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Response of Linear Structures to Classes of Pressure Fields. I. Deterministic Nonconvecting Fields

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Pressure fields are classified according to the analyticity of the input-function wavenumber and frequency spectrum. This property is sufficiently general to allow for grouping of a host of different input functions into a class, and yet provides sufficient information for the calculation of the response. A methodology thus results for the theoretical prediction of the linear response of structures. Within the framework of this methodology, a detailed mathematical description of the excitation is not required; rather, knowledge of the nature of the singularities of the input-function spectrum suffices. This paper considers the important class of input functions whose spectra are meromorphic functions of wavenumber and frequency. The methodology and computational procedure are exemplified by the calculation of the response of simply supported uniform beams and plates to deterministic excitation; the excitation is described by input functions of the foregoing class. With a single calculation, response solutions result that are applicable to a variety of input functions. It is shown that the dependence of the response on the physical parameters of interest is dictated by the location and order of the poles of the input-function spectrum. The usefulness of the derived solutions in engineering applications is shown through a simple example.

LIST OF SYMBOLS

a_n	structural damping in the nth mode
$p(\mathbf{r},t)$	external pressure field
$\mathbf{r}$	position vector (x for beam, $\{x_1, x_2\}$ for plate)
t	time variable
$U(x)$	unit step function ($1, x>0$; $0, x<0$)
$w(\mathbf{r},t)$	structural vibratory displacement
$\delta(t)$	delta function
ν_i	poles of the input-function frequency spectrum
κ	poles of the input-function wavenumber spectrum
τ	$t-t_1$
$\bar{\omega}_n$	complex structural modal frequencies
Ω	stands for Ω_1 or Ω_2
P^+	denotes a function whose singularities are in the upper half-plane
P^-	denotes a function whose singularities are in the lower half-plane
$\mathcal{P}\int$	denotes principal-value integral

INTRODUCTION

Sufficient information for the theoretical calculation of structural response consists of the dynamic characteristics of the structure and the appropriate description of the excitation. The structure is characterized by its normal modes and frequencies and the excitation is specified by an appropriate function of the spatial and time variables, which we refer to as the input function. With this information at hand, one can calculate the response of a given structure to a given pressure field utilizing any of the classical methods of analysis.

The undesirable feature of the above procedure is that, given a structure, the entire response calculation has to be repeated anew every time the input function is changed (depending on the loading conditions) and, conversely, given an input function, the response has to be recalculated every time the structural configuration and/or boundary conditions are altered. This

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aspect of response calculations, coupled with uncertainties in the proper mathematical definition of pressure fields, presents formidable difficulties, for the calculations involved are, in general, lengthy and cumbersome. One would like, therefore, to have available response solutions that are applicable to a variety of input functions and structural configurations; that is, one would like to formulate the response problem of linear structures in a manner that affords a measure of generality. In order to introduce such generality, one is led to consider the mathematical classification of pressure fields according to some general property that may be shared by a variety of input functions.[1]

This paper presents a methodology for classifying pressure fields. The classification scheme is based on the analytic properties of the wavenumber and frequency spectrum of the input function. Since a host of functions of space and time have Fourier transforms of similar analyticity, spectral space provides a particularly convenient language for the grouping of input functions into classes. In this paper, we consider the class of input functions whose k and ω spectrum has any number of poles of any order, i.e., input functions whose Fourier transforms are meromorphic functions of k and ω. This class is by far the most important to consider, for it adequately describes most physically realizable pressure fields one encounters in applications.

The analyticity of the input-function spectrum is a property that, on the one hand, is sufficiently general to allow the classification of the pressure fields, and, on the other, is sufficiently specific to make possible the calculation of the response. In the Sections that follow, we calculate the response of uniform finite beams and plates to deterministic excitation that is describable by input functions of the foregoing class. The cases of convecting/nonconvecting random pressure fields, as well as the response of cylindrical shells to either deterministic or random fields, are considered in subsequent papers.

In what follows, we use the notation $f(x)\leftrightarrow \hat{f}(k)$ to indicate the Fourier pair

$$\hat{f}(k)=\int_{-\infty}^{\infty} dx e^{-ikx} f(x);\quad f(x)=(2\pi)^{-1}\int_{-\infty}^{\infty} dk e^{ikx}\hat{f}(k).$$

I. RESPONSE INTEGRAL REPRESENTATION IN SPECTRAL SPACE

We consider linear structures having an equation of motion of the form

$$[\mathfrak{L}_r+\mu(\partial^2/\partial t^2)+\beta(\partial/\partial t)]w(\mathbf{r},t)=p(\mathbf{r},t), \tag{1}$$

where $\mathfrak{L}_r$ is the linear operator in the spatial variables appropriate to the particular structure at hand; we assume it to be self-adjoint under the boundary conditions satisfied by $w(\mathbf{r},t)$.[2] The mass density μ and viscous-damping coefficient β are assumed constant. With $p(\mathbf{r},t)=0$, Eq. 1 admits a solution of the form

$$w_n(\mathbf{r},t)=\psi_n(\mathbf{r})\exp(i\bar{\omega}_n t);\quad \bar{\omega}_n=\omega_n+ia_n;\quad a_n\simeq\beta/2\mu;\quad a_n^2\ll\omega_n^2, \tag{2}$$

where n stands for the mode numbers required to specify the mode shapes $\psi_n(\mathbf{r})$. When $p(\mathbf{r},t)$ is a point force in both space and time, Eq. 1 becomes the equation of motion for the Green's function (impulse response), $G(\mathbf{r},t;\mathbf{r}',t')$. It can be shown that the Green's function solution can always be written in the following form of an eigenfunction expansion[3]:

$$G(\mathbf{r},\mathbf{r}';t-t')=\sum_n \frac{\psi_n(\mathbf{r})\psi_n^*(\mathbf{r}')}{\mu\omega_n} g_n(t-t') \tag{3}$$

and

$$g_n(t-t')=\exp[-a_n(t-t')]\sin[\omega_n(t-t')]U(t-t'),$$

where $\psi_n(\mathbf{r})$ satisfies the equation

$$[\mathfrak{L}_r-\bar{\omega}_n^2\mu+i\bar{\omega}_n\beta]\psi_n(\mathbf{r})=0. \tag{4}$$

Note that $g_n(t-t')$ is form invariant with respect to the structural characteristics—i.e., with respect to the operator $\mathfrak{L}_r$.

Assuming that initially the displacement and velocity are zero and taking advantage of the Hermitian property of $\mathfrak{L}_r$, one can easily show that Eq. 1 admits the solution[3]

$$w(\mathbf{r},t)=\int_S d\mathbf{r}'\int_0^t dt' G(\mathbf{r},\mathbf{r}';t-t')p(\mathbf{r}',t'), \tag{5}$$

where the spatial integration is over the extent of the structure. We now wish to transform the integrations in Eq. 5 to the wavenumber and frequency variables, allowing for the possibility of truncation in $p(\mathbf{r},t)$. Let

$$p(\mathbf{r},t)=p_1(\mathbf{r})p_{2>}(t);\quad p_{2>}(t)=p_2(t)U(t). \tag{6}$$

With Eqs. 3 and 6, the one-dimensional version of

[1] A similar classification for the structures according to some property of the mode shapes is suggested, but we concentrate here on the input function.

[2] This assumption is true in the case of most structures of concern in practical applications.

[3] P. M. Morse and H. Feshback, *Methods of Theoretical Physics* (McGraw-Hill Book Co. New York, 1953), Chap. 7.

Eq. 5 is

$$w(x,t)=\sum_n \frac{\psi_n(x)}{\mu\omega_n}\int_{-\infty}^{\infty} dx'\psi_n{}^*(x')p_1(x')p_L(x') \times\int_{-\infty}^{\infty} dt' g_n(t-t')p_{2>}(t'), \quad (7)$$

where $p_L(x)=U(x)-U(x-L)$. Introducing the transforms $p_1(x)\leftrightarrow \hat{p}_1(k)$, $p_{2>}(t)\leftrightarrow \hat{p}_2{}^+(\omega)$, and making use of the convolution theorem, Eq. 7 can be cast in the form

$$w(x,t)=\sum_n \frac{\psi_n(x)}{(2\pi)^3 i\mu\omega_n}\int_{-\infty}^{\infty} dk\hat{p}_1(k) \times\int_{-\infty}^{\infty} dk'\hat{\psi}_n{}^*(-k')\frac{[e^{i(k+k')L}-1]}{k+k'} \times\int_{-\infty}^{\infty} d\omega e^{i\omega t}\hat{g}_n{}^+(\omega)\hat{p}_2{}^+(\omega), \quad (8)$$

where

$$\psi_n(x)\leftrightarrow\hat{\psi}_n(k),$$
$$g_n(t-t')\leftrightarrow \hat{g}_n{}^+(\omega)=\tfrac{1}{2}[(\omega-\Omega_2)^{-1}-(\omega-\Omega_1)^{-1}], \quad (9)$$

and

$$\begin{matrix}\Omega_1\\ \Omega_2\end{matrix}=\pm\omega_n+ia_n.$$

We note that, in the new variables, the single x integration of Eq. 7 has been replaced by a double integration in wavenumber. This is the price one has to pay in order to achieve the desired classification of pressure fields. However, for simple structures under ideal boundary conditions, the mode-shape spectrum involves delta functions, so that one of the integrations becomes trivial. The appearance of a double k integration in Eq. 8 is a consequence of the finite spatial extent of the structure.[4] In general, structural discontinuities (e.g., beams attached to infinite plates, boundaries, etc.) always give rise to convolution integrals in wavenumber space. The physical meaning of this feature of spectral representations is that structural discontinuities act as "wavenumber convertors"; that is, a single wavenumber is scattered by a discontinuity into a spectrum of wavenumbers.[5] This point is worth noting, for it is of considerable consequence not only to the calculation of structural response, but also to the definition of the acoustic field radiated from vibrating structures.

Using Eq. 9, we define the functions

$$\mathbf{D}_n(L,0)=\mathbf{A}_n(L)-\mathbf{A}_n(0), \quad (10)$$

$$\mathbf{C}_n(t)=\mathbf{B}_n(t,\Omega_2)-\mathbf{B}_n(t,\Omega_1), \quad (11)$$

$$\mathbf{A}_n(y)=\int_{-\infty}^{\infty} dk e^{iky}\hat{p}_1(k)\mathcal{P}\int_{-\infty}^{\infty} dk' e^{ik'y}\frac{\hat{\psi}_n{}^*(-k')}{k+k'}, \quad (12)$$

and

$$\mathbf{B}_n(t,\Omega)=\int_{-\infty}^{\infty} d\omega e^{i\omega t}\frac{\hat{p}_2{}^+(\omega)}{\omega-\Omega}, \quad (13)$$

so that Eq. 8 assumes the form

$$w(x,t)=\sum_n A_n\psi_n(x)\mathbf{D}_n(L,0)\mathbf{C}_n(t), \quad (14)$$

where $A_n=(16\pi^3 i\mu\omega_n)^{-1}$. Note that the integrand in Eq. 8 is regular at $k=-k'$. However, when we write the k, k' integral as the difference of two integrals (Eq. 10), each has a simple pole on the contour of integration at $k=-k'$ that gives rise to the principal-value definition in Eq. 12. That the function $\mathbf{A}_n(y)$ must be defined as a principal-value integral can be seen by writing Eq. 7 as the difference of two integrals, inserting the explicit expression for $p_L(x)$; upon transformation to the k variable, the integrand will involve the Fourier transform of a step function. Of course, in the evaluation of Eq. 10, the contributions from the poles on the contour cancel.

In the event that the pressure field is truncated in time at t_1 and vanishes in space outside the interval $a<x<b$, $p(x,t)=p_{1,ab}(x)p_{2,t_1}(t)$, $p_{1,ab}(x)=p_1(x)[U(x-a)-U(x-b)]$, $p_{2,t_1}(t)=p_{2>}(t)U(t_1-t)$. The wavenumber–frequency spectrum is given by

$$\hat{p}_{1,ab}(k)=-\frac{1}{2\pi i}\int_{-\infty}^{\infty} dk''\frac{\hat{p}_1(k'')}{k-k''} \times[e^{-i(k-k'')b}-e^{-i(k-k'')a}] \quad (15)$$

and

$$\hat{p}_{2,t_1}(\omega)=\tfrac{1}{2}\hat{p}_2{}^+(\omega)-\frac{1}{2\pi i}\mathcal{P}\int_{-\infty}^{\infty} d\omega' e^{-i(\omega-\omega')t_1}\frac{\hat{p}_2{}^+(\omega')}{\omega-\omega'}. \quad (16)$$

Inserting Eqs. 15 and 16 in Eq. 8, performing the k and ω integrations, and redefining the dummy variables of integration, we obtain the following response rep-

[4] Indeed, in the case of infinite uniform structures, the Green's function is a homogeneous function of the spatial variables as well and the x' integration of Eq. 7 is a convolution integral. Transformation to the k variable gives rise to a single integration.

[5] G. Maidanik, "Surface Impedance Nonuniformities as Wavevector Convertors," J. Acoust. Soc. Amer. 46 (1969) (to be published).

resentations for all possible combinations of truncation in the pressure field:

$$w(x,t)=\sum_n A_n\psi_n(x)[\mathbf{C}_n(t)U(-\tau)+\mathbf{E}_n(t,t_1)U(\tau)]\begin{bmatrix} 0 \\ \mathbf{D}_n(b,0) \\ \mathbf{D}_n(b,a) \\ \mathbf{D}_n(L,0) \\ \mathbf{D}_n(L,a) \\ 0 \end{bmatrix}\begin{array}{l} ;\ L>0>b>a \\ ;\ L>b>0>a \\ ;\ L>b>a>0 \\ ;\ b>L>0>a \\ ;\ b>L>a>0 \\ ;\ b>a>L>0, \end{array} \tag{17}$$

where

$$\mathbf{E}_n(t,t_1)=e^{i\Omega_2\tau}\mathbf{B}_n(t_1,\Omega_2)-e^{i\Omega_1\tau}\mathbf{B}_n(t_1,\Omega_1), \quad \tau=t-t_1, \tag{18}$$

and

$$\mathbf{D}_n(y,z)=\mathbf{A}_n(y)-\mathbf{A}_n(z). \tag{19}$$

The product of any of the $\mathbf{D}_n$ with the $\mathbf{C}_n$ or $\mathbf{E}_n$ functions yields the appropriate response solution. For instance, the response of a beam to a pressure field that is truncated in both space and time and such that $L>b>0>a$, at $t>t_1$ is given by

$$w(x,t)=\sum_n A_n\psi_n(x)\mathbf{D}_n(b,0)\mathbf{E}_n(t,t_1).$$

The extension of the above representations to the case of a rectangular plate of dimensions L_1, L_2 along the Cartesian axes x_1, x_2, is quite straightforward, provided that we assume

$$p_1(\mathbf{r})=p_{11}(x_1)p_{12}(x_2). \tag{20}$$

Furthermore, $\psi_n(\mathbf{r})=\psi_n(x_1)\psi_m(x_2)$, so that the spatial integrations in Eq. 5 uncouple to yield two integrals over x_1', x_2' of the form appearing in Eq. 7. We can, therefore, easily deduce the two-dimensional counterpart of Eq. 21 in the event that the pressure field is nonzero in the intervals $a_1<x_1<b_1$, $a_2<x_2<b_2$:

$$w(\mathbf{r},t)=\sum_{n,m} A_{nm}\psi_{nm}(\mathbf{r})[\mathbf{C}_{nm}(t)U(-\tau)+\mathbf{E}_{nm}(t,t_1)U(\tau)]\begin{bmatrix} 0 \\ \mathbf{D}_n(b_1,0) \\ \mathbf{D}_n(b_1,a_1) \\ \mathbf{D}_n(L_1,0) \\ \mathbf{D}_n(L_1,a_1) \\ 0 \end{bmatrix}\begin{bmatrix} 0 \\ \mathbf{D}_m(b_2,0) \\ \mathbf{D}_m(b_2,a_2) \\ \mathbf{D}_m(L_2,0) \\ \mathbf{D}_m(L_2,a_2) \\ 0 \end{bmatrix}\begin{array}{l} ;\ L_i>0>b_i>a_i \\ ;\ L_i>b_i>0>a_i \\ ;\ L_i>b_i>a_i>0 \\ ;\ b_i>L_i>0>a_i \\ ;\ b_i>L_i>a_i>0 \\ ;\ b_i>a_i>L_i>0 \end{array} \quad (i=1,2), \tag{21}$$

where $A_{nm}=[-2(2\pi)^5\mu\omega_{nm}]^{-1}$. The $\mathbf{C}$ and $\mathbf{E}$ functions are given by Eqs. 11 and 18, respectively, with n replaced by nm; the $\mathbf{D}_n$ functions are defined in Eqs. 12 and 19 with $\hat{p}_1(k)$ replaced by $\hat{p}_{11}(k_1)$ $[p_{11}(x_1)\leftrightarrow\hat{p}_{11}(k_1)]$, while the $\mathbf{D}_m$ functions are defined in the above equations with $\hat{p}_1(k)$ replaced by $\hat{p}_{12}(k_2)[p_{12}(x_2)\leftrightarrow\hat{p}_{12}(k_2)]$ and $\hat{\psi}_n(k)$ replaced by $\hat{\psi}_m(k_2)$. Thus, for instance, the response to a pressure field which is truncated in both spatial directions and in time and is such that $L_1>b_1>0>a_1$, $L_2>b_2>a_2>0$, at $t<t_1$, is given by

$$w(\mathbf{r},t)=\sum_{n,m} A_{nm}\psi_{nm}(\mathbf{r})\mathbf{D}_n(b_1,0)\mathbf{D}_m(b_2,a_2)\mathbf{C}_{nm}(t).$$

Inspection of Eqs. 17 and 21 shows that the evaluation of the response, for all possible combinations of truncation in the pressure field, rests on the evaluation of two integrals given in Eqs. 12 and 13. An analogous statement can be made, of course, if one works directly with Eq. 5. However, the wavenumber–frequency description in the format of this paper makes it possible to formulate a methodology for the classification of pressure fields. This classification can be simply affected by allowing k and ω to be complex variables.[6]

II. MATHEMATICAL CLASSIFICATION OF PRESSURE FIELDS

The classification scheme adopted in this paper is based upon the analyticity of the input-function spectrum $\hat{p}(k,\omega)$. This property is sufficiently general to allow the grouping of different input functions into a class, and, yet, provides sufficient information for the calculation of the response. As a function of the complex variables k and ω, the spectra $\hat{p}(k,\omega)$ fall into three main classes: those that have any number of poles of any order, those that have essential singularities, and those that have branch-point singularities. We consider the first class, that is, input functions whose spectra are

[6] In letting k and ω be complex, it is assumed that the input-function and mode-shape spectra possess an analytic continuation in the complex plane.

meromorphic functions of k and ω. In view of Eq. 6, we consider the frequency and wavenumber spectra separately.

Since $\hat{p}_2^+(\omega)$ is assumed meromorphic in the extended ω plane, it must necessarily be a rational function. Furthermore, its poles must be in the upper half-plane.[7] Accordingly, we write

$$\hat{p}_2^+(\omega)=\alpha_0+\sum_{i=1}^{N}\sum_{s=1}^{M_i}\alpha_{i,s}(\omega-\nu_i)^{-s};\quad \nu_i=\nu_i'+i\nu_i'',\qquad(22)$$

where α_0, $\alpha_{i,s}$ are constants, N is the number of poles, M_i the order of the ith pole, and $\nu_i''>0$. Normally, a meromorphic function would also involve terms of positive powers of ω. These terms are omitted in Eq. 22, for they correspond to terms in $p_{2>}(t)$ involving derivatives of $\delta(t)$; such terms are not an acceptable description of physically realizable pressure fields.[8]

The wavenumber spectrum $\hat{p}_1(k)$ must have a form analogous to that of Eq. 22 if it is to be a meromorphic function of k. However, since $p_1(x)$ is not required to be causal, the poles of its Fourier transform may be in both the upper and lower half-planes. Accordingly, we write

$$\hat{p}_1(k)=\sum_{j=1}^{N'}\sum_{r=1}^{M_j}\beta_{j,r}(k-\kappa_j)^{-r}+\sum_{h=1}^{N''}\sum_{l=1}^{M_h}\gamma_{h,l}(k-\kappa_h)^{-l}$$

$$\equiv P_1^+(k)+P_1^-(k),\qquad(23)$$

$$\kappa_j=\kappa_j'+i\kappa_j'';\quad \kappa_h=\kappa_h'-i\kappa_h'',$$

where $\beta_{j,r}$, $\gamma_{h,l}$ are constants and $\kappa_j''>0$, $\kappa_h''>0$, and where we have omitted terms of positive powers of k and have neglected any constant part of the spectrum.[8]

The inverse of Eqs. 22 and 23 is

$$p_{2>}(t)=i\sum_{i=1}^{N}\sum_{s=1}^{M_i}\frac{\alpha_{i,s}}{(s-1)!}e^{i\nu_it}(it)^{s-1}U(t)\qquad(24)$$

and

$$p_1(x)=\begin{cases} i\displaystyle\sum_{j=1}^{N'}\sum_{r=1}^{M_j}\frac{\beta_{j,r}}{(r-1)!}e^{i\kappa_jx}(ix)^{r-1}; & x>0,\\ -i\displaystyle\sum_{h=1}^{N''}\sum_{l=1}^{M_h}\frac{\gamma_{h,l}}{(l-1)!}e^{i\kappa_hx}(ix)^{l-1}; & x<0.\end{cases}\qquad(25)$$

Thus, the class of pressure fields considered in this paper is describable by input functions whose spatial and time distribution involves various combinations of exponential, trigonometric functions and polynomials of any degree in the variables. This class of functions adequately describes the majority of pressure fields with which one is concerned in applications. The details of a particular pressure field within the class are incorporated in the poles ν and κ (i.e., their precise location, order, number, and strength). Knowledge of these details is not necessary, however, for the calculation of the response. In the next Sections, we calculate the response of uniform, finite beams and plates to any pressure field of the class given by Eqs. 24 and 25.

III. TIME DEVELOPMENT OF THE RESPONSE

We consider first the integral $\mathbf{B}_n(t,\Omega)$ of Eq. 13. With Eq. 22, the integration can be carried out by closing the contour with a semicircle at infinity in the upper half-plane. Summing the residues at the simple pole Ω and at all the poles ν_i, we obtain the result[9]

$$\mathbf{B}_n(t,\Omega)=2\pi i\left[e^{i\Omega t}\hat{p}_2^+(\Omega)-\sum_{i,s}\sum_{q=0}^{s-1}\alpha_{i,s}e^{i\nu_it}\frac{(it)^q}{q!(\Omega-\nu_i)^{s-q}}\right]U(t).\qquad(26)$$

It follows from Eqs. 11, 17, 18, and 21 that the full time dependence of the response is given by

$$\mathbf{C}_n(t)=2\pi i\left[e^{i\Omega_2t}\hat{p}_2^+(\Omega_2)-e^{i\Omega_1t}\hat{p}_2^+(\Omega_1)-\sum_{i,s}\sum_{q=0}^{s-1}\alpha_{i,s}e^{i\nu_it}\frac{(it)^q}{q!}\{(\Omega_2-\nu_i)^{q-s}-(\Omega_1-\nu_i)^{q-s}\}\right]U(t);\quad \tau<0,\qquad(27)$$

[7] The fact that $p_2^+(\omega)$ is analytic in the lower half-plane is the mathematical statement of causality.

[8] The special case of a constant pressure field giving rise to a delta-function spectrum is not treated here for, in this instance, $\mathbf{B}_n(t,\Omega)$ can be evaluated trivially. Equally simple is the evaluation of the response to a delta-function load using Eq. 5 and, therefore, we henceforth set $\alpha_0=0$.

[9] Equation 26 has been derived on the assumption that none of the poles ν_i coincide with the pole at Ω for all modes n. The coalescence of some pole ν_j of order m, with Ω for some n, implies perfect matching of that complex frequency in the excitation with the nth complex modal frequency—a highly improbable circumstance in practice. Nonetheless, from the mathematical viewpoint, coalescence of the above singularities simply raises the order of the pole at Ω from 1 to $(m+1)$. The residue is readily evaluated to yield the result

$$\mathbf{B}_n(t,\Omega)=2\pi i\left[\alpha_{j,m}e^{i\Omega t}\frac{(it)^m}{m!}-\sum_{\substack{i\neq j\\ s\neq m}}\sum_{q=0}^{s-1}\alpha_{i,s}e^{i\nu_it}\frac{(it)^q}{q!(\Omega-\nu_i)^{s-q}}\right]U(t).$$

and

$$\mathbf{E}_n(t,t_1)=2\pi i\Bigg[e^{i\Omega_2 t}\hat{p}_2^{+}(\Omega_2)-e^{i\Omega_1 t}\hat{p}_2^{+}(\Omega_1)-e^{i\Omega_2 t}\sum_{i,s}\sum_{q=0}^{s-1}\alpha_{i,s}e^{-i(\Omega_2-\nu_i)t_1}\frac{(it_1)^q}{q!(\Omega_2-\nu_i)^{s-q}}$$

$$+e^{i\Omega_1 t}\sum_{i,s}\sum_{q=0}^{s-1}\alpha_{i,s}e^{-i(\Omega_1-\nu_i)t_1}\frac{(it_1)^q}{q!(\Omega_1-\nu_i)^{s-q}}\Bigg]U(t);\quad \tau>0. \quad (28)$$

These results are independent of the structural characteristics, save for the value of the constants Ω_1,Ω_2. For instance, in the case of a flat plate (Eq. 21), Eqs. 27 and 28 hold with n replaced by nm. This is a consequence of the form invariance of $g_n(t-t')$ with respect to the operator $\mathfrak{L}_r$, which was noted in Sec. I, and of Eq. 6.

The physical meaning of the poles ν_i is now clear. Consider Eq. 27. The first two terms on the right-hand side (the transient portion of the solution) represent a damped oscillation that is controlled by the structural complex frequencies Ω. All remaining terms (the steady-state solution) are in the form of a product of a constant, an exponential, and a polynomial in t. *The location of the poles ν_i controls the type of exponential, while their order dictates the degree of the polynomial in t*, viz., a pole in the first or second quadrants gives rise to a damped oscillating term, while poles on the imaginary axis correspond to damped motion without oscillation. The degree of the polynomial in t is one less than the order of the contributing pole. Table I summarizes the type of terms a pole ν_i may contribute to the response. Note that the constant coefficients depend on inverse powers of the separation between the singularities $\Omega-\nu_i=(\pm\omega_n+\nu_i')+i(a_n-\nu_i'')$. The case of $(\Omega-\nu_i)$ pure imaginary corresponds to the resonance condition $(\pm\omega_n=\mp|\nu_i'|)$; terms in the response solution for which this is true will assume their maximum value. In the case of Eq. 28, the dependence on t is that of a damped oscillation controlled by the structural parameters alone. The dependence on t_1 is dictated by the nature of the poles ν_i. Terms for which $\pm\omega_n\neq\mp|\nu_i'|$, $a_n>\nu_i''$ give rise to an exponentially *growing* oscillation in t_1. Resonance terms in the steady-state solution will either grow exponentially or be damped out with t_1, depending on whether $a_n\gtrless\nu_i''$, respectively. In that ν_i'' represents the temporal decay of the pressure field (Eq. 24), this condition relates structural damping to the dissipation of the load. The degree of the polynomial in t_1 is dictated, as before, by the order of the contributing pole.

We see, then, that in the framework of the methodology employed here, knowledge of the spectrum singularities ν_i completely determines the character of the response as a function of the time variables t and t_1. We have thus effected a classification for the loads where, with a single calculation, the temporal portion of the response solution for a whole class of input functions is easily determined. To obtain the response given any one input function that belongs to the class (Eq. 24) is now a simple matter of identification of symbols (see Sec. V).

IV. COMPLETE RESPONSE SOLUTION

To complete the response solutions, we need calculate the **D** functions which appear in Eqs. 17 and 21. This calculation rests on the evaluation of the wavenumber integrations of Eq. 12. For simple support end conditions, the normalized eigenfunctions are given by

$$\psi_n(x)=(2/L)^{\frac{1}{2}}\sin k_n x,\quad k_n=n\pi/L$$

and the corresponding k spectra

$$\hat{\psi}_n(k)=i\pi(2/L)^{\frac{1}{2}}[\delta(k+k_n)-\delta(k-k_n)]=\hat{\psi}_n^*(-k). \quad (29)$$

By substituting Eq. 29 in Eq. 12, the k' integration is simply carried out to yield the result

$$\mathbf{A}_n(y)=i\pi(2/L)^{\frac{1}{2}}[e^{-ik_n y}I(k_n,y)-e^{ik_n y}I(-k_n,y)], \quad (30)$$

where

$$I(k_n,y)=\mathcal{P}\int_{-\infty}^{\infty}dk\frac{e^{iky}}{k-k_n}\hat{p}_1(k).$$

Using Eq. 23, we evaluate $I(k_n,y)$ by closing the contour

TABLE I. Type of terms contributed to the time development of the response by the poles of the excitation frequency spectrum.

Location of pole	Order	Contributing term
$\nu_i=\nu_i'+i\nu_i''$	1	$\alpha_{i1}\exp[i\nu_i't-\nu_i''t]A_0$
	2	$\alpha_{i2}\exp[i\nu_i't-\nu_i''t](A_0+A_1t)$
	⋮	
	s	$\alpha_{is}\exp[i\nu_i't-\nu_i''t](A_0+A_1t+\cdots+A_{s-1}t^{s-1})$
	⋮	
$\nu_i=i\nu_i''$	1	$\alpha_{i1}\exp(-\nu_i''t)A_0$
	2	$\alpha_{i2}\exp(-\nu_i''t)(A_0+A_1t)$
	⋮	
	s	$\alpha_{is}\exp(-\nu_i''t)(A_0+A_1t+\cdots+A_{s-1}t^{s-1})$
	⋮	

with a semicircle at infinity in the upper or lower half-plane depending on whether $y>0$ or $y<0$,respectively. In the first instance, the poles κ_j contribute; whereas in the second, the poles κ_h contribute. Adding one-half the residue at the pole $k=k_n$ and the residues at the poles κ, we obtain

$$\mathbf{A}_n(y>0)=-2\pi^2\left(\frac{2}{L}\right)^{\frac{1}{2}}\left[\tfrac{1}{2}\{\hat{p}_1(k_n)-\hat{p}_1(-k_n)\}-\sum_{j,r}\sum_{q=0}^{r-1}(-1)^{r-q}\beta_{j,r}\frac{(iy)^q}{q!}\left\{\frac{e^{i(\kappa_j-k_n)y}}{(\kappa_j-k_n)^{r-q}}-\frac{e^{i(\kappa_j+k_n)y}}{(\kappa_j+k_n)^{r-q}}\right\}\right], \quad (31)$$

$$\mathbf{A}_n(y<0)=2\pi^2\left(\frac{2}{L}\right)^{\frac{1}{2}}\left[\tfrac{1}{2}\{\hat{p}_1(k_n)-\hat{p}_1(-k_n)\}-\sum_{h,l}\sum_{q=0}^{l-1}(-1)^{l-q}\gamma_{h,l}\frac{(iy)^q}{q!}\left\{\frac{e^{i(\kappa_h-k_n)y}}{(\kappa_h-k_n)^{l-q}}-\frac{e^{i(\kappa_h+k_n)y}}{(\kappa_h+k_n)^{l-q}}\right\}\right], \quad (32)$$

and

$$\mathbf{A}_n(y=0)=\pi^2(2/L)^{\frac{1}{2}}[\{P_1^+(k_n)-P_1^-(k_n)\}-\{P_1^+(-k_n)-P_1^-(-k_n)\}], \quad (33)$$

where, of course, Eq. 33 results whether we approach $y=0$ from positive or negative values. With Eqs. 31 and 33 we obtain, from Eq. 19,

$$\mathbf{D}_n(y,z)=2\pi^2\left(\frac{2}{L}\right)^{\frac{1}{2}}\left[\sum_{j,r}\sum_{q=0}^{r-1}(-1)^{r-q}\beta_{j,r}\frac{(iy)^q}{q!}\left\{\frac{e^{i(\kappa_j-k_n)y}}{(\kappa_j-k_n)^{r-q}}-\frac{e^{i(\kappa_j+k_n)y}}{(\kappa_j+k_n)^{r-q}}\right\}\right.$$
$$\left.-\sum_{j,r}\sum_{q=0}^{r-1}(-1)^{r-q}\beta_{j,r}\frac{(iz)^q}{q!}\left\{\frac{e^{i(\kappa_j-k_n)z}}{(\kappa_j-k_n)^{r-q}}-\frac{e^{i(\kappa_j+k_n)z}}{(\kappa_j+k_n)^{r-q}}\right\}\right];\quad y>0,\quad z>0, \quad (34)$$

and

$$\mathbf{D}_n(y,0)=-2\pi^2\left(\frac{2}{L}\right)^{\frac{1}{2}}\left[P_1^+(k_n)-P_1^+(-k_n)-\sum_{j,r}\sum_{q=0}^{r-1}(-1)^{r-q}\beta_{j,r}\frac{(iy)^q}{q!}\left\{\frac{e^{i(\kappa_j-k_n)y}}{(\kappa_j-k_n)^{r-q}}-\frac{e^{i(\kappa_j+k_n)y}}{(\kappa_j+k_n)^{r-q}}\right\}\right];\quad y>0. \quad (35)$$

The parameters y, z stand for the physical constants L, a, b, which appear in the response solutions, Eq. 17. We observe that the dependence on these parameters is controlled by the singularities κ of the input-function wavenumber spectrum in a manner identical to that in which the dependence on the time variables is controlled by the poles ν. Consider, for instance, response solutions that involve the function $\mathbf{D}_n(y,z)$, e.g., $\mathbf{D}_n(b,a)$ of Eq. 17. Each term is in the form of a constant times an exponential and a polynomial in y or z. The order of the poles κ_j dictates the degree of the polynomials, while their location determines the type of exponentials. Complex poles give rise to damped oscillations of frequency $(\kappa_j'\pm k_n)$, while pure imaginary poles correspond to damped oscillations of frequency $\pm k_n$. The degree of the polynomials in y and z is one less than the order of the contributing pole. Response solutions that involve $\mathbf{D}_n(y,0)$, e.g., $\mathbf{D}_n(b,0)$ of Eq. 17, involve the sum of a constant part (namely, the values of $p_1^+(k)$ at the modal wavenumbers $\pm k_n$) and terms in y whose form is controlled by the poles κ_j in the manner discussed above. In the event that $y=L$, e.g., $\mathbf{D}_n(L,a)$ of Eq. 17, the dependence on L is not fully determined according to the rules given above. This is so because L appears in k_n and the normalization constant of the eigenfunctions $\psi_n(x)$. To extract the full L dependence in the generality of the present results is cumbersome; it can best be determined in specific cases.

The circumstance $\pm|\kappa_j'|=\mp k_n$ corresponds to the matching of a particular wavelength in the excitation to the structural wavelength associated with the nth mode. It is to those spectral components of the excitation that the structure will respond most favorably. Note, however, that, in contrast to the resonance condition in the frequency domain, "resonance in wavenumber" gives rise to terms in the response that damp out exponentially with y or z; there is no oscillation.

The above comments with regard to the rôle played by the poles κ are, of course, applicable to the two-dimensional case with the proper change of symbols. In this instance, Eq. 23 with the appropriate change of notation describes the wavenumber spectra $\hat{p}_{11}(k_1)$, $\hat{p}_{12}(k_2)$. The poles of these spectra dictate the behavior of the functions $\mathbf{D}_n$, $\mathbf{D}_m$ (Eq. 21) with respect to the physical parameters L_i, a_i, b_i, $(i=1, 2)$ in the manner discussed above.

With a single calculation, we have thus completed the evaluation of the response for the class of input functions given in Eqs. 24 and 25. We have also noted the physical significance of the poles of the input-function spectrum in the sense that they control the behavior of the response as a function of the physical parameters t, t_1, L, a, b and the corresponding ones in the two-dimensional case; the location and order of the poles ν, κ determines the dependence of the response on these parameters. With the above solutions at hand, the response to any particular pressure field of the class considered can be deduced by inspection. To show the use of the derived solutions, we consider next a simple example.

V. EXAMPLE

In this Section, we attempt to show, through a simple example, the manner in which the response solutions of this paper can be used in applications. We consider the time dependence of the response. Considerations similar to those which follow can be applied in connection with the other parameters in the response solutions.

Consider the pressure field

$$p(\mathbf{r},t)=P_0p_1(\mathbf{r})p_{2>}(t,a),$$

where P_0 is a constant amplitude and

$$p_{2>}(t,a)=\exp(-at)\cos btU(t). \qquad (36)$$

We write

$$p_{2>}(t,a)=(\tfrac{1}{2})\{\exp[i(b+ia)t]+\exp[i(-b+ia)t]\}U(t),$$

so that, by comparison with Eq. 24, $s=1$, $i=1$, 2, $\alpha_{1,1}=\alpha_{2,1}=1/2i$, $\nu_1=b+ia$, $\nu_2=-b+ia$. The input-function spectrum has two complex simple poles which, according to Table I, contribute to the response in the nth-mode damped oscillating terms. In particular, we immediately deduce, by substitution of the above constants in Eq. 27, the response solution:

$$\mathbf{C}_n(t,a)=\pi\{e^{i\Omega_2t}\sum_{i=1}^{2}(\Omega_2-\nu_i)^{-1}-e^{i\Omega_1t}\sum_{i=1}^{2}(\Omega_1-\nu_i)^{-1}$$
$$-\sum_{i=1}^{2}e^{i\nu_it}[(\Omega_2-\nu_i)^{-1}-(\Omega_1-\nu_i)^{-1}]\}U(t). \qquad (37)$$

Of course, Eq. 37 would result if one were to substitute Eq. 36 in Eq. 5 and perform the indicated time integration. Consider, however, another pressure field given by

$$p_{2>}'(t,a')=\exp(-a't)(t^2+1)\cos btU(t). \qquad (38)$$

Comparison with Eq. 24 yields $\alpha_{1,1}=\alpha_{2,1}=1/2i$, $\alpha_{1,2}=\alpha_{2,2}=0$, $\alpha_{1,3}=\alpha_{2,3}=i$, $\nu_1=b+ia'$, $\nu_2=-b+ia'$. In this case, the poles ν_1, ν_2 are in a different location in the complex plane and of third order. According to Table I, each contributes a term in the form of the product of a constant, an exponential, and a polynomial in t of second degree. Equation 27 now reads

$$\mathbf{C}_n'(t,a')=\mathbf{C}_n(t,a')-2\pi\Big[e^{i\Omega_2t}\sum_{i=1}^{2}(\Omega_2-\nu_i)^{-3}$$
$$-e^{i\Omega_1t}\sum_{i=1}^{2}(\Omega_1-\nu_i)^{-3}-\sum_{i=1}^{2}\sum_{q=0}^{2}e^{i\nu_it}\frac{(it)^q}{q!}\{(\Omega_2-\nu_i)^{q-3}$$
$$-(\Omega_1-\nu_i)^{q-3}\}\Big]U(t). \qquad (39)$$

Again, we arrive at the response solution by inspection of the general result, Eq. 27, while, according to traditional computational procedures, one would have to substitute Eq. 38 in Eq. 5 and perform the time integration anew.

We can use this example to illustrate the usefulness of the methodology developed here in two problem areas that arise in applications. First, let us suppose that $p_{2>}(t,a)$ is a field that can be produced in the laboratory and in which a structure is to be tested, while $p_{2>}'(t,a')$ represents the actual environment that the structure will encounter in operation. We ask: How well is the response simulated in the laboratory? To answer, we only need identify the singularities of the corresponding frequency spectra. With this information, we immediately deduce the difference in the two cases. For instance, we note from Eq. 39 that $\mathbf{C}_n(t,a)$, $\mathbf{C}_n'(t,a')$ differ primarily by terms that damp out owing to dissipation in the structure and by terms which involve second-degree polynomials in t. We expect the latter to dominate for some range of t, particularly so at resonance.

Similar considerations are involved in another circumstance of common occurrence in engineering applications. Suppose that measurements of a particular pressure field have sufficient scatter that either Eq. 36 or 38 could be used as reasonable descriptions for some value of the parameters a and a'. The question is: How sensitive is the response to the particular choice? Again, we need only identify the poles of the spectra in order to arrive at qualitative and semiquantitative answers.

VI. CONCLUDING REMARKS

A methodology has been presented for the calculation of the response of linear structures to classes of pressure fields. The methodology is based on a classification scheme for the pressure fields according to the analyticity of the input-function wavenumber and frequency spectrum. This property is sufficiently general to allow the grouping of a host of input functions into a class. In particular, this paper has considered the class of input functions whose spectra are meromorphic functions of k and ω. This class is by far the most common that one is likely to encounter in applications. Functions $p(\mathbf{r},t)$, whose spectra have branch-point singularities, are either unacceptable descriptions of physically realizable loads (e.g., functions with logarithmic branch points), or not particularly convenient representations of data.

Specification of the analytic properties of the input-function spectrum provides sufficient information for the calculation of the response. Furthermore, one is able to bring to bear the powerful tool of contour integration and derive, with a single calculation, response solutions which are applicable to a host of input functions. To exemplify the methodology in its simplest application, we have considered here the response of simply supported beams and plates to nonconvecting deterministic excitation. The more complicated cases of nonconvecting and convecting random excitation are considered in subsequent papers of this series.

It has been noted in Sec. V that, having calculated the response once and for all for the class of input functions given by Eqs. 24 and 25, the response to any particular one can be immediately deduced by inspection. The usefulness of such versatility in the various stages of the design of aerospace vehicles, in the response-simulation problem and in establishing the general behavior of the response as a function of the various parameters of interest, is clear. On the other hand, from the point of view of numerical evaluation, a single computer program of the response solutions derived here can serve to evaluate the response to a particular load of interest and to perform detailed parametric studies. The input to such a program essentially consists of the structural parameters and the number, location, and order of the poles of the input-function spectrum.

Finally, it may be mentioned in passing that the solutions derived here can be simply extended by application of certain theorems of Fourier transforms. Consider, for instance, the temporal part of the input function. Let it be shifted by a constant t_0, scaled by a positive constant a, and modulated at ω_0:

$$p_{>}(t)=\exp(i\omega_0 t)p_{2>}(at-t_0).$$

Substituting the corresponding spectrum

$$\hat{p}^{+}(\omega)=\frac{1}{a}\exp\left[-i\left(\frac{\omega-\omega_0}{a}\right)t\right]\hat{p}_2{}^{+}\left(\frac{\omega-\omega_0}{a}\right)$$

in Eq. 13, we obtain

$$\mathbf{B}_n{}^a(t,\Omega)=(1/a)e^{i\omega_0 t}\mathbf{B}_n(T,\Omega_0),$$

where $T=at-t_0$, $\Omega_0=(\Omega-\omega_0)/a$. The function $\mathbf{B}_n(T,\Omega_0)$ is given by Eq. 26 with t, Ω replaced by T, Ω_0, respectively. Thus, the effects on the response of time shifting, scaling, and modulation in the pressure field become transparent by inspection of Eqs. 27 and 28 under the above substitutions.

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