General relativity in two- and three-dimensional space–times

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We consider the general relativity field equations in two- and three-dimensional space–times. We find that in a two-dimensional space–time we can have curvature but not matter. In a three-dimensional space–time we find that empty space must be flat, that a de Sitter solution exists, and that finite mass distributions with constant surface density must have zero "surface tension." Finally, an expanding dust-filled universe turns out to be like Milne's model.

INTRODUCTION

Because of the complexity of general relativity theory, there are few problems that are both simple enough for a first course in the subject, and also interesting and instructive to both students and teachers. We have found, however, that solving the general relativity field equations in a space–time of reduced dimensionality[1] is rather simple but yields some amusing results that are of pedagogical and scientific interest, and yet are apparently unfamiliar to most physicists. Although naturally the models are unphysical, the mathematical techniques and general reasoning are the same as for the full four-dimensional space–time. Therefore, the exercise is very instructive while at the same time it offers some insight into the effect of changing the dimensionality.

We begin by examining the simplest nontrivial case, namely, the two-dimensional space–time. We investigate the field equations for the most general Riemannian metric. Then we proceed to the three-dimensional space–time, where, after deriving a result which holds for any metric, we restrict ourselves to metrics possessing central (or "circular") symmetry. We consider exterior and interior solutions to a surface mass distribution, and finally we look at a cosmological model.

I. TWO-DIMENSIONAL SPACE–TIME

The most general two-dimensional Riemannian metric is given by the line element

$$ds^2 = a(x,t)dt^2 + b(x,t)dx^2 + 2c(x,t)dxdt, \qquad (1)$$

where a, b, and c are functions of x and t. We can eliminate c the standard way by defining a new time

$$d\bar{t} = \tau(x,t)[a(x,t)dt + c(x,t)dt] \qquad (2)$$
$$\equiv (\partial\bar{t}/\partial t)dt + (\partial\bar{t}/\partial x)dx,$$

where τ is chosen to make the right-hand side of Eq. (2) an exact differential; thus,

$$\partial^2\bar{t}/\partial x\partial t = \frac{\partial}{\partial x}[\tau(x,t)a(x,t)]$$
$$= \frac{\partial}{\partial t}[\tau(x,t)c(x,t)] \qquad (3)$$

Equation (3) can be solved to yield τ, so that in conjunction with Eq. (2) line element (1) becomes

$$ds^2 = \tau^{-2}a^{-1}d\bar{t}^2 - (c^2a^{-1} - b)dx^2;$$

or, letting, for simplicity,

$$\bar{t} = t, \quad e^{2\nu} = \tau^{-2}a^{-1}, \quad e^{2\lambda} = c^2a^{-1} - b,$$

we write

$$ds^2 = e^{2\nu}dt^2 - e^{2\lambda}dx^2, \qquad (4)$$

where $\nu = \nu(x,t)$, and $\lambda = \lambda(x,t)$.

The nonvanishing elements of the metric tensor and its inverse are

$$g_{tt} = e^{2\nu}, \; g_{xx} = -e^{2\lambda},$$
$$g^{tt} = e^{-2\nu}, \; g^{xx} = -e^{-2\lambda}. \qquad (5)$$

Using the formula

$$\Gamma^i_{kn} = \frac{1}{2}g^{im}\left(\frac{\partial g_{mk}}{\partial x^n} + \frac{\partial g_{mn}}{\partial x^k} - \frac{\partial g_{kn}}{\partial x^m}\right), \qquad (6)$$

where each of the latin indices runs through (t,x), and Eq. (5), we obtain the nonvanishing elements of the affine connection:

$$\Gamma^x_{xx} = \lambda', \quad \Gamma^t_{tx} = \Gamma^t_{xt} = \nu', \quad \Gamma^t_{tt} = \dot{\nu},$$
$$\Gamma^x_{tt} = \nu' e^{2\nu-2\lambda}, \quad \Gamma^x_{xt} = \Gamma^x_{tx} = \dot{\lambda}, \quad \Gamma^t_{xx} = \dot{\lambda}e^{2\lambda-2\nu}. \qquad (7)$$

