

Acoustic radiation force on a particle in a temperature gradient

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A general expression is derived for the acoustic radiation force on a small spherical particle of radius R in a standing wave field in a temperature gradient. The case of a particle inside a long tube chamber with a temperature gradient along the axis of symmetry is examined in more detail. The analysis is considerably simplified by the introduction of the mass flux density potential ψ . Assuming $kR \ll 1$, and neglecting convection, acoustic streaming, heat conduction, and viscosity effects, an expression is obtained for the force that consists of a "local" version of Gor'kov's result as well as correction terms of order β/k , where $\beta \sim (1/T)(dT/dz)$.

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INTRODUCTION

The forces exerted by sound waves on particles in a fluid or on bubbles in a liquid have been the subject of extensive research both theoretical and experimental.¹ Nonzero average forces arise as a consequence of second-order effects. These acoustic radiation forces play an important role in several areas such as the degassing of liquids, coagulation processes, and aerosol dispersion¹ as well as biological systems.² Recently the theory has been extended by Guz' and Zhuk,³ and Danilov⁴ to include the effects of viscosity and acoustic streaming. We have been concerned with the use of acoustic radiation forces as a positioning (or levitation) technique for the processing of materials mainly in the low gravity environment of space.⁵ For the development of these acoustic processing techniques, it is important to determine the effects of a temperature gradient on the equilibrium position and restoring force on a sample in the sound field.

In this article we consider the acoustic radiation forces on a small spherical object in an inhomogeneous fluid where the inhomogeneity is due to a steady-state temperature gradient. Our analysis parallels that of Gor'kov⁶ for a homogeneous fluid. We first discuss our assumptions, derive the equations which apply to the inhomogeneous case, and go into the details of the approximations made. Then we present a general expression for the time-averaged acoustic force. Our results for the new sample positions and restoring forces for a plane-wave mode are compared to the homogeneous case.

I. THEORY

We derive a general expression for the acoustic force on a small spherical particle of radius R in a monochromatic standing wave field. The particle is inside a gas-filled chamber where a temperature gradient is established. We will examine in more detail the case of a temperature gradient along the chamber axis of symmetry (z axis). We assume that $R \ll \lambda$. We neglect convection, streaming effects, and viscosity, and since the Prandtl number for gases is ~ 1 it will be consistent to also neglect heat conduction.⁷ It is therefore

assumed that the inhomogeneous gaseous medium surrounding the particle is in a state of "local (thermodynamic) equilibrium" with the ambient pressure being constant both in space and time. Thus the inhomogeneity is due to the spatial change in the temperature and the corresponding change in density. This description of course makes sense only in weakly conducting media, and then only relative to the rapid acoustic processes.^{7,8}

We expand the pressure, density, and particle velocity,

$$\begin{aligned} p &= p_0 + p_1 + p_2 + \dots, \\ \rho &= \rho_0 + \rho_1 + \rho_2 + \dots, \\ \mathbf{v} &= \mathbf{v}_1 + \mathbf{v}_2 + \dots, \end{aligned} \quad (1)$$

where the subscripts denote increasing orders of smallness. Here, p_0 and ρ_0 are the ambient pressure and density; thus, for the present analysis, $p_0 = \text{const}$, $\rho_0 = \rho_0(\mathbf{r})$, and $\partial \rho_0 / \partial t = 0$. By using the ideal gas equation of state, continuity equation, Euler's equation, and the adiabatic condition,⁹ we obtain to first order the modified wave equation for the acoustic pressure p_1 (see Refs. 8 and 9)

$$\text{div} \left(\frac{\text{grad } p_1}{\rho_0} \right) - \left(\frac{1}{\rho_0 c_0^2} \right) \frac{\partial^2 p_1}{\partial t^2} = 0. \quad (2)$$

