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Algebraic Solution of the Kepler Problem Using the Runge–Lenz Vector

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The Runge–Lenz vector is used to obtain the equation for the Kepler orbits algebraically and the frequency of small oscillations of a particle about a stable circular orbit. The invariance of the Hamiltonian under the Lie algebras generated by the components of the Runge–Lenz and angular momentum vectors is discussed.

I. INTRODUCTION

In this paper we present a new and simple algebraic way of obtaining the equation of the orbit for the Kepler problem. The method requires only elementary Newtonian mechanics and does not involve solving a differential equation or performing any integrations.[1]

This method has been used in mechanics courses at all levels but is particularly suited to a junior level course, in which students will be solving a number of orbit problems by the standard techniques. Apart from its simplicity, the method is intrinsically interesting because it makes use of an unusual conserved quantity, the so-called Runge–Lenz vector. In the corresponding quantum mechanical case, the hydrogen atom, the conservation of the Runge–Lenz vector and the associated invariance under the group $O(4)$ have been used to obtain the spectrum algebraically and to understand the so-called accidental degeneracy.[2] Thus the student may encounter generalizations of the ideas considered here in more advanced courses.

In Sec. II we show that the Runge–Lenz vector is a constant of the motion and subsequently use this fact to obtain the equation of the orbit and the frequency of small oscillations about a perturbed circular orbit. In Sec. III we outline briefly the connection between the conservation of the Runge–Lenz vector and the invariance of the Kepler problem Hamiltonian under the groups $O(4)$ or $O(3, 1)$. The contents of Sec. III are in no way necessary for the results of Sec. II and are included for the benefit of readers who might wish to pursue the subject more deeply.

II. KEPLER'S PROBLEM

We begin by showing that in the case of a potential

$$V(r) = -\alpha/r,$$

the Runge–Lenz vector

$$\mathbf{A}=\mathbf{v}\times\mathbf{l}-\alpha\mathbf{r}/r, \tag{1}$$

is a constant of the motion.[3] In Eq. (1), $\mathbf{v}$ and $\mathbf{l}$ are, respectively, the velocity and orbital angular momentum of a particle of mass μ. Since for central forces $\dot{\mathbf{l}}=0$, the total time derivative of $\mathbf{A}$ is

$$\dot{\mathbf{A}}=\dot{\mathbf{v}}\times\mathbf{l}-\alpha\mathbf{v}/r+\alpha(\mathbf{r}\cdot\mathbf{v})\mathbf{r}/r^3,$$

where we have used also the fact that $\mathbf{r}\cdot\mathbf{v}=r\dot{r}$. Substituting $\mathbf{l}=\mu\mathbf{r}\times\mathbf{v}$ above and using the vector identity $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\mathbf{c}$, we obtain

$$\dot{\mathbf{A}}=(\mu\dot{\mathbf{v}}\cdot\mathbf{v})\mathbf{r}-(\mu\dot{\mathbf{v}}\cdot\mathbf{r})\mathbf{v}-\alpha\mathbf{v}/r+\alpha(\mathbf{r}\cdot\mathbf{v})\mathbf{r}/r^3.$$

The Newtonian equation of motion is

$$\mu\dot{\mathbf{v}}=-\alpha\mathbf{r}/r^3,$$

and upon substitution in the expression for $\dot{\mathbf{A}}$ we find that $\dot{\mathbf{A}}=0$.

We now proceed to find the equation for the orbit. In cylindrical coordinates we have

$$\mathbf{l}=\mu r^2\dot{\theta}\mathbf{e}_z, \tag{2}$$

and

$$\mathbf{A}=(\mu r^3\dot{\theta}^2-\alpha)\,\mathbf{e}_r-\mu r^2\dot{r}\dot{\theta}\mathbf{e}_\theta, \tag{3}$$

where $\mathbf{e}_r$, $\mathbf{e}_\theta$, $\mathbf{e}_z$ are unit vectors. Clearly the constant vector $\mathbf{A}$ lies in the plane of the motion and determines a direction which for simplicity we choose as our polar axis.