We have used prime to denote $\partial/\partial x$, and the dot to denote $\partial/\partial t$. Now, from

$$R_{ik} = \frac{\partial\Gamma^n_{ik}}{\partial x^n} - \frac{\partial\Gamma^n_{in}}{\partial x^k} + \Gamma^n_{ik}\Gamma^m_{nm} - \Gamma^m_{in}\Gamma^n_{km} \qquad (8)$$

and Eq. (7), we obtain the nonzero components of the Ricci tensor:

$$R^t_t = R^x_x = [\nu'' + \nu'(\nu' - \lambda')]e^{-2\lambda}$$
$$- [\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu})]e^{-2\nu}. \qquad (9)$$

Finally, the field equations in mixed components are

$$R^k_i - \tfrac{1}{2}\delta^k_i R = 8\pi T^k_i, \qquad (10)$$

where we use $G = c = 1$ and the scalar curvature

$$R = R^t_t + R^x_x. \qquad (11)$$

It is easy to see using Eqs. (9)–(11) that

$$T^t_t = T^x_x = 0. \qquad (12)$$

Since the energy-momentum tensor T^k_i vanishes, we can have *no matter in this space–time*. It is also easy to verify by direct calculation that the energy-momentum pseudotensor,[2]

$$t^{ik} = (1/16\pi)\,[(2\Gamma^n_{lm}\Gamma^p_{np} - \Gamma^n_{lp}\Gamma^p_{mn} - \Gamma^n_{ln}\Gamma^p_{mp})(g^{il}g^{km} - g^{ik}g^{lm}) + g^{il}g^{mn}(\Gamma^k_{lp}\Gamma^p_{mn} + \Gamma^k_{mn}\Gamma^p_{lp} - \Gamma^k_{np}\Gamma^p_{lm} - \Gamma^k_{lm}\Gamma^p_{np}) + g^{kl}g^{mn}(\Gamma^i_{lp}\Gamma^p_{mn} + \Gamma^i_{mn}\Gamma^p_{lp} - \Gamma^i_{np}\Gamma^p_{lm} - \Gamma^i_{lm}\Gamma^p_{np}) + g^{lm}g^{np}(\Gamma^i_{ln}\Gamma^k_{mp} - \Gamma^i_{lm}\Gamma^k_{np})],$$

vanishes identically so that, as expected, there is no gravitational field either. On the other hand, R need not vanish since ν and λ are arbitrary functions of x and t. Hence *we can have curvature* (the Gaussian curvature $K = R/2$).

An alternative, quick way to illustrate how the vanishing of T^k_i does not necessitate the vanishing of the Ricci tensor (as it does in higher dimensional space-times) is to contract the field equations (10) on the indices i and k, recalling that in a two-dimensional space-time the trace $\delta^k_k = 2$; thus we obtain

$$R - (2)R/2 = 0 = 8\pi(T^t_t + T^x_x).$$

II. THREE-DIMENSIONAL SPACE-TIME

We begin by considering the field equations (10) for an arbitrary metric. Contracting the field equations on the indices i and k (which now run over t and the polar coordinates r,ϕ), we obtain (recall that in a three-dimensional space-time the trace $\delta^k_k = 3$)

$$R = -16\pi T, \tag{13}$$

where $T = T^t_t + T^r_r + T^\phi_\phi$. Therefore, the field equations can also be written in the form

$$R_{ik} = 8\pi(T_{ik} - g_{ik}T). \tag{14}$$

Here, for empty space ($T_{ik} = 0$), Eq. (14) becomes

$$R_{ik} = 0, \tag{15}$$

and since the curvature tensor R_{ijkm} for a three-dimensional space-time is given by[3]

$$R_{ijkm} = R_{ik}g_{jm} - R_{im}g_{jk} + R_{jm}g_{ik} - R_{jk}g_{im} + (R/2)\,(g_{im}g_{jk} - g_{ik}g_{jm}),$$

we see that Eq. (15) implies $R_{ijkm} = 0$. Therefore, *empty space,* in a three-dimensional space-time, *is flat,* and there can be no gravitational field in it (the energy-momentum pseudotensor also has to vanish).

