

Quivers

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1 Introduction

This is our project.



Throughout this paper, \mathbf{k} will indicate an algebraically closed field of characteristic 0.

2 Quivers

2.1 basic definitions and Path algebra

Definition 2.1 A quiver is directed graph $Q = (Q_0, Q_1)$ where Q_0 is the set of vertices and Q_1 set of arrows with maps $h, t : Q_1 \rightarrow Q_0$ which assign to each arrow its head and tail, respectively.

Definition 2.2 A path is a sequence of arrows $p = a_1 a_2 \cdots a_n$ such that $h(a_{k+1}) = t(a_k)$ for every $k = 1, \dots, n-1$. The head of the path is $h(a_1)$, and the tail of the path is $t(a_n)$.

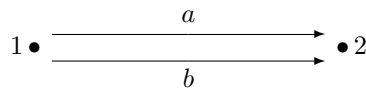
Definition 2.3 An oriented cycle is a path p such that $h(p) = t(p)$, and $h(a_i) \neq t(a_j)$ for any other $i \neq j+1$.

Note that: this definitions implies that a quiver like an “eight” is not an oriented cycle.

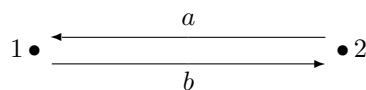
Proposition 2.1 A quiver with an oriented cycle has an infinite set of paths.

Example 2.1 The Jordan quiver This quiver does not appear in this program.

Example 2.2 The 2-Kronecker

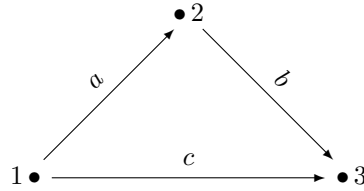


has no oriented cycle, but

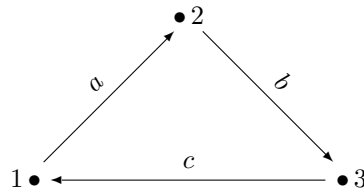


has an oriented cycle.

Example 2.3

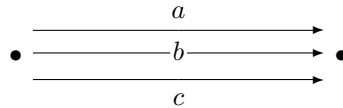


has no oriented cycle, but



has an oriented cycle.

Example 2.4 The 3-Kronecker quiver



Path algebra is an algebra build up from the paths. To form the path algebra, we must define a multiplication rule between the paths of the quiver:

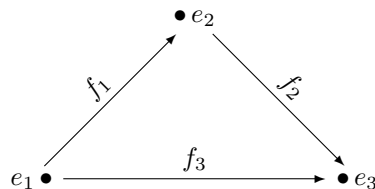
Definition 2.4 Given a quiver Q , the path algebra kQ is the k -vector space generated by all paths in Q with multiplication rule:

$$p * q = \begin{cases} pq & \text{if } h(q) = t(p) \\ 0 & \text{otherwise} \end{cases}$$

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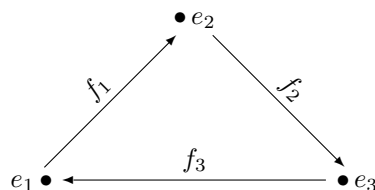
Now on we will denote the points in Q_0 as its respective null path.

Example 2.5



This quiver has path algebra with basis $\{e_1, e_2, e_3, f_1, f_2, f_3, f_2f_1\}$.

Example 2.6



This quiver has an infinite dimensional path algebra with basis: $\{e_1, e_2, e_3, f_1, f_2, f_3, f_2f_1, f_3f_2, f_1f_3, f_3f_2f_1, f_1f_3f_2, f_2f_1f_3, \dots\}$ since the quiver contains an oriented cycle.

2.2 quivers with relations

We can impose further relations on the composition of the arrows to make paths.

3 Quiver Representations

Definition 3.1 A quiver representation is a collection $\{V_i | i \in Q_0\}$ of finite dimensional \mathbf{k} -vector spaces together with a collection $\{\phi_a : V_{t(a)} \rightarrow V_{h(a)} | a \in Q_1\}$ of \mathbf{k} -linear maps.

From now on, we will denote a representation $\mathcal{R} = (V_i, \phi_a)$.

Definition 3.2 Suppose $\mathcal{R} = (V_i, \phi_a)$ and $\mathcal{S} = (W_i, \psi_a)$ are representations of Q . The representation \mathcal{R}' is called a subrepresentation of \mathcal{R} if

- for every $i \in Q_0$, W_i is a subspace of V_i and
- for every $a \in Q_1$, the restriction of $\phi_a : V_{t(a)} \rightarrow V_{h(a)}$ to $W_{t(a)}$ is equal to $\psi_a : W_{t(a)} \rightarrow W_{h(a)}$.

Definition 3.3 A non-zero representation V is called simple representation if the only subrepresentations of V are the zero representation and V itself.

Definition 3.4 If $\mathcal{R} = (V_i, \phi_a)$ and $\mathcal{S} = (W_i, \psi_a)$ are representations of a quiver Q then we can define the direct sum representation $\mathcal{R} \oplus \mathcal{S} = (\mathcal{U}_i, \rho_a)$ by:

- $U_i = V_i \oplus W_i$ for every $i \in Q_0$, and
- $\rho_a : V_{t(a)} \oplus W_{t(a)} \rightarrow V_{h(a)} \oplus W_{h(a)}$ is defined by the matrix $\begin{pmatrix} V_a & 0 \\ 0 & W_a \end{pmatrix}$

Definition 3.5 If \mathcal{R} and \mathcal{S} are two representation of a quiver Q , then a representation morphism $\Phi : \mathcal{R} \rightarrow \mathcal{S}$ is a collection \mathbf{k} -linear maps $\{\varphi_i : V_i \rightarrow W_i | i \in Q_0\}$ such that the diagram

$$\begin{array}{ccc}
 V_{t(a)} & \xrightarrow{\phi_a} & V_{h(a)} \\
 \varphi_{t(a)} \downarrow & & \varphi_{h(a)} \downarrow \\
 W_{t(a)} & \xrightarrow{\psi_a} & W_{h(a)}
 \end{array}$$

commutes for all $a \in Q_1$.

If φ_i is invertible for every $i \in Q_0$, then the morphism Φ is called an isomorphism and \mathcal{R} and \mathcal{S} are called isomorphic representations.

Definition 3.6 A representation \mathcal{R} of a quiver Q is called decomposable if $\mathcal{R} \cong \mathcal{S} \oplus \mathcal{T}$ where \mathcal{S} and \mathcal{T} are nonzero representations of Q .

A nonzero representation is called indecomposable if it is not decomposable.

3.1 Isomorphism Classes of Representations

The study of quiver representations will be simplified significantly if we can consider isomorphism classes of quiver representations, rather than specific representations.

To find a representative element of each isomorphism class, we apply representation isomorphisms to change the basis of the vector space at each vertex in order to simplify the matrices for the maps between the vector spaces. For square matrices and representations over \mathbb{C} , this process is the same as the theory of the Jordan normal form.

Example 3.1 For the so-called Jordan quiver with one vertex and one arrow, every isomorphism class of representations has a representative element of the form $\mathcal{R} = (\{V_1\}, \{J_1\})$ where J_1 is a matrix in Jordan normal form.

This is a direct result of the theorem that every square matrix $M = P^{-1}JP$, where J is a Jordan-form matrix, and P is an invertible matrix, corresponding to the change of basis required to isolate the eigenspace of each eigenvalue. {reference: Halmos}.

If we restrict ourselves to representations with invertible maps at each arrow, we may simultaneously describe the isomorphism classes of representations of quivers which differ from each other only in the orientation of their arrows.

Note that the invertibility condition implies that the representation must have equidimensional vector spaces at all vertices.

The isomorphism classes of these representations can often be described neatly, by analogy to the case of the Jordan quiver.

Example 3.2 For a quiver with two vertices and two arrows, given any representation $\mathcal{R} = (\{V_1, V_2\}, \{A, B\})$, where A and B are both invertible, we can find an isomorphic representation of the form:

$$V_1 \bullet \begin{array}{c} \xleftarrow{A} \\ \xrightarrow{B} \end{array} \bullet V_2 \cong V'_1 \bullet \begin{array}{c} \xleftarrow{Id} \\ \xrightarrow{J} \end{array} \bullet V'_2$$

To find this isomorphic representation, let \mathcal{B}_1 and \mathcal{B}_2 be bases for V_1 and V_2 , respectively. Take P_0 to be the change-of-basis matrix taking \mathcal{B}_2 to $A\mathcal{B}_1$. This is possible because invertibility implies equidimensionality.

Then the representation isomorphism $\Phi_0 = (Id, P_0)$ yields the isomorphic representation $\mathcal{R}' = (\{V_1, V'_2\}, \{Id, BA\})$.

By Jordan's Theorem {Halmos}, we can find P_1 invertible such that $BA = P_1^{-1}JP_1$, where J is a Jordan-form matrix. Applying the representation isomorphism $\Phi_1 = (P_1, P_1)$ yields the desired isomorphic representation.

Example 3.3 Given a representation $\mathcal{R} = (\{V_1, V_2, V_3\}, \{A, B, C\})$ (with A, B, C all invertible) of the triangular quiver with three vertices and three arrows, we may find an isomorphic representation of the form:

$$\begin{array}{ccc} & \bullet 2 & \\ & \nearrow A & \searrow B \\ \bullet 1 & & \bullet 3 \\ & \xleftarrow{C} & \end{array} \cong \begin{array}{ccc} & \bullet 2 & \\ & \nearrow Id & \searrow Id \\ \bullet 1 & & \bullet 3 \\ & \xleftarrow{J} & \end{array}$$

where J is a Jordan-form matrix.

The process for finding this representation is similar to the one described in Example 3.2 above.

Example 3.4 *The case of the 3-Kronecker quiver is more complicated, because we may not be able to simultaneously put the maps on the second and third arrows in Jordan normal form.*

However, in the case $\dim(V_1) = \dim(V_2) = 2$, we will always be able to conjugate bases and obtain the form:

$$\begin{array}{ccc}
 & \xrightarrow{A} & \\
 V_1 \bullet & \xrightarrow{B} & \bullet V_2 \\
 & \xrightarrow{C} &
 \end{array}
 \cong
 \begin{array}{ccc}
 & \xrightarrow{Id} & \\
 V'_1 \bullet & \xrightarrow{J} & \bullet V'_2 \\
 & \xrightarrow{\begin{pmatrix} a & 0 \\ b & c \end{pmatrix}} &
 \end{array}$$

where $A = Id$, B is in Jordan normal form, and C is a lower-triangular matrix.

3.2 Simple Representations

Definition 3.7 *A canonical representation \mathcal{R} for the quiver $Q = (Q_0, Q_1)$ is a representation of the form*

$$\mathcal{R} = \{V_i = \begin{cases} \mathbf{k} & \text{for one } i \in Q_0 \\ \{0\} & \text{otherwise} \end{cases}, \phi_a = 0 \text{ for all } a \in Q_1\}.$$

A canonical representation \mathcal{R} must be simple, because the only subspace of it's V_i is the null space at each vertex. This implies that the only proper subrepresentation is the trivial one, so \mathcal{R} is simple

Proposition 3.1 *Let Q be a quiver with no oriented cycles. A representation \mathcal{R} of Q is simple if and only if it is canonical.*

Proof

(\Leftarrow) This follows from the definition of canonical representations.

(\Rightarrow) We will show that every representation not of the form described above has a proper subrepresentation of that form. First, we begin with the following lemma. ■

Lemma 3.1 *If $Q = (Q_0, Q_1)$ is a quiver with no oriented cycles, then there is some vertex $x \in Q_0$ that is not the tail of any arrow. Such an arrow is called a sink.*

Proof: Proof by contrapositive. Suppose for any $v_i \in Q_0$, $v_i = t(a)$ for some $a \in Q_1$. Choose some $v_1 \in Q_0$ and form a path as follows: Choose a_n such that $t(a_n) = v_n$. Write $v_{n+1} = h(a_n)$, and repeat. Since Q_0 is a finite set, eventually we will get $v_{n+1} = v_i$ for some $i \leq n$. Then $p = a_i \cdots a_n$ is an oriented cycle in Q . ■

Lemma 3.2 *Let Q be a quiver with no oriented cycle, write $x_1 \in Q_0$ is a vertex such that $t(a) \neq x_1$ for all $a \in Q_1$.*

Proof: Given an arbitrary representation $R = (V_i, \rho_a)$, if $V_{x_1} \neq \{0\}$, then label this vertex x . ($V_{h(a)} = \{0\}$ for all a such that $t(a) = x$). If $V_{x_1} = \{0\}$, define $Q^1 = (Q_0^1 = Q_0/\{x_1\}, Q_1^1 = Q_1/\{a \in Q_1 | h(a) = x_1\})$. Since Q contained no oriented cycles, and $Q_0^1 \subset Q_0$, $Q_1^1 \subset Q_1$, Q^1 contains no oriented cycle, so we may apply the lemma. Write $x_2 \in Q_0^1$ is a vertex such that $t(a) \neq x_2$ for all $a \in Q_1^1$. Define the representation R' of Q' by restricting the representation R of Q . If R is a non-trivial representation of Q , we will eventually find $x_n \in Q_0$ such that $V_{x_n} \neq \{0\}$ but $V_i = \{0\}$ for all i such that $i = h(a)$ for some $a \in Q_1$ such that $t(a) = x_n$. ■

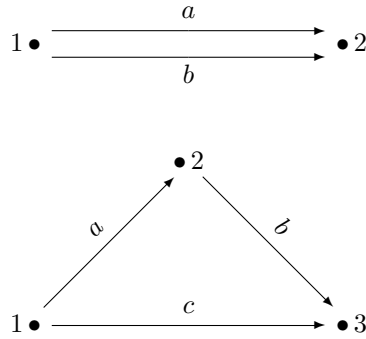
Proposition 3.2 *Construct a representation*

$$\mathcal{S} = \{W_i = \begin{cases} \mathbf{k} & i = x \\ \{0\} & i \neq x \end{cases}, \phi_a = 0 \text{ for all } a \in Q_1\} \text{ for the quiver } Q.$$

\mathcal{S} is a subrepresentation of \mathcal{R} .

Proof: Clearly, for all $i \neq x$ $\{0\} = w_i \subseteq v_i$. Since v_x is a non-zero k -vector space, $k = w_x \subseteq v_x$. Define $p = \{p_i : i \in \mathbb{Q}_0\}$ a representation morphism such that $p_i : w_i \rightarrow v_i$ is the inclusion map. To check that all maps commute, first note that for $a \in \mathbb{Q}_1$ such that $t(a) \neq x$, $W_{t(a)} = \{0\}$. So $\psi_a : W_{t(a)} \rightarrow W_{h(a)}$ has $\{0\}$ as its domain, i.e. $\psi_a = 0$. Similarly $P_{t(a)} : W_{t(a)} \rightarrow V_{t(a)}$ is the inclusion of $\{0\}$ implies $P_{t(a)} = 0$. Hence, for all $a \in \mathbb{Q}_1$ such that $t(a) \neq x$ we have: $p_{h(a)} \circ \psi_a = p_{h(a)} \circ 0 = 0$ and $\varphi_a \circ p_{t(a)} = \varphi_a \circ 0 = 0$ so the diagram commutes. Now, for all a such that $t(a) = x$, we know that $V_{h(a)} = \{0\}$. So $\varphi_a : V_{t(a)} \rightarrow V_{h(a)}$ is $\varphi_a : V_x \rightarrow \{0\}$, i.e. $\varphi_a = \{0\}$. Similarly, $\psi_a = 0$ and $p_{h(a)} : \{0\} \rightarrow \{0\}$ is also the zero map. So for all $a \in \mathbb{Q}_1$ such that $t(a) = x$, we have $p_{h(a)} = \psi_a = 0 \circ 0 = 0$ so the diagram commutes. Therefore, \mathcal{S} is a subrepresentation of \mathcal{R} of the desired form. ■

Example 3.5



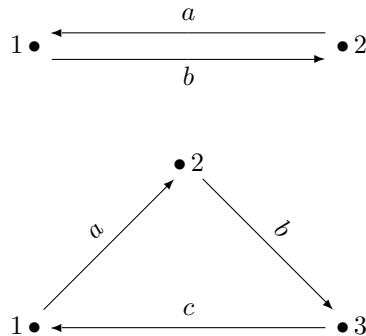
These quivers have no oriented cycles, and only canonical simple representations.

Definition 3.8 A unique cycle is a cycle that has no other cycle on its vertices.

Proposition 3.3 Every unique cycle generates a 1-parameter family of simple representations.

Sketch of proof: Form the identity around the quiver. Get the first column of the quiver and it will be one parameter.

Example 3.6



these quivers have unique oriented cycles, and 1-parameter families of simple representations.

3.3 Indecomposable Representations

Here we will work by the examples we have shown above. The invertability of maps and the dimensions vectors will play an important role in order to give all indecomposable representations for some given quiver representation.

Example 3.7 *For the jordan quiver, if the map is invertible and there is an isomorphic representation with the canonical Jordan matrix, then it is indecomposable.*

Example 3.8 *For the example 2.3(?), we have the following classification:*

Proposition 3.4 *A representation $\mathcal{R} = (\{V_1 = \mathbb{C}^m, V_2 = \mathbb{C}^n\}, \{A, B\})$ (such that wlog $m > n$) of the quiver Q is indecomposable if and only if one of the following holds:*

- $\mathcal{R} \cong \mathcal{R}' = (\{W_1, W_2\}, \{Id, J_\lambda\})$ where J_λ is a matrix in Jordan normal form with only one eigenblock.
- $(BA)^k = 0$ for some $k \in \mathbb{Z}$ and $\dim \ker BA = 1$.

Corollary: The set of dimension vectors of indecomposable representations of Q is $\mathcal{D}_Q = \{(n, n), (n, n \pm 1) | n \in \mathbb{Z}_+\}$.

Proof of Proposition: We describe the possible cases, and prove decomposability or indecomposability for each case.

1. BA invertible
 - \mathcal{R} is indecomposable \leftrightarrow it is isomorphic to a representation with one Jordan block.
2. BA non-invertible
 - (a) BA is nilpotent, i.e. $(BA)^k = 0$ for some $k \in \mathbb{Z}_+$
 - i. $\text{null} BA = 1$
 - \mathcal{R} is indecomposable.
 - ii. $\text{null} BA \geq 1$
 - \mathcal{R} is decomposable.
 - (b) BA is not nilpotent. i.e. $(BA)^k \neq 0 \forall k \in \mathbb{Z}_+$
 - \mathcal{R} is decomposable.

Proof of Claim:

1. BA invertible
 - $m=n$, A, B are both invertible
 - as shown above \mathcal{R} is indecomposable \iff it is isomorphic to a representation with one Jordan block.

2(a)i. $(BA)^k = 0$ for some $k \in \mathbb{Z}_+$, $\text{null}(BA)=1$.
 Let $x \in \ker BA$. Then $BAy = 0 \rightarrow y = \lambda x$ for some $\lambda \in \mathbb{Z}_+$
 Suppose $\mathcal{R} = \mathcal{R}' \oplus \mathcal{R}''$ where

$$\mathcal{R}' = (\{W_1, W_2\}, \{A|_{W_1}, B|_{W_2}\}) , \mathcal{R}'' = (\{U_1, U_2\}, \{A|_{U_1}, B|_{U_2}\})$$

are both non-trivial. wlog, $x \in W_1$.
 Suppose $y \in U_1, y \neq 0$.
 → by definition of decomposability, $(BA)^i \in U_1 \forall i \in \mathbb{Z}_+$

But $(BA)^k = 0$

→ pick the least $j \in \mathbb{Z}_+$ such that $(BA)^j = 0$.

→ $(BA)^{j-1} \in \ker BA$

→ $(BA)^{j-1}y = \lambda x \in W_1$.

This yields a contradiction → $V_1 = W_1$.

Suppose $y \in U_2$, $y \neq 0$

→ because of a dimension argument, either $y = Ax$ for some $x \in V_1$ or $By = x$ for some nonzero $x \in V_1$.

→ in either case, $y \in W_2$ because of the invariance of subrepresentations.

This yields a contradiction → $V_2 = W_2$

→ \mathcal{R}'' is the trivial subrepresentation.

→ \mathcal{R} is indecomposable.

2(a)ii. $\text{null}(BA) \not\subseteq 1$

write $\ker BA = W_0 \oplus U_0$ both of which are non-zero.

for $x \in V_1$, write $j_x \in \mathbb{Z}_+$ is the minimal integer such that $(BA)^{j_x} = 0$, and define:

$$W_1 = \{x \in V_1 | (BA)^{j_x} x - 1x \in W_0\}, \quad U_1 = \{x \in V_1 | (BA)^{j_x-1} x \in U_0\}$$

These two sets define a decomposition of \mathcal{R} .

→ \mathcal{R} is decomposable.

2(b) $(BA)^k \neq 0 \forall k \in \mathbb{Z}_+$

→ $\exists j \in \mathbb{Z}_+$ s.t. $V_1 = \ker(BA)^j \oplus W_1$ and $(BA)^j|_{W_1}$ is invertible.

These sets define a decomposition of \mathcal{R}

→ \mathcal{R} is decomposable. ■

3.4 Representation of Path Algebras

Given a representation $\mathcal{R} = (V_i, \phi_a)$ of the quiver $Q = (Q_0, Q_1)$, we can construct a representation $\rho: \mathbf{k}Q \rightarrow \text{End}(\bigoplus_{i \in Q_0} V_i)$ of the path algebra $\mathbf{k}Q$.

It suffices to define the representation on the e_i 's and f_j 's, as these generate the basis as a ring.

$$\begin{aligned} \rho(e_i) &:= \text{Id}|_{V_i}, & \rho(f_j) &: V_{t(j)} \rightarrow V_{h(j)} \\ & & & x \mapsto \phi_j(x) \end{aligned}$$

This extends to a representation on all elements of $\mathbf{k}Q$.

4 Lie Algebras and Their Representations

Definition 4.1 A Lie Algebra \mathfrak{g} is a non-associative algebra with the multiple rule given by the bilinear map $[\cdot, \cdot]$ which satisfies

- $[x, x] = 0$ for all $x \in \mathfrak{g}$,
- $[x, [y, x]] + [y, [z, x]] + [z, [x, y]] = 0 \forall x, y, z \in \mathfrak{g}$.

These two properties imply that the $[\cdot, \cdot]$ operation is anti-symmetric, i.e. $[x, y] = -[y, x] \forall x, y \in \mathfrak{g}$.

We can construct a Lie algebra from any associative algebra by defining the bracket operation as the commutator $[a, b] = ab - ba$.

Example 4.1 Let V be a vector space. this is an associative algebra. we can define the commutator bracket. because V is abelian, $[a, b] = 0$ always.

4.1 Representation of $\mathfrak{sl}_2(\mathbf{k})$

The simple linear algebra $\mathfrak{sl}_2(\mathbf{k}) = \{A \in M_2(\mathbf{k}) | \text{tr}A = 0\}$ of traceless 2×2 matrices is a Lie Algebra with bracket operation defined by the commutator $[A, B] = AB - BA$ and basis:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We will describe the equivalence classes of simple and indecomposable representations of $\mathfrak{sl}_2(\mathbf{k})$ and show that these correspond to simple and indecomposable representations of the quiver with relations described in Example ???.

We will restrict ourselves to representations of $\mathfrak{sl}_2(\mathbf{k})$ such that

- $V = \bigoplus_{k \in \mathbb{Z}} V_k$,
 where $V_k = \{v \in V | hv = kv\}$ is the eigenspace with eigenvalue k for the action of h on V .
- $V_k = 0$ for $k \gg 0$.
- Each V_k is finite dimensional.

Given $v \in V_k$, we calculate to find the following properties:

- $h(f(v)) = (k - 2)f(v)$
- $h(e(v)) = (k + 2)e(v)$

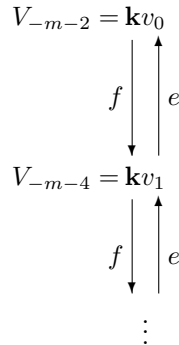
In other words, the action of f takes the eigenspace V_k with eigenvalue k to V_{k-2} with eigenvalue $k - 2$, and the action of e takes V_k to V_{k+2} .

Since $m \in \mathbb{Z}$ is the maximal eigenvalue, we observe that

$$V_k = \begin{cases} \mathbf{k} & \text{if } k = m - 2i \text{ for } i \in \mathbb{Z}_+ \\ 0 & \text{otherwise} \end{cases}$$

Thus, if we take $v_0 \in V_m$, we can define a basis $\mathcal{B} = \{v_i | i \in \mathbb{Z}_+\}$ for $V(m) = \bigoplus_{k \leq m} V_k$ by the rule $v_i = f^i(v_0)$. From the equations above, we can calculate that $e(v_i) = i(m - i + 1)v_{i+1}$.

M(-m-2) If the highest eigenvalue of a representation is negative, then the map e does not annihilate any of the eigenspaces, and we have an infinite-dimensional, simple representation called $M(-m-2)$. In this notation, m is positive, so $-m-2$ is the desired negative integer.



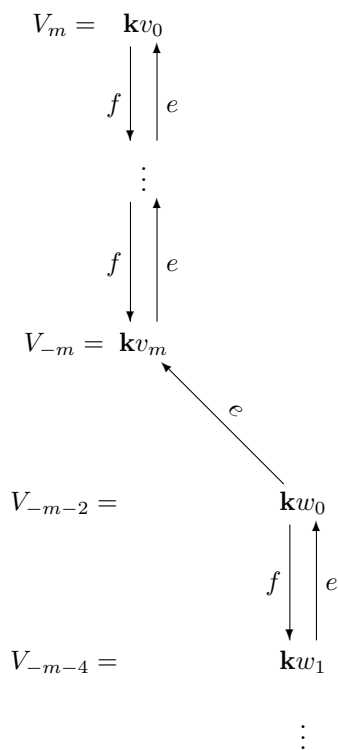
V(m) However, if the highest eigenvalue is positive, the action of e will annihilate the eigenspace V_{-m-2} , so the representation will not be simple, in fact, it has $M(-m-2)$ for a subrepresentation. It will, however, be indecomposable, as the subspace $\bigoplus_{-m/2 \leq k \leq m} V_k$ is not invariant under the action of f . This representation is called $V(m)$.

$$\begin{array}{c}
 V_m = \mathbf{k}v_0 \\
 \downarrow f \quad \uparrow e \\
 \vdots \\
 \downarrow f \quad \uparrow e \\
 V_{-m} = \mathbf{k}v_m \\
 \downarrow f \\
 V_{-m-2} = \mathbf{k}v_{m+1} \\
 \downarrow f \quad \uparrow e \\
 \vdots
 \end{array}$$

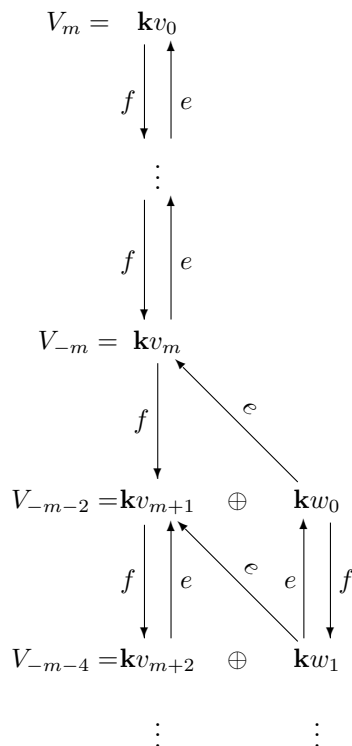
M(m) Taking the quotient representation $V(m)/M(-m-2)$ gives the second simple representation, and the only one with positive maximal eigenvalue, $M(m)$:

$$\begin{array}{c}
 V_m = \mathbf{k}v_0 \\
 \downarrow f \quad \uparrow e \\
 \vdots \\
 \downarrow f \quad \uparrow e \\
 V_{-m} = \mathbf{k}v_m
 \end{array}$$

M*(m) Because we have $e(v_{-m-2}) = 0$, we may define another linearly independent eigenvector w_0 with eigenvalue $-m-2$, such that $e(w_0) = v_{-m}$. This can be extended by the rule $w_i = f^i(w_0)$ to find a second eigenvector for each eigenvalue $\lambda = -m - 2(i + 1)$. Taking $V_k = \text{span}\{v_{1/2(m-k)}\}$ for $m > k > -m$ and $V_k = \text{span}\{w_{1/2(-m-2-k)}\}$ for $k \leq -m - 2$, we find the representation $M^*(m)$:



P(-m-2) However, for $k \leq -m-2$, we may also allow V_k to be two-dimensional, that is, $V_k = \text{span}\{v_{1/2(m-k)}, w_{1/2(-m-2-k)}\}$. This describes the representation called $P(-m-2)$.



In this case, note that $e(w_i) = \mathbf{a}w_{i-1} + \mathbf{b}vi - m - 2$

5 Cartan Matrix and Kac-Moody Algebras

Definition 5.1 A Cartan matrix is a matrix $A = (a_{ij})$ such that:

- $a_{ii} = 2$
- $a_{ij} \leq 0$ if $i \neq j$ and $a_{ij} \in \mathbb{Z}$
- $a_{ij} = 0$ if and only if $a_{ji} = 0$.

Let Q be a quiver without loops, $\#Q_0 = n$. We can construct an $n \times n$ Cartan matrix

$$A = (a_{ij}) \text{ where } \begin{cases} a_{ii} = 2 & (i = 1, \dots, n) \\ -a_{ij} = \#\{\text{arrows btw vertices } i \text{ and } j\} & (i \neq j) \end{cases}$$

The orientation of the arrows in Q does not matter in the construction of the Cartan matrix, so we may disregard the orientation.

Definition 5.2 Given an $n \times n$ Cartan matrix, we may construct a Lie algebra, called the Kac-Moody algebra, with the following generators and relations:

generators: $\{e_i, f_i, h_i \mid i = 1, \dots, n\}$

relations:

- $[h_i, e_j] = a_{ij}e_j$
- $[h_i, f_j] = -a_{ij}f_j$
- $[e_i, f_j] = \delta_{i,j}h_i$
- $ad^{1-a_{ij}}(e_i)e_j = 0$
- $ad^{1-a_{ij}}(f_i)f_j = 0$.

where $ad(\alpha) = ad_\alpha$, $\alpha \in \mathfrak{g}(A)$ is the adjoint action $ad_\alpha(\beta) = [\alpha, \beta]$.

Definition 5.3 The simple roots of a Kac-Moody algebra $\mathfrak{g}(A)$ are the functionals $\{\alpha_j\}$, $j = 1, \dots, n$, defined by: $\alpha_j(h_i) = a_{ij}$.

Consider $\Gamma = \bigoplus \mathbb{Z}\alpha_i$ the set of all \mathbb{Z} -linear combinations of simple roots.

Claim: If α is a root of $\mathfrak{g}(A)$ then $\alpha \in \Gamma$.

Definition 5.4 For $(\alpha_i, \alpha_j) = a_{ij}$ the maps $r_i : \mathfrak{h}^* \rightarrow \mathfrak{h}^*$, such that $r_i(\lambda) = \lambda - (\lambda, \alpha_i)\alpha_i$ are called simple reflections.

Remark: Since $(r_i) = 1_{\mathfrak{h}^*}$, r_i is invertible for every i and the set of all compositions of r_i is indeed a group, denoted by \mathcal{W} , called the *Weyl Group*. The Weyl group is a very important group as we will see in the next proposition when we want to find the real and imaginary roots, as the following definitions:

Definition 5.5 Let α be an element of Γ , so α is a real root if $\exists w \in \mathcal{W}$ such that $w(\alpha) = \alpha_i$ for some $i \in \{1, 2, \dots, n\}$. The others are called imaginary roots.

Proposition 5.1 The real and imaginary roots may be calculate by the following formulas:

- $\mathcal{R}^{re} = \bigcup_{w \in \mathcal{W}} w(\alpha_i)$
- $\mathcal{R}^{im} = \bigcup_{w \in \mathcal{W}} w(M)$, where $M = \{\alpha \in \Gamma^+ \mid (\alpha, \alpha_i) \leq 0, \forall i\}$

remark Indeed, the definition of M contains one more property which does not make difference in the examples we are working with.

From now on we will find the simple roots, simple reflections, real and imaginary roots of the five examples above.

5.1 $\mathfrak{sl}_2(\mathbf{k})$

The quiver $Q_A = \bullet$ gives Cartan matrix $A = (2)$ The $\mathfrak{g}(A)$ is the Lie algebra generated by e, f, h such that:

- $[h, e] = 2e$
- $[h, f] = -2f$
- $[e, f] = h$

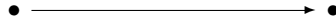
So $\mathfrak{g}(A)$ has one simple root, namely $\alpha(h) = 2$.

The simple reflection is $r(\lambda\alpha) = \lambda\alpha - (\lambda\alpha, \alpha)\alpha = -\lambda\alpha$.

Then the Weyl group is $\mathcal{W} = \{e, r\}$. Now the real roots are $\mathcal{R}^{re} = \{\alpha, -\alpha\}$, and since $M = \{\alpha \in \Gamma^+ \setminus \{0\} \mid (\alpha, \alpha) \leq 0\}$, this implies that $M = \emptyset$, which means that all the roots are real.

5.2 $\mathfrak{sl}_3(\mathbf{k})$

The quiver $Q_B =$



gives Cartan matrix $B = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$\mathfrak{g}(B)$ is the Lie Algebra generated by $\{e_1, e_2, f_1, f_2, h_1, h_2\}$ with relations:

- $[h_1, e_1] = 2e_1, [h_1, e_2] = -e_2$
- $[h_1, f_1] = -2f_1, [h_1, f_2] = f_2$
- $[e_1, f_1] = h_1, [e_1, f_2] = 0, [e_2, f_1] = 0, [e_2, f_2] = h_2$
- $[e_1, [e_1, e_2]] = 0, [e_2, [e_2, e_1]] = 0$
- $[f_1, [f_1, f_2]] = 0, [f_2, [f_2, f_1]] = 0$
- $[h_1, h_2] = 0$.

Indeed, $\mathfrak{g}(B) \simeq \mathfrak{sl}_3(\mathbf{k})$, the algebra of 3×3 traceless matrices with bracket operation defined by the commutator.

The simple roots are $\{\alpha_1, \alpha_2\}$ where $\alpha_1(h_1) = 2, \alpha_2(h_1) = -1, \alpha_1(h_2) = -1, \alpha_2(h_2) = 2$ and any root is of the form $\alpha = (z_1, z_2) \in \Gamma$.

The simple reflections are $r_1(z_1, z_2) = (z_2 - z_1, z_2)$ and $r_2(z_1, z_2) = (z_1, z_1 - z_2)$. The Weyl Group is $\mathcal{W} = \{I_d, r_1, r_2, r_1r_2, r_2r_1, r_1r_2r_1\}$ where:

- $r_1r_2(z_1, z_2) = (-z_2, z_1 - z_2),$
- $r_2r_1(z_1, z_2) = (z_2 - z_1, -z_1)$
- $r_1r_2r_1(z_1, z_2) = r_2r_1r_2(z_1, z_2) = (-z_2, -z_1)$

So, \mathcal{W} is a finite group. The characterization of \mathcal{W} makes possible to find all real and imaginary roots, as follows:

$$\mathcal{R}^{re} = \{\alpha_1, -\alpha_1, \alpha_2, -\alpha_2, \alpha_1 + \alpha_2, -\alpha_1 - \alpha_2\}$$

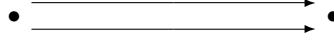
.

$$\mathcal{R}^{im} = \emptyset$$

since $M = \emptyset$.

5.3 $\widehat{\mathfrak{sl}}_2$

The quiver $Q_C =$



gives Cartan matrix $C = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

$\mathfrak{g}(C)$ is the Lie Algebra generated by $\{e_1, e_2, f_1, f_2, h_1, h_2\}$ with relations:

- $[h_1, e_1] = 2e_1, [h_1, e_2] = -2e_2$
- $[h_1, f_1] = -2f_1, [h_1, f_2] = 2f_2$
- $[e_1, f_1] = h_1, [e_1, f_2] = 0, [e_2, f_1] = 0, [e_2, f_2] = h_2$
- $[e_1, [e_1, [e_1, e_2]]] = 0, [e_2, [e_2, [e_2, e_1]]] = 0,$
- $[f_1, [f_1, [f_1, f_2]]] = 0, [f_2, [f_2, [f_2, f_1]]] = 0$
- $[h_1, h_2] = 0.$

Indeed, $\mathfrak{g}(C) \simeq \mathfrak{sl}_2[t, t^{-1}] \oplus \mathfrak{kc}$ for some constant c .

The simple roots are $\{\alpha_1, \alpha_2\}$ where

$$\alpha_1(h_1) = 2 = \alpha_2(h_2), \alpha_2(h_1) = -2 = \alpha_1(h_2)$$

and any root is of the form $\alpha = (z_1, z_2) \in \Gamma$.

The simple reflections are $r_1(z_1, z_2) = (-z_1 + 2z_2, z_2)$ and $r_2(z_1, z_2) = (z_1, 2z_1 - z_2)$. The Weyl Group $\mathcal{W} = \langle r_1, r_2 \rangle / \langle r_1^2 = 1, r_2^2 = 1 \rangle$ is an infinite dimensional group with the following properties, for $r \in \mathcal{W}$:

$$r(\alpha_1) = \begin{cases} (-2j+1, -2j) & \text{if } r = (r_2 r_1)^j \\ (2j+1, 2j) & \text{if } r = (r_1 r_2)^j \\ (-2j-1, -2j) & \text{if } r = r_1 (r_2 r_1)^j \\ (2j+1, 2(j+1)) & \text{if } r = r_2 (r_1 r_2)^j \end{cases}$$

$$r(\alpha_2) = \begin{cases} (2j, 2j+1) & \text{if } r = (r_2 r_1)^j \\ (-2j, -2j+1) & \text{if } r = (r_1 r_2)^j \\ (2(j+1), 2j+1) & \text{if } r = r_1 (r_2 r_1)^j \\ (-2j, -2j-1) & \text{if } r = r_2 (r_1 r_2)^j \end{cases}$$

where $j \in \mathbb{N}$.

So by choosing any positive integer j , we find that the real roots are:

$$\mathcal{R}^{re} = \{(m, n) \mid m, n \in \mathbb{Z}, |m - n| = 1\}$$

To calculate the imaginary roots, we note that:

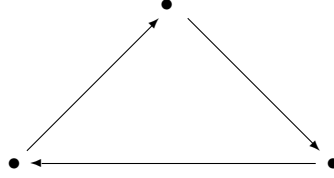
$$\begin{aligned} M &= \{(z_1, z_2) \mid z_1, z_2 \in \mathbb{Z}_+, 2z_1 - 2z_2 \leq 0, -2z_1 + 2z_2 \leq 0\} \\ &= \{(z_1, z_2) \mid z_1, z_2 \in \mathbb{Z}_+, z_1 = z_2\} \\ &= \mathbb{Z}_+(1, 1) \end{aligned}$$

These elements are preserved by the action of \mathcal{W} , so we have:

$$\mathcal{R}^{im} = \mathbb{Z}_+(\alpha_1 + \alpha_2) = \{(n, n) \mid n \in \mathbb{Z}\}$$

5.4 $\widehat{\mathfrak{sl}}_3\mathbf{k}$

The quiver $Q_D =$



gives Cartan matrix $D = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$.

$\mathfrak{g}(D)$ is the Lie Algebra generated by $\{e_1, e_2, e_3, f_1, f_2, f_3, h_1, h_2, h_3\}$ with relations:

- $[h_1, e_1] = 2e_1, [h_1, e_2] = -e_2, [h_1, e_3] = -e_3,$
- $[h_2, e_1] = -e_1, [h_2, e_2] = 2e_2, [h_2, e_3] = -e_3,$
- $[h_3, e_1] = -e_1, [h_3, e_2] = -e_2, [h_3, e_3] = 2e_3,$
- $[h_1, f_1] = -2f_1, [h_1, f_2] = f_2, [h_1, f_3] = f_3,$
- $[h_2, f_1] = f_1, [h_2, f_2] = -2f_2, [h_2, f_3] = f_3,$
- $[h_3, f_1] = f_1, [h_3, f_2] = f_2, [h_3, f_3] = -2f_3,$
- $[e_1, f_1] = h_1, [e_1, f_2] = 0, [e_1, f_3] = 0,$
- $[e_2, f_1] = 0, [e_2, f_2] = h_2, [e_2, f_3] = 0,$
- $[e_3, f_1] = 0, [e_3, f_2] = 0, [e_3, f_3] = h_3,$
- $[e_1, [e_1, e_2]] = 0, [e_1, [e_1, e_3]] = 0,$
- $[e_2, [e_2, e_1]] = 0, [e_2, [e_2, e_3]] = 0,$
- $[e_3, [e_3, e_1]] = 0, [e_3, [e_3, e_2]] = 0,$
- $[f_1, [f_1, f_2]] = 0, [f_1, [f_1, f_3]] = 0,$
- $[f_2, [f_2, f_1]] = 0, [f_2, [f_2, f_3]] = 0,$
- $[f_3, [f_3, f_1]] = 0, [f_3, [f_3, f_2]] = 0$
- $[h_1, h_2] = 0, [h_2, h_3] = 0.$

Indeed, $\mathfrak{g}(D) \simeq \widehat{\mathfrak{sl}}_3[t, t^{-1}] \oplus \mathbb{C}c$ for some constant c .

The simple roots are $\{\alpha_1, \alpha_2, \alpha_3\}$ where:

$$\alpha_1(h_1) = \alpha_2(h_2) = \alpha_3(h_3) = 2$$

$$\alpha_2(h_1) = \alpha_3(h_1) = \alpha_1(h_2) = \alpha_3(h_2) = \alpha_1(h_3) = \alpha_2(h_3) = -1$$

and any root is of the form $\alpha = \alpha = (z_1, z_2, z_3) \in \Gamma$.

The simple reflections are $r_1(z_1, z_2, z_3) = (-z_1 + z_2 + z_3, z_2, z_3),$
 $r_1(z_1, z_2, z_3) = (z_1, z_1 - z_2 + z_3, z_3),$ $r_1(z_1, z_2, z_3) = (z_1, z_2, z_1 + z_3 - z_3).$

The Weyl Group $\mathcal{W} = \langle r_1, r_2 \rangle / \langle r_1^2 = 1, r_2^2 = 1 \rangle$ is an infinite dimensional group with the following properties:

- $r_1(k \pm 1, k, k) = (k \mp 1, k, k)$

- $r_1(k, k \pm 1, k) = (k \pm 1, k \pm 1, k)$
- $r_1(k, k, k \pm 1) = (k \pm 1, k, k \pm 1)$
- $r_2(k \pm 1, k, k) = (k \pm 1, k \pm 1, k)$
- $r_2(k, k \pm 1, k) = (k, k \mp 1, k)$
- $r_2(k, k, k \pm 1) = (k, k \mp 1, k \pm 1)$
- $r_3(k \pm 1, k, k) = (k \pm 1, k, k \pm 1)$
- $r_3(k, k \pm 1, k) = (k, k \