Taking the curl of the first-order Euler's equation, we find that $\text{curl}(\rho_0 \mathbf{v}_1) = \mathbf{f}$, where $\partial \mathbf{f} / \partial t = 0$. Since, however, in the present case the time average $\langle \rho_0 \mathbf{v}_1 \rangle = 0$, we have that $\mathbf{f} = 0$. Therefore, to first order, we may write the mass flux density as the gradient of a mass flux density potential ψ

$$\rho_0 \mathbf{v}_1 = \text{grad } \psi. \quad (3)$$

Using Euler's equation and Eq. (3) and requiring that $p_1 = 0$ when $\psi = 0$, we obtain

$$p_1 = - \frac{\partial \psi}{\partial t}. \quad (4)$$

The potential ψ introduced by Eq. (3) is very useful in simplifying the calculations and in making the analogies between the present case and the uniform temperature case

very clear. From Eqs. (2)–(4), we obtain the modified wave equation for ψ

$$\Delta\psi - \frac{1}{\rho_0} \text{grad } \rho_0 \cdot \text{grad } \psi - \frac{1}{c_0^2} \frac{\partial^2 \psi}{\partial t^2} = 0. \quad (5)$$

The components of the time average of the acoustic force on the sphere are given by¹⁰

$$\langle F_i \rangle = - \int_S \langle \Pi_{ik} \rangle dS_k = - \int_V \left\langle \frac{\partial \Pi_{ik}}{\partial x_k} \right\rangle dV, \quad (6)$$

where the closed surface S encloses the particle. To second order the momentum flux density tensor Π_{ik} is given by

$$\Pi_{ik} = (p_1 + p_2)\delta_{ik} + \rho_0 v_i v_k \quad (7)$$

and using Euler's equation we obtain

$$-\frac{\partial \Pi_{ik}}{\partial x_k} = \rho_0 \frac{\partial v_{1i}}{\partial t} + \rho_0 \frac{\partial v_{2i}}{\partial t} + \rho_1 \frac{\partial v_{1i}}{\partial t} - v_{1i} \frac{\partial (\rho_0 v_{1k})}{\partial x_k}. \quad (8)$$

In the present case, because of the temperature gradient, $p = p(\rho, s)$, where s is the entropy per unit mass, and so the fluid is not barotropic. It is thus far simpler to use Eq. (8) to calculate the force components $\langle F_i \rangle$ than to find an expression for p_2 in Eq. (7). The first two terms on the right-hand side of Eq. (8) vanish after time averaging for the case of monochromatic standing waves.¹¹ The particle velocity and mass flux density potential are composed of an incident (in) and scattered (sc) term

$$\mathbf{v}_1 = \mathbf{v}_1^{\text{in}} + \mathbf{v}_1^{\text{sc}}, \quad \psi = \psi^{\text{in}} + \psi^{\text{sc}}. \quad (9)$$

It can be shown that for any incident acoustic field (with the exception of plane traveling waves) the main contribution to Eq. (6) comes from the interference terms between the incident and scattered radiation,¹² the neglected terms being of order $(R/\lambda)^3$. Thus we restrict our attention to the interference terms only. From the equation of state and adiabatic condition, we have⁸

$$\frac{\partial p_1}{\partial t} = \frac{1}{c_0^2} \frac{\partial p_1}{\partial t} - \mathbf{v}_1 \cdot \text{grad } \rho_0. \quad (10)$$

Integration by parts yields $\langle \rho_1 \partial v_{1i} / \partial t \rangle = - \langle v_{1i} \partial p_1 / \partial t \rangle$, and using Eq. (10) we obtain

$$\langle \mathbf{F} \rangle = - \int_V \left\langle \mathbf{v}_1^{\text{in}} \left[\Delta \psi^{\text{sc}} - \frac{1}{\rho_0} \text{grad } \rho_0 \cdot \text{grad } \psi^{\text{sc}} + \left(\frac{\omega}{c_0} \right)^2 \psi^{\text{sc}} \right] \right\rangle dV. \quad (11)$$