Let us now restrict ourselves to centrally (or "circularly") symmetric metrics. The most general centrally symmetric expression for the line element is

$$ds^2 = a(r,t)dt^2 - b(r,t)dr^2 + 2c(r,t)drdt - f(r,t)r^2d\phi^2, \tag{16}$$

where a, b, c, and f are functions of r and t. The transformation $\bar{r} = rf^{1/2}$ changes the coefficient of $d\phi^2$ to $\bar{r}^2$, while a transformation like the one given by Eq. (2) removes the cross term $d\bar{r}dt$. Dropping the bars and introducing new functions $\nu = \nu(r,t)$ and $\lambda = \lambda(r,t)$, we rewrite Eq. (16) as

$$ds^2 = e^{2\nu}dt^2 - e^{2\lambda}dr^2 - r^2d\phi^2. \tag{17}$$

The nonvanishing elements of the metric tensor and its inverse are

$$g_{tt} = e^{2\nu},\ g_{rr} = -e^{2\lambda},\ g_{\phi\phi} = -r^2,$$
$$g^{tt} = e^{-2\nu},\ g^{rr} = -e^{-2\lambda},\ g^{\phi\phi} = -r^{-2}. \tag{18}$$

Using Eqs. (6) and (18), we obtain the nonvanishing elements of the affine connection:

$$\Gamma^r_{rr} = \lambda',\quad \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = 1/r,\quad \Gamma^t_{tt} = \dot{\nu},$$
$$\Gamma^r_{\phi\phi} = -re^{-2\lambda},\quad \Gamma^t_{tr} = \Gamma^t_{rt} = \nu',\quad \Gamma^t_{rr} = \dot{\lambda}e^{2\lambda-2\nu},$$
$$\Gamma^r_{tt} = \nu'e^{2\nu-2\lambda},\quad \Gamma^r_{rt} = \Gamma^r_{tr} = \dot{\lambda}. \tag{19}$$

The primes and dots stand for $\partial/\partial r$ and $\partial/\partial t$, respectively. From Eqs. (8) and (19) we obtain the independent nonzero components of the Ricci tensor:

$$R^t_t = [\nu'' + \nu'(\nu' - \lambda' + 1/r)]e^{-2\lambda} + [\dot{\nu}\dot{\lambda} - \ddot{\lambda} - (\dot{\lambda})^2]e^{-2\nu},$$
$$R^r_r = [\nu'' + \nu'(\nu' - \lambda') - \lambda'/r]e^{-2\lambda} + [\dot{\nu}\dot{\lambda} - \ddot{\lambda} - (\dot{\lambda})^2]e^{-2\nu},$$
$$R^\phi_\phi = e^{-2\lambda}(\nu' - \lambda')/r,$$
$$R^r_t = -e^{-2\lambda}\dot{\lambda}/r. \tag{20}$$

Finally, using the field equations (10) with $R = R^t_t + R^r_r + R^\phi_\phi$ and Eq. (20), we obtain for the energy-momentum tensor

$$8\pi T^t_t = e^{-2\lambda}\lambda'/r,$$
$$8\pi T^r_r = -e^{-2\lambda}\nu'/r,$$
$$8\pi T^\phi_\phi = -e^{-2\lambda}[\nu'' + \nu'(\nu' - \lambda')] + e^{-2\nu}[\ddot{\lambda} + \dot{\lambda}(\dot{\lambda} - \dot{\nu})],$$
$$8\pi T^r_t = -e^{-2\lambda}\dot{\lambda}/r. \tag{21}$$

We now examine some specific solutions that we found interesting to compare with their well-known four-dimensional counterparts.

A. Exterior solutions

These have, of course, been obtained at the beginning of this section, where we showed that empty space must be flat. It is easy to see that this result (for the centrally symmetric case) also follows immediately from Eq. (21). Setting $T^i_k = 0$ implies that λ must be independent of r,t and that ν must be independent of r. However, T^ϕ_ϕ will vanish even if $\nu = \nu(t)$; this is no problem since we can transform to a new time $\bar{t} = \int e^\nu\, dt$.