Using $\mathbf{r}=r\mathbf{e}_r$ and Eq. (3), we have

$$\mathbf{r}\cdot\mathbf{A}=rA\cos\theta=r(\mu r^3\dot{\theta}^2-\alpha). \tag{4}$$

Finally, making use of Eq. (2) to eliminate $\dot{\theta}$ from Eq. (4), we obtain the equation for the orbit

$$(l^2/\alpha\mu)(1/r)=(A/\alpha)\cos\theta+1. \tag{5}$$

The eccentricity ϵ and the length of the latus

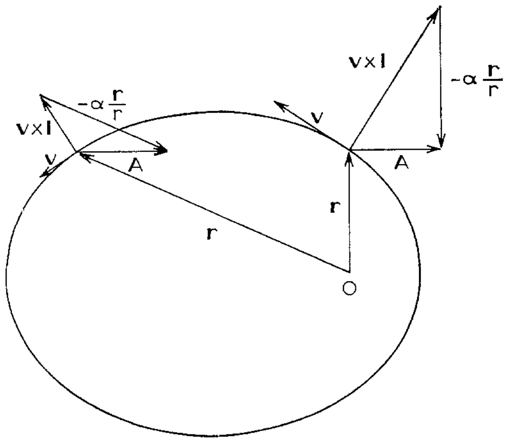

FIG. 1. Arrangement of the vectors for two positions on an elliptical orbit. $A=\alpha\epsilon$, $|\alpha\mathbf{r}/r|=\text{const}$, $|\mathbf{v}\times\mathbf{l}|=vl\sim v$, l being constant.

rectum ξ are given by

$$\epsilon=A/\alpha, \qquad \xi=2l^2/(\alpha\mu).$$

From Eq. (4) we see that $\mathbf{A}$ is parallel to $\mathbf{r}(\theta=0)=\mathbf{r}_{\min}$; we illustrate the situation in Fig. 1.

We can also obtain the frequency of small oscillations about a stable circular orbit which results if the particle is given a small radial impulse changing its energy but not its angular momentum.[4]

From Eqs. (2) and (3) we have

$$A^2=(l^2/\mu r-\alpha)^2+l^2\dot{r}^2, \tag{6}$$

and since for a circular orbit, at $r=r_0$, A and $\dot{r}_0$ must vanish, we obtain from Eq. (6)

$$l^2/(\mu r_0)=\alpha. \tag{7}$$

We perturb the orbit, letting $r\to r_0+x$ with $\dot{r}=\dot{x}$ in Eq. (6):

$$A^2=\{[l^2/\mu(r_0+x)]-\alpha\}^2+l^2\dot{x}^2. \tag{8}$$

Assuming that $x/r_0\ll 1$ and using Eq. (7) twice, we approximate Eq. (8) as

$$A^2\approx\alpha^2(x/r_0)^2+\alpha\mu r_0\dot{x}^2. \tag{9}$$

Since the particle is still moving in the same force field, we must have $\dot{\mathbf{A}}=0$; thus from Eq. (9) we have

$$2A\dot{A}\approx 2\alpha\mu r_0\dot{x}[\ddot{x}+\alpha x/(\mu r_0^3)]=0,$$

or

$$\ddot{x}+\alpha x/(\mu r_0^3)=0,$$

which gives us

$$\omega^2=\alpha/(\mu r_0^3).$$

III. CONNECTION WITH LIE ALGEBRAS

The existence of the additional constant of the motion is due to the fact that the Hamiltonian, H, for the Kepler problem is invariant not only under the 3-dimensional rotation group $O(3)$ (angular momentum conservation) but also under the 4-dimensional rotation group $O(4)$ if $H<0$, or the group $O(3,1)$ if $H>0$.[5] It is somewhat tedious but not difficult to show that the components of the orbital angular momentum $\mathbf{l}$ along with the components of the vector

$$\mathbf{N}=[\mu/(2|H|)]^{1/2}\mathbf{A}$$

are the generators of the Lie algebra of $O(4)$ or $O(3,1)$. In particular the following Poisson brackets hold

$$(l_x,l_y)=l_z,\ (l_y,l_z)=l_x,\ (l_z,l_x)=l_y,$$
$$(l_x,N_y)=(N_x,l_y)=N_z,$$
$$(l_y,N_z)=(N_y,l_z)=N_x,$$
$$(l_z,N_x)=(N_z,l_x)=N_y,$$
$$(l_x,N_x)=(l_y,N_y)=(l_z,N_z)=0,$$
$$(N_x,N_y)=\pm l_z,\ (N_y,N_z)=\pm l_x,\ (N_z,N_x)=\pm l_y,$$

the $(+)$ sign for $H<0$ and the $(-)$ sign for $H>0$.