The only contribution to the integral of Eq. (11) comes from a singularity at $r = 0$. This singularity, arising from the scattering terms, is a delta function, $\delta(\mathbf{r})$. Note the close analogy between Eq. (11) and Gor'kov's⁶ Eq. (11) to which the above reduces for uniform temperature. The first and third terms in Eqs. (5) and (11) have the following orders of magnitude:

$$|\Delta \psi^{\text{sc}}| \sim |\psi^{\text{sc}}|/R^2, \quad (\omega/c_0)^2 |\psi^{\text{sc}}| \sim |\psi^{\text{sc}}|/\lambda^2. \quad (12)$$

Since we have assumed $R^2/\lambda^2 \ll 1$, the term $(\omega/c_0)^2 \psi^{\text{sc}}$ can be neglected in Eq. (5) and in the integrand of Eq. (11).

For an ideal gas at constant pressure, we have

$$(1/\rho_0) \text{grad } \rho_0 = - (1/T) \text{grad } T, \quad (13)$$

where T is the local absolute temperature. Using Eq. (13) we may rewrite Eq. (5) to obtain the equation for ψ^{sc}

$$\Delta \psi^{\text{sc}} + (1/T) \text{grad } T \cdot \text{grad } \psi^{\text{sc}} = 0. \quad (14)$$

Comparing the orders of magnitude of the two terms of Eq. (14) we have

$$|(1/T) \text{grad } T \cdot \text{grad } \psi^{\text{sc}}| / |\Delta \psi^{\text{sc}}| \sim (1/T) |\text{grad } T| R. \quad (15)$$

In general, the temperature gradient as viewed from the sphere will be anisotropic. Let us now assume that the temperature gradient sufficiently close to the sphere satisfies $(1/T) |\text{grad } T| R \ll 1$. For example, consider a gradient $T = T(z)$ that is locally linear about the sphere's position ($z_s = 0$), i.e., $T = T_s (1 + \beta z_s)$. Then the estimate (15) becomes $\sim \beta R$. Thus, if $T_s = 300^\circ \text{K}$, $\beta = 0.3 \text{ cm}^{-1}$, and $R = 0.1 \text{ cm}$, then $\beta R \approx 0.03$. In general, the second term in Eq. (14) will thus be significant only for rather steep temperature gradients and for larger R . Consequently, we have neglected this term and the corresponding term in the integrand of Eq. (11) in the present calculation.

The expression for the force on the particle finally reduces to

$$\langle \mathbf{F} \rangle = - \int_V \langle \mathbf{v}_1^{\text{in}} \Delta \psi^{\text{sc}} \rangle dV, \quad (16)$$

where ψ^{sc} is a solution of Laplace's equation $\Delta \psi^{\text{sc}} = 0$. We seek a solution to Laplace's equation in a region $R \ll r \ll \lambda$ because we wish to neglect terms in ψ^{sc} which decrease like $(R/r)^3$ or faster. Therefore, our solution should be valid around $r \gtrsim 5R$. By retaining in ψ^{sc} the "monopole" and "dipole" contributions only, we are able to take into account both the effect of the sphere's compressibility and that of its translational oscillation under the influence of the sound wave. In applying the boundary conditions,^{6,12} we have made use of Eq. (10) that connects ρ_1 to p_1 , since the familiar $p_1 = c_0^2 \rho_1$ does not hold in our case. We find

$$\psi^{\text{sc}} = - \left(\frac{R^3}{3r} \right) \left(f_1 \frac{\partial p_1^{\text{in}}}{\partial t} + (f_1 - 1) \mathbf{v}_1^{\text{in}} \cdot \text{grad } \rho_0 \right) - (\rho_0 R^3/2) f_2 \mathbf{v}_1^{\text{in}} \cdot \text{grad } (1/r), \quad (17)$$