It is interesting at this point to recall Birkhoff's theorem, namely, that any spherically symmetric solution of Einstein's empty-space equations is equivalent to the Schwarzschild solution. (Note that staticness and asymptotic flatness follow in this case from spherical symmetry.) In Newtonian gravitation, the analog of Birkhoff's theorem follows from the spherically symmetric part of Laplace's equation, namely,

$$\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = 0,$$

which has a solution $\Phi = A/r + B$, with A and B arbitrary constants. Letting $B = 0$ and $A = -M$, we obtain the Newtonian gravitation potential outside a spherical mass M. We see that in our three-dimensional space-time we have something stronger than Birkhoff's theorem, in that any solution—no symmetry required—of the empty-space equations is a flat space–time solution. We remark also that in our case this result does not agree with the corresponding Newtonian result. Consider, for example, the radial part of the two-dimensional Laplace's equation

$$\frac{d}{dr}\left(r\frac{d\Phi}{dr}\right) = 0,$$

which has as a solution $\Phi = A \log r + B$.

B. Interior solutions

We shall be concerned only with *perfect fluids*, that is, matter distributions whose energy-momentum tensor is given by

$$T^{ik} = (\sigma + s)u^i u^k - sg^{ik}, \tag{22}$$

where u^i is the velocity four-vector σ is the proper energy (surface) density, and s is the (negative) surface tension; σ and s correspond to the density ρ and pressure p, respectively, of four-dimensional space–time perfect fluids. We also have from $u^i u_i = 1$ and Eq. (18), the relation

$$e^{2\nu}(u^t)^2 - e^{2\lambda}(u^r)^2 - r^2(u^\phi)^2 = 1. \tag{23}$$

Making $s = -\sigma$ is unphysical because this kind of surface tension will act in the same direction as gravity. Suppose, however, that we allow a surface tension $s = -\sigma$; then from Eq. (22) we obtain the nonzero components of the energy-momentum tensor

$$T_t^t = T_r^r = T_\phi^\phi = \sigma.$$

Substituting this result in Eq. (21), we have that $\dot{\lambda} = 0$ and

$$8\pi\sigma = e^{-2\lambda}\lambda'/r, \tag{24}$$

$$8\pi\sigma = -e^{-2\lambda}\nu'/r, \tag{25}$$

$$8\pi\sigma = -e^{-2\lambda}[\nu'' + \nu'(\nu' - \lambda')]. \tag{26}$$

Equations (24) and (25) imply that

$$\nu' = -\lambda', \tag{27}$$

while Eq. (27) along with Eq. (26) yield

$$\nu'' + 2\nu'^2 - \nu'/r = 0.$$

The substitution $\nu' = y$ converts the above differential equation into Bernoulli's equation[4]:

$$y' - y/r = -2y^2,$$

or, dividing by y^2,

$$y^{-2}dy - y^{-1}dr/r = -2dr. \tag{28}$$

Finally, the substitution $z = y^{-1}$ converts Eq. (28) into a linear equation in standard form, in fact an exact equation:

$$rdz + zdr = 2rdr,$$

or,

$$d(rz) = d(r^2).$$

Therefore,

$$\nu = (1/2) \log [b([r^2 + a)],$$

and from Eq. (27)

$$\lambda = -(1/2) \log [bc(r^2 + a)],$$

where a, b, and c are constants of integration. From Eq. (18) we see that

$$g_{tt} = b(r^2 + a), \quad g_{rr} = -[bc(r^2 + a)]^{-1}.$$

Now if our mass distribution is a circle of radius r_0, we must require that

$$g_{tt}(r_0) = 1, \quad g_{rr}(r_0) = -1, \quad \begin{matrix} \sigma > 0 & \text{for } r < r_0 \\ \sigma = 0 & \text{for } r > r_0 \end{matrix}. \tag{29}$$

It is easy to verify using, say, Eq. (25) that the last of conditions (29) cannot be satisfied. We get

$$\sigma = -bc/8\pi.$$

If $bc = 0$, then there is no mass; if, on the other hand, $bc \neq 0$, then the entire space is filled (a cosmology!). In fact the choice $b < 0$, $ba = 1$, $c = 1$, gives us a de Sitter space-time.

Next we seek an "interior Schwarzschild" solution. We let $u^r = u^\phi = 0$ and obtain from Eq. (23).

$$u^t = e^{-\nu}. \tag{30}$$

Substituting Eq. (30) in Eq. (22), we obtain

$$T_t^t = \sigma, \quad T_r^r = T_\phi^\phi = -s, \quad T_t^r = 0. \tag{31}$$

Equations (31) and (21) give

$$8\pi\sigma = e^{-2\lambda}\lambda'/r, \tag{32}$$

$$8\pi s = e^{-2\lambda}\nu'/r, \tag{33}$$

$$8\pi s = e^{-2\lambda}[\nu'' + \nu'(\nu' - \lambda')], \tag{34}$$

where we have discarded the terms containing time derivatives since the problem is static. We can now use the vanishing of the covariant derivative

$$T_{i;k}^k = (\partial T_i^k/\partial x^k) + \Gamma_{mk}^k T_i^m - \Gamma_{ik}^m T_m^k = 0, \tag{35}$$

expressing energy and momentum conservation, with $i = r$ the radial coordinate, to obtain the useful equation

$$(ds/dr) + (\sigma + s)(d\nu/dr) = 0. \tag{36}$$

This equation can also be obtained by differentiating Eq. (33), equating Eq. (33) to Eq. (34), and combining all of these and Eq. (32) to eliminate ν'', λ, and λ'.

We begin by integrating Eq. (32), which yields

$$e^{-2\lambda(r)} = 2A - 16\pi \int_0^r \sigma r dr, \tag{37}$$

where A is the integration constant. Let the radius of the mass distribution be r_0; then since the exterior metric is Minkowski, we must require that

$$e^{-2\lambda(r_0)} = 2A - 16\pi \int_0^{r_0} \sigma r dr = 1. \tag{38}$$

For given σ and r_0 Eq. (38) determines A. In particular if we choose σ to be constant, we have from Eq. (38)

$$A = \tfrac{1}{2} + 4\pi r_0^2\sigma,$$

so that together with Eq. (37) we obtain

$$e^{-2\lambda} = 1 + 8\pi\sigma(r_0^2 - r^2). \tag{39}$$

Next we eliminate the $e^{-2\lambda}$ and the ν' from Eq. (33), using Eqs. (39) and (36), respectively; thus, we obtain

$$8\pi s = -\frac{[1 + 8\pi\sigma(r_0^2 - r^2)]}{r(\sigma + s)}\frac{ds}{dr},$$

which can be integrated to give $s = s(r)$. We have

$$s = \frac{B\sigma[1 + 8\pi\sigma(r_0^2 - r^2)]^{1/2}}{\{(1 + 8\pi\sigma r_0^2)^{1/2} - B[1 + 8\pi\sigma(r_0^2 - r^2)]^{1/2}\}}. \quad (40)$$

We require, as usual, that $s(r_0) = 0$. This, however, implies that $s = 0$ for all r, and hence from Eq. (33) we find that ν must be a constant. To satisfy the boundary condition at r_0, we let $\nu = 0$. Therefore, we see that we cannot have constant σ and $s \neq 0$. This result one may have suspected, since in the absence of gravitational attraction between the particles, a nonvanishing surface tension would make the mass distribution unstable.

C. Cosmology

As a final example, we consider a dust-filled universe; that is, we let $s = 0$, so that

$$T^{ik} = \sigma u^i u^k \quad (41)$$

We shall use comoving coordinates[5] and assume again central symmetry (isotropy). Thus we can write the line element in the form

$$ds^2 = dt^2 - A(r,t)dr^2 - B(r,t)d\phi^2, \quad (42)$$

so that

$$g_{tt} = 1, \ g_{rr} = -A, \ g_{\phi\phi} = -B,$$
$$g^{tt} = 1, \ g^{rr} = -1/A, \ g^{\phi\phi} = -1/B. \quad (43)$$

Since the matter is at rest in the coordinates we are using,

$$u^r = u^\phi = 0, \quad u^t = 1, \quad (44)$$

and therefore Eq. (41) gives simply

$$T_t^t = \sigma, \quad (45)$$

as the only nonvanishing component. Proceeding as usual, we obtain the nonvanishing elements of the affine connection,

$$\Gamma^r_{rr} = A'/2A, \ \Gamma^t_{rr} = \dot{A}/2, \ \Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = B'/2B,$$
$$\Gamma^r_{\phi\phi} = -B'/2A, \ \Gamma^t_{\phi\phi} = \dot{B}/2, \ \Gamma^\phi_{t\phi} = \Gamma^\phi_{\phi t} = \dot{B}/2B,$$
$$\Gamma^r_{rt} = \Gamma^r_{tr} = \dot{A}/2A, \quad (46)$$

and hence the independent nonvanishing elements of the Ricci tensor are

$$R_{tt} = -\frac{\ddot{A}}{2A} - \frac{\ddot{B}}{2B} + \frac{\dot{A}^2}{4A^2} + \frac{\dot{B}^2}{4B^2},$$
$$R_{rr} = -\frac{B''}{2B} + \frac{B'^2}{4B^2} + \frac{A'B'}{4AB} + \frac{\ddot{A}}{2} - \frac{\dot{A}^2}{4A} + \frac{\dot{A}\dot{B}}{4B},$$
$$R_{\phi\phi} = -\frac{B''}{2A} + \frac{A'B'}{4A^2} + \frac{B'^2}{4AB} + \frac{\ddot{B}}{2} + \frac{\dot{A}\dot{B}}{4A} - \frac{\dot{B}^2}{4B},$$
$$R_{tr} = -\frac{\dot{B}'}{2B} + \frac{B'\dot{B}}{4B^2} + \frac{\dot{A}B'}{4AB}. \quad (47)$$

Substituting Eqs. (41) and (45) in Eq. (14), we have

$$R_{ik} = 8\pi\sigma(u_i u_k - g_{ik}),$$

from which we find, using Eqs. (43) and (44),

$$R_{tt} = 0, \ R_{rr} = 8\pi\sigma A, \ R_{\phi\phi} = 8\pi\sigma B, \ R_{tr} = 0. \quad (48)$$

Combining Eqs. (47) and (48), we obtain the four field equations:

$$\left(-B'' + \frac{B'^2}{2B} + \frac{A'B'}{2A}\right)\frac{1}{2AB} + \left(\ddot{A} - \frac{\dot{A}^2}{2A} + \frac{\dot{A}\dot{B}}{2B}\right)\frac{1}{2A} = 8\pi\sigma, \quad (49)$$

$$\left(-B'' + \frac{B'^2}{2B} + \frac{A'B'}{2A}\right)\frac{1}{2AB} + \left(\ddot{B} - \frac{\dot{B}^2}{2B} + \frac{\dot{A}\dot{B}}{2A}\right)\frac{1}{2B} = 8\pi\sigma, \quad (50)$$

$$-\frac{\ddot{A}}{2A} - \frac{\ddot{B}}{2B} + \frac{\dot{A}^2}{4A^2} + \frac{\dot{B}^2}{4B^2} = 0, \quad (51)$$

$$-\frac{\dot{B}'}{2B} + \frac{\dot{B}B'}{4B^2} + \frac{\dot{A}B'}{4AB} = 0. \quad (52)$$

We now assume that σ is a function of time only and let

$$A = a^2(t)\alpha(r), \ B = b^2(t)\beta(r). \quad (53)$$

Substituting Eq. (53) in Eq. (52), we get

$$\dot{a}/a = \dot{b}/b;$$

therefore we can choose α and β so that $a = b$. We also transform to another radial coordinate by $\bar{r} = \beta^{1/2}$. Finally, dropping the bars, we can rewrite Eq. (53) as

$$A = a^2(t)\alpha(r), \quad B = a^2(t)r^2. \quad (54)$$

Equations (49) and (50) become identical when we substitute the A and B of Eq. (54), and so we obtain

$$\frac{\alpha'}{2r\alpha^2} + \ddot{a}a + \dot{a}^2 = 8\pi\sigma a^2. \quad (55)$$

The first term of Eq. (55) must be a constant, so we define

$$k \equiv \alpha'/2r\alpha^2, \quad (56)$$

which gives us

$$\alpha = (h - kr^2)^{-1}, \quad (57)$$

h being an integration constant.

We can now rewrite the line element (42), using Eqs. (54) and (57), as

$$ds^2 = dt^2 - a^2(t)\left[\frac{dr^2}{(h - kr^2)} + r^2 d\phi^2\right]. \quad (58)$$

Now substituting Eq. (54) in Eq. (51), we obtain

$$\ddot{a} = 0, \quad (59)$$

and using Eqs. (55), (56), and (59), we have

$$k + \dot{a}^2 = 8\pi\sigma a^2. \quad (60)$$

The conservation law of Eq. (35), with $i = t$ the time coordinate, gives us

$$d(\sigma a^2)\,dt = 0,$$

which is satisfied by

$$\sigma = \sigma_0 a_0^2/a^2, \quad (61)$$

where σ_0 and a_0 are constants. Finally, substituting Eq. (61) in Eq. (60) yields

$$\dot{a} = \pm(8\pi\sigma_0 a_0^2 - k)^{1/2}. \tag{62}$$

Integrating Eq. (62) subject to the condition

$$a(0) = 0, \tag{63}$$

we get

$$a = \pm(8\pi\sigma_0 a_0^2 - k)^{1/2} t. \tag{64}$$

We note that only one kind of expanding universe is possible, namely *open,* and it always satisfies Hubble's law, that is, there is no deceleration. This last fact was to be expected since the point masses cannot affect each other gravitationally in this space–time. This cosmological model is essentially the same as Milne's model.[6]

It is easy to calculate the scalar curvature R using Eqs. (43) and (48). We find

$$R = -16\pi\sigma, \tag{65}$$

so that if we use Eq. (61) for σ and Eq. (64) for a in Eq. (65), we have

$$R = -16\pi\sigma_0 a_0^2/(8\pi\sigma_0 a_0^2 - k)t^2. \tag{66}$$

We see that Eq. (66) is singular for $t = 0$.

CONCLUSIONS

We have investigated general relativity in two- and three-dimensional space–times. We found that in a two-dimensional space–time—for a general metric—there is a "decoupling" of the energy–momentum tensor and the Ricci tensor; and although the energy–momentum tensor vanishes, the Ricci tensor remains arbitrary. Therefore, this space–time is *empty, but with arbitrary curvature.*

In a three-dimensional space–time we began by showing that empty space must be flat and therefore masses cannot interact with each other gravitationally. Then we restricted ourselves to metrics with central symmetry. We saw that the correspondence between general relativity and Newtonian gravitation does not hold in these space–times.

We sought interior solutions to circular mass distributions and found for nonvanishing surface tension a de Sitter space–time solution; we also found that a static mass distribution must have zero surface tension, as expected, since otherwise it would be unstable.

Finally, we examined a dust-filled universe cosmology. It turned out that the expansion function $a(t)$ is linear in the time t, and the cosmology has the essential features of Milne's model[6]; that is, we have an expanding universe without gravity and hence without deceleration.

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[1]The excellent papers by R. W. Brehme, Am. J. Phys. **33**, 383 (1965); **33**, 713 (1965); and by R. W. Brehme and W. E. Moore, Am. J. Phys. **37**, 683 (1969), although dealing with two-dimensional space–times, do not examine the topics of this paper.

[2]See, for example, L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields,* 3rd ed. (Pergamon, Oxford, 1971), p. 306.

[3]See Ref. 2, prob. 1, p. 263.

[4]We wish to thank Dr. T. Azzarelli for pointing this out to us.

[5]See Ref. 2, p. 336.

[6]E. A. Milne, Nature **130**, 9 (1932); W. Rindler, *Essential Relativity* (Van Nostrand Reinhold, New York, 1969), pp. 227–232.

[7]See, G. Clement, Nucl. Phys. B **114**, 437 (1976), in particular the Appendix for some additional properties of three-dimensional space-times.