions,'' Celest. Mech. **32**, 73–86 (1984).

[5] H. Yoshida, ''Existence of exponentially unstable periodic solutions and the nonintegrability of homogeneous Hamiltonian systems,'' Physica D **21**, 163–170 (1986).

[6] H. Yoshida, ''A criterion for the non-existence of an additional integral in Hamiltonian systems with a homogeneous potential,'' Physica D **29**, 128–142 (1987).

[7] L. D. Landau and E. M. Lifshitz, *Mechanics*, 3rd ed. (Butterworth-Heinemann, Oxford, 1976), p. 22.

[8] M. V. Berry, ''Regularity and chaos in classical mechanics, illustrated by three deformations of a circular billiard,'' Eur. J. Phys. **2**, 91–102 (1981).

[9] S. L. Ziglin, ''Branching of solutions and non-existence of first integrals in Hamiltonian mechanics I,'' Funct. Anal. Appl. **16**, 181–189 (1983); ''Branching of solutions and non-existence of first integrals in Hamiltonian mechanics II,'' *ibid*. **17**, 6–17 (1983).

[10] A. Ramani, B. Dorizzi, and B. Grammaticos, ''Painlevé conjecture revisited,'' Phys. Rev. Lett. **49**, 1539–1541 (1982).