where $f_1 = 1 - \rho_0 c_0^2 / \rho_s c_s^2$, $f_2 = 2(\rho_s - \rho_0) / (2\rho_s + \rho_0)$, and $\rho_0, c_0, \partial p_1^{\text{in}} / \partial t$, and \mathbf{v}_1^{in} are evaluated at the sphere's position. The subscript s denotes properties of the sphere. The surrounding medium is usually a gas; thus $\rho_s \gg \rho_0$ and $c_s^2 \gg c_0^2$ and consequently $f_1 \sim f_2 \sim 1$. Substituting Eq. (17) for ψ^{sc} in Eq. (16) and evaluating the integral yield the general expression for the force on a sphere

$$\langle \mathbf{F} \rangle = \pi R^3 \rho_0 \left\{ - [2f_1/3(\rho_0 c_0)^2] \text{grad} \langle (p_1^{\text{in}})^2 \rangle + f_2 \text{grad} \langle (v_1^{\text{in}})^2 \rangle + 2(f_2 - 2/3)(1/T) \langle (\mathbf{v}_1^{\text{in}} \cdot \text{grad } T) v_1^{\text{in}} \rangle - 2f_2(1/T) \langle (v_1^{\text{in}})^2 \text{grad } T \rangle \right\}. \quad (18)$$

After time averaging, Eq. (18) involves only the space-dependent parts of p_1^{in} and \mathbf{v}_1^{in} . The spatial dependence of the acoustic pressure p_1 is obtained from Eq. (2). The particle velocity is calculated using p_1 and Euler's equation. All of the quantities in Eq. (18) are evaluated at the sphere's posi-

tion. Equation (18) was written in a form which facilitates comparison with Gor'kov's result. The first two terms are formally identical to Gor'kov's expression except that in the present case $\rho_0 = \rho_0(r)$. The grad T terms represent corrections related to the particle's translational oscillations and the modified adiabatic condition Eq. (10). The above way of writing $\langle F \rangle$ is somewhat deceptive because there is a hidden temperature gradient term in the derivative of v_1^{in} . This term is of the same order of magnitude as the other temperature gradient terms. Clearly, Eq. (18) reduces to Gor'kov's result in the absence of a temperature gradient.

II. DISCUSSION

To illustrate the results of this analysis we will now consider the case of a plane standing wave in a long tube chamber with a temperature gradient along the axis of symmetry (z axis), i.e., $T = T(z)$. For the case of a $(0,0,n_z)$ plane-wave mode, we write for the space-dependent parts $p_1^{in} = p(z)$, and $v_1^{in} = v(z)e_z$, where e_z is the unit vector in the z direction. Euler's equation then gives $v(z) = -(1/\omega\rho_0)dp/dz$. For the $(0,0,n_z)$ mode and $f_1 = f_2 = 1$, Eq. (18) becomes

$$\langle F_z \rangle = \frac{\pi R^3}{\rho_0 c_0^2} \frac{dp}{dz} \left(-\frac{2}{3}p + \frac{1}{k^2} \frac{d^2 p}{dz^2} + \frac{1}{3k^2 T} \frac{dT}{dz} \frac{dp}{dz} \right). \quad (19)$$

The first two terms of Eq. (19) are a "local" version of Gor'kov's result and the third term represents a correction proportional to the temperature gradient. Since $dp/dz \sim pk$, and $\beta \sim (1/T)(dT/dz)$, the dT/dz -dependent term of Eq. (19) will be of order β/k relative to the other terms. Thus the correction term will be more important for the lower n_z modes.

We have solved the acoustic pressure wave equation, Eq. (2), numerically for a temperature gradient of the form¹³

$$T = T_0 \{1 + b \tanh[\alpha(z - z_0)]\}. \quad (20)$$

This is a phenomenological gradient that approximates the gradient in a long chamber of length l_z having a cold end at a temperature T_c and a hot end at a temperature T_h . Here, $z = z_0$ is the position where the slope of the gradient is maximum, $(dT/dz)_{max} = T_0 \alpha b$, and T_0 and b are determined by the conditions $T(z=0) = T_c$, and $T(z=l_z) = T_h$.

