LETTERE AL NUOVO CIMENTO VOL. 21, N. 2 14 Gennaio 1978

Simple Tests for Proposed Interior Kerr Metrics.

P. COLLAS

Department of Physics and Astronomy
California State University - Northridge, Cal. 91330

(ricevuto il 24 Agosto 1977; manoscritto revisionato ricevuto il 21 Novembre 1977)

An extremely important problem in astrophysics is that of obtaining a metric for the interior of a rotating body. This is, for example, necessary for the description of the gravitational field (along with pressure and energy density) inside a rotating star. A special case of interest is to seek a *physical*, stationary and axissymmetric solution which can be matched—more or less smoothly [1]—to the Kerr (exterior) solution and which describes a rotating *perfect* or *nonperfect* (anisotropic pressure) *fluid*. Although straightforward in principle, the problem is very difficult [2].

HERNANDEZ [1] outlined a method for constructing *exact* interior solution which might serve as sources for the Kerr metric; he later [3] reformulated his method making use of Boyer-Lindquist co-ordinates [4] and proposed a solution. Briefly, the Hernandez method consists of guessing certain arbitrary functions which appear in a generalization of the Kerr metric; the guessed metric matches the Kerr metric on a suitable surface [5] and, in the limit of no rotation, goes into the interior Schwarzschild solution. Other generalizations of the Kerr metric which do not have the interior Schwarzschild as a limiting case may, of course, be written and examined [5,6].

Ultimately, one must calculate the physical components of the stress-energy tensor in a locally nonrotating frame (LNRF) [7] to find out whether one is dealing with a physically meaningful fluid. This calculation is very tedious even with the aid of a computer.

In this note we wish to point out simple tests which enable one to investigate and if necessary discard bad metrics without any loss of time. This method consists of

(1) For the precise conditions, see, W. C. HERNANDEZ *jr.*: *Phys. Rev.*, **159**, 1070 (1967).

(2) J. B. HARTLE: *Astrophys. Journ.*, **150**, 1005 (1967) gives the components of the Ricci tensor in his appendix.

(3) W. C. HERNANDEZ *jr.*: *Phys. Rev.*, **167**, 1180 (1968).

(4) R. H. BOYER and R. LINDQUIST: *Journ. Math. Phys.*, **8**, 265 (1967).

(5) A. KRASINSKI: Institute of Astronomy, Polish Academy of Sciences preprint No. 63, Warsaw (May 1976), has shown that the surface of a source of the Kerr metric should be given by $r = \text{const}$ in Boyer-Liquist co-ordinates.

(6) P. COLLAS and J. K. LAWRENCE: *General Relativity and Gravitation*, **7**, 715 (1976).

(7) J. M. BARDEEN: *Astrophys. Journ.*, **162**, 71 (1970), sect. **6**, and appendices A, B, and C; J. M. BARDEEN and R. V. WAGONER: *Astrophys. Journ.*, **167**, 359 (1971), sect. **2** and **9**; J. M. BARDEEN, W. H. PRESS and S. A. TEUKOLSKY: *Astrophys. Journ.*, **178**, 347 (1972), sect. **3**.

examining the behavior of the red-shift observed at infinity for photons emitted in an LNRF as this frame approaches the center of symmetry from different directions. LNRF are defined so as to cancel out, as much as possible, the frame-dragging effects due to mass' rotation, and consequently physical processes analysed in such frames appear far simpler than in other kinds of co-ordinate frames. As can be seen in the examples below singularities, event horizons, and other anomalies show up readily.

Consider the *stationary* and *axissymmetric* line element

$$\mathrm{d}s^2 = -\exp[2\nu]\,\mathrm{d}t^2 + \exp[2\psi](\mathrm{d}\varphi - \omega\,\mathrm{d}t)^2 + \exp[2\mu_2]\,\mathrm{d}r^2 + \exp[2\mu_3]\,\mathrm{d}\theta^2\,, \tag{1}$$

where the five metric functions ν, ψ, ω, μ_2 and μ_3 depend only on r and θ. In addition, the metric is assumed to be *asymptotically flat*; thus at spatial infinity ν, ω, and μ_2 must vanish. It is easy to show ([7]) that the red-shift observed at infinity for photons emitted in an LNRF at (r, θ) is given by

$$z = \exp[-\nu]\left(1 - \omega\frac{L}{E}\right) - 1\,, \tag{2}$$

where L is the photon's component of angular momentum parallel to the symmetry axis and E is the photon's energy (both conserved a long null geodesics) ([8]). Furthermore, ω is the angular velocity of the LNRF as seen from infinity, while ν can be considered the general-relativistic gravitational potential ([7]).