There is only one invariant[6] (Casimir operator)

$$F=\tfrac{1}{2}(l^2+N^2),$$

the other $G=\mathbf{l}\cdot\mathbf{N}=0$. Finally the Hamiltonian can be written as

$$H=-(\mu/2)[\alpha^2/(N^2+l^2)]$$

and is thus manifestly invariant under $O(4)$ or $O(3,1)$.

ACKNOWLEDGMENTS

I wish to thank T. Azzarelli and R. W. Huff for enlightening conversations on the algebra of $O(4)$ and its application to the hydrogen atom.

[1] For a geometric approach to the problem see W. C. Parke and O. Bergmann, Amer. J. Phys. **35**, 1131 (1967).

[2] W. Pauli, Z. Physik **36**, 336 (1926); V. Fock **98**, 145 (1935); V. Bargmann **99**, 576 (1936); L. D. Landau and E. M. Lifshitz, *Quantum Mechanics—Non-Relativistic Theory* (Pergamon Press, Inc., New York, 1965), 2nd ed., p. 128.

[3] L. D. Landau and E. M. Lifshitz, *Mechanics* (Pergamon Press, Inc., New York, 1960), p. 39. Although we consider here an attractive potential our results, with appropriate sign changes, hold for the case of a repulsive potential also.

[4] The particle moves onto a nearly circular ellipse, which intersects the circular orbit and has as one of its foci the center of the circle. See, for example, L. A. Pars, *Introduction to Dynamics* (Cambridge University Press, Cambridge, England, 1953), p. 273.

[5] It can be shown that all classical dynamic systems with f degrees of freedom are invariant under the $O(f+1)$ algebras. See N. Mukunda, Phys. Rev. **155**, 1383 (1967); P. Stehle and M. Y. Han, Phys. Rev. **159**, 1076 (1967), and references therein. For additional examples of Runge type constants of the motion and how to find them in certain cases, see V. A. Dulock and H. V. McIntosh, Amer. J. Phys. **33**, 109 (1965); and Ref. 3, problems on p. 154.

[6] An invariant is a quantity whose Poisson bracket with any generator vanishes.

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Core Course for Science Majors Combining Material from Physics, Chemistry, and Biology*

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In view of the increasing interdependence of the sciences and of the growing curricular demands upon students of science, a new program encompassing material from introductory majors-level courses in physics, chemistry, and biology has been developed. This core course is a two-year sequence which is designed to prepare students to undertake an advanced undergraduate program in any of these fields beginning in their sophomore year. The syllabus and special features of the course are discussed and several of the ongoing problems are reviewed.

INTRODUCTION

An emerging concern among college science teachers in recent years has been with the development of multidisciplinary courses at the undergraduate level.[1–6] A common combination has been physics and chemistry, but a number of significant attempts have been made involving the biological sciences as well. The incentives underlying these developments are manifold. One can cite: (a) the rapidly growing body of interdisciplinary scientific research such as biophysics, environmental science, and molecular biology; (b) the desire to eliminate overlap among the disciplines in the teaching of certain subject matter in the face of expanding curricular demands; (c) a need to serve the science student who has not committed himself to a major, as well as the nonscience student who increasingly requires a broad background in science; (d) the multidisciplinary background required of science-oriented professionals such as medical doctors and oceanographers; (e) the recent development of multidisciplinary high school courses,[7,8] which argues for a similar orientation in the training of secondary school science teachers; and finally, (f) a growing ecumenical feeling that new channels and dialog should be created among heretofore fragmented areas of science.[9]

The record of success in these undertakings has been spotty. Courses for nonscience majors have, in many cases, prospered. An outstanding example is the baccalaureate science program developed at the Rensselaer Polytechnic Institute.[10] Despite powerful reasons for the creation of combined courses designed for science majors, these students have not in general been the beneficiaries of the multidisciplinary fervor. Of the 520 multidisciplinary courses reported upon by Fuller,[2] only 51 were designated as being suitable for science majors. Of these, 11 are concerned with the three core sciences: physics, chemistry, and biology.