To illustrate the results of this analysis we show in Fig. 1 the force F_z for a (002) plane wave propagating along the z axis.¹⁴ The equilibrium sample positions for the homogeneous temperature case (dashed line) are $z_e/l_z = 0.25$ and 0.75 . The main effects of the temperature gradient are: (a) the sample equilibrium positions are shifted to lower chamber positions ($z_e/l_z = 0.192, 0.676$) due primarily to the increase of the sound wavelength as a function of temperature, (b) the restoring force at z_e ($\propto dF_z/dz$) is increased at the lower equilibrium position and is decreased at the upper equilibrium position, and (c) the maximum restoring forces in the high-temperature region are considerably less than in the low-temperature region. A more in-depth analysis shows that the sample equilibrium positions

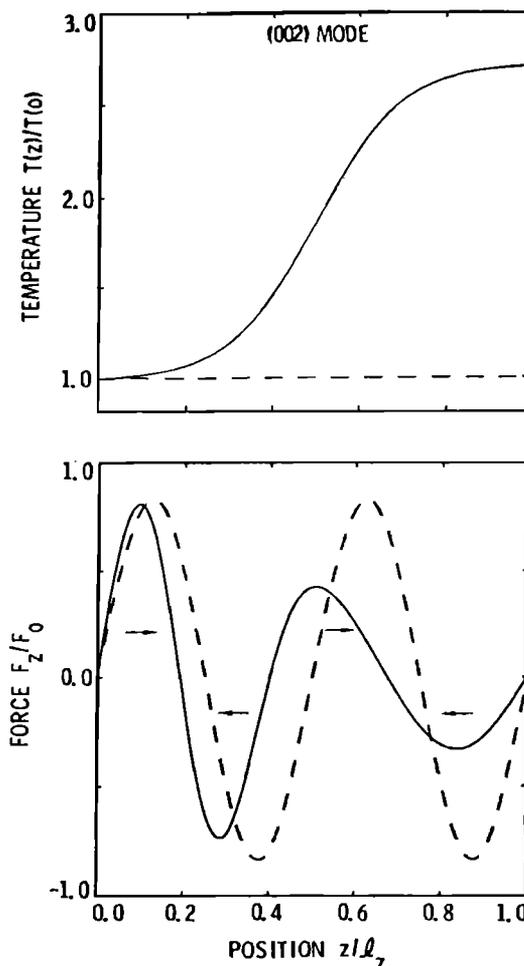


FIG. 1. Position dependence of the temperature gradient and force on a rigid sphere for the (002) z -axis plane wave. The dashed and solid curves correspond to uniform temperature and a 500 °K gradient, respectively. The temperature gradient corresponds to Eq. (20) with $z_0/l_z = 0.5$, $\alpha l_z = 0.2$, $T_0/T_c = 1.92$, $T_h = 800$ °K, and $T_c = 300$ °K.

and the associated restoring forces in a temperature gradient depend critically on the magnitude and sign of the quantity $z_0 - z_i$, where z_i is the initial sample equilibrium position in the homogeneous temperature case and z_0 is the steepest temperature gradient position.

III. CONCLUSION

In conclusion, we have developed a general expression for the acoustic radiation force on a particle in a temperature gradient using an approach similar to Gor'kov's.⁶ The novel features of this theoretical analysis are: (a) the introduction of the mass flux density potential ψ , since the usual velocity potential does not hold, and (b) the analysis of the scattering from a sphere in an anisotropic medium. Finally, we would like to point out that while an analysis was carried out for a z -dependent temperature gradient, the force expression in Eq. (18) is quite general and can be used to easily obtain expressions for the force valid for other types of temperature gradients. A more detailed and complete analysis of the force in various temperature gradients will be published later.

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