Example I. The Kerr metric. – Let us consider the well-known case of the Kerr metric in Boyer-Lindquist ([4]) co-ordinates

$$\mathrm{d}s^2 = -\frac{\Delta - a^2\sin^2\theta}{\varrho^2}\,\mathrm{d}t^2 + \frac{\varrho^2}{\Delta}\,\mathrm{d}r^2 + \varrho^2\,\mathrm{d}\theta^2 + \frac{B\sin^2\theta}{\varrho^2}\,\mathrm{d}\varphi^2 - \frac{4amr\sin^2\theta}{\varrho^2}\,\mathrm{d}\varphi\,\mathrm{d}t\,, \tag{3}$$

where

$$\begin{cases} \Delta = r^2 - 2mr + a^2\,, \\ \varrho^2 = r^2 + a^2\cos^2\theta\,, \\ B = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta\,. \end{cases} \tag{4}$$

In this case eq. (2) for the red-shift is

$$z = \sqrt{\frac{B}{\varrho^2\Delta}}\left(1 - \frac{2amrL}{BE}\right) - 1\,. \tag{5}$$

When $a^2 < m^2$ the larger of the roots of $\Delta = 0$, $r_+ = m + (m^2 - a^2)^{\frac{1}{2}}$, is an event horizon ($z \to \infty$). The vanishing of ϱ^2 for $\theta = \pi/2$, $r = 0$, gives us the ring singula-, rity ([4]). The function B is positive in the physical range of the variables except at $\theta = \pi/2$, $r = 0$, where it vanishes linearly, and therefore it does not affect z. The stationary limit surface does not appear in eq. (5) since it is not an event horizon for our photons.

([8]) Note that since the energy of a physical particle must always be positive as measured by the LNRF observer, we must require $E - \omega L > 0$; for the origin of this condition see the second of ref. ([7]).

When $a^2 > m^2$, $\Delta \neq 0$ in the physical range of the variables, and so there is no event horizon. We have a naked singularity, and the red-shift at $r = 0$ is discontinuous with respect to the angle; *e.g.*, for $L = 0$ photons

$$z\Big|_{\substack{\theta=\pi/2 \\ r=0}} \to \infty\,, \qquad z\Big|_{\substack{\theta\neq\pi/2 \\ r=0}} = 0\,.$$

Example II. Hogan's metric. – HOGAN [9] proposed an interior Kerr solution which, in Boyer-Lindquist co-ordinates, is given by

$$\text{(6)} \qquad \mathrm{d}s^2 = -\mathrm{d}t^2 + (1-f)(\mathrm{d}t + a\sin^2\theta\,\mathrm{d}\varphi)^2 + (r^2+a^2)\sin^2\theta\,\mathrm{d}\varphi^2 + (\varrho^2/\chi)\,\mathrm{d}r^2 + \varrho^2\,\mathrm{d}\theta^2\,,$$

where ϱ^2 is given by eq. (4),

$$\chi = r^2 - qR^2\varrho^2 + a^2\,, \qquad f = \left(\tfrac{3}{2}\sqrt{1-qb^2} - \tfrac{1}{2}\sqrt{1-qR^2}\,\right)^2, \qquad R = \varrho^2/r\,,$$

b is a constant, $q = 2m/b^3$ and $b > 2m$. Line element (6) matches continuously to (3) on the closed 2-surface

$$r = \frac{b}{2}\left(1 + \sqrt{1 - \frac{4a^2\cos^2\theta}{b^2}}\right), \qquad 0 \leqslant \varphi < 2\pi\,,$$

where

$$(b/2)^2 > m^2 \geqslant a^2\,.$$

Moreover for $a = 0$ it becomes the interior Schwarzschild solution for a homogeneous sphere of perfect fluid of radius b.

Unfortunately it is easy to see that line element (6) becomes complex in the physical range of the variables. The function f can be rewritten as follows:

$$\text{(7)} \qquad f = \frac{1}{2}\left[5 - \frac{qm}{b} - \frac{m(r^2+a^2\cos^2\theta)^2}{b^3r^2} - 3\sqrt{\left(1-\frac{2m}{b}\right)\left(1 - \frac{2m(r^2+a^2\cos^2\theta)^2}{b^3r^2}\right)}\,\right].$$

The term $-2ma^4\cos^4\theta/(b^3r^2)$ in the square root will grow, for $\theta \neq \pi/2$, as $r \to 0$ and f will become complex. The red-shift (we consider $L = 0$ photons for simplicity) is given by

$$z = \left[\frac{r^2 + a^2 + (1-f)a^2\sin^2\theta}{\varrho^2 f + a^2\sin^2\theta}\right]^{\frac{1}{2}} - 1\,,$$

and, likewise, becomes complex (it is interesting that for $\theta = \pi/2$, z is well-behaved all the way to $r = 0$).

Example III. Another interior Kerr metric. Consider the line element (again in Boyer-Lindquist co-ordinates)

$$\text{(8)} \qquad \mathrm{d}s^2 = -\frac{\Delta_0 - a^2\sin^2\theta}{\varrho^2}\,\mathrm{d}t^2 + \frac{\varrho^2}{\Delta_1}\,\mathrm{d}r^2 + \varrho^2\,\mathrm{d}\theta^2 + \frac{B\sin^2\theta}{\varrho^2}\,\mathrm{d}\varphi^2 - \frac{4af_0\sin^2\theta}{\varrho^2}\,\mathrm{d}\varphi\,\mathrm{d}t\,,$$

---

[9] P. A. HOGAN: *Lett. Nuovo Cimento*, **16**, 33 (1976).

where ϱ^2 is given by eq. (4), and

$$(9) \quad \begin{cases} \Delta_i = r^2 + a^2 - 2f_i\,, & i = 0, 1\,, \\ B = (r^2 + a^2)^2 - a^2 \Delta_0 \sin^2\theta\,, \\ f_0 = \dfrac{r^2}{4}\left[\dfrac{mr^2}{r_0^3} + \dfrac{qm}{r_0} + 3\sqrt{\left(1 - \dfrac{2m}{r_0}\right)\left(1 - \dfrac{2mr^2}{r_0^3}\right)} - 3\right], \\ f_1 = \dfrac{mr^4}{r_0^3}\,. \end{cases}$$

We require that $r_0 > qm/4$. Line element (8) is a slight generalization of the line element investigated by LAWRENCE and the author [6]; it matches continuously the Kerr metric (3) on the ellipsoid $r = r_0$ [5] since

$$f_i(r_0) = mr_0\,, \qquad i = 0, 1$$

and

$$\left.\frac{\mathrm{d}f_0}{\mathrm{d}r}\right|_{r=r_0} = m$$

(in fact it satisfies Hernandez' continuity conditions [1]). Furthermore line element (8) for $a = 0$ becomes the interior Schwarzschild solution for a sphere of radius r_0.

The red-shift (again $L = 0$ for simplicity) is now given by

$$(10) \qquad z = \sqrt{\frac{B}{\varrho^2 \Delta_0}} - 1\,.$$

In this case $\Delta_0 > 0$, and $B \geqslant 0$ in the physical range of the variables; B vanishes only at $\theta = \pi/2$, $r = 0$, where, now, it vanishes quadratically and again it does not affect z. Therefore there is no event horizon. The ring singularity at $\varrho^2 = 0$ is still there, unfortunately, and gives rise to finite red-shift at $r = 0$, but discontinuous in θ:

$$(11) \qquad z\Big|_{\substack{\theta\neq\pi/2\\ r=0}} = 0\,, \qquad z\Big|_{\substack{\theta=\pi/2\\ r=0}} = \sqrt{1 + 2\alpha} - 1\,,$$

where

$$\alpha = \frac{3}{4}\left(\frac{3m}{r_0} - 1 + \sqrt{1 - \frac{2m}{r_0}}\right).$$

One may, of course, choose a different f_0; for example in f_0 with the property $f_0 = O(r^c)$ as $r \to 0$, $c > 2$, would make $z = 0$ at $r = 0$ and continuous. However, that does not eliminate the trouble. This is because $z(r, \theta)$ for any given r, θ *must either depend on the parameter a, or, otherwise, be equal to the nonrotating limit z at that point.* For the present case the nonrotating limit is the interior Schwarzschild metric, and we obtain

$$z|_{r=0}=\frac{2}{3\sqrt{1-\dfrac{2m}{r_0}}-1}-1\,,$$

which is positive definite (and becomes infinite for $r_0 = qm/4$ as expected).

HERNANDEZ [8] gives an example of an interior Kerr metric which for sufficiently small angular-momentum parameter a has only a co-ordinate singularity along the symmetry axis ($\theta = 0$). It can be shown that this singularity arises from the vanishing of $g_{\varphi\varphi} = \exp[2\psi]$. The circumference of a circle around the axis of symmetry as measured in a LNRF is [7] $2\pi(g_{\varphi\varphi})^{\frac{1}{2}} = 2\pi\exp[\psi] \equiv 2\pi r \sin\theta \exp[\alpha(r,\theta)]$, and therefore we see that it is expected to vanish at $\theta = 0$.

In conclusion, we would like to emphasize that these tests can be applied to *any* metric of the form (1) which is *asymptotically flat.*

* * *

I am grateful to J. K. LAWRENCE for stimulating and helpful discussions.

Direttore responsabile: CARLO CASTAGNOLI

Stampato in Bologna della Tipografia Compositori coi tipi della Tipografia Monograf
Questo fascicolo è stato licenziato dai torchi il 9-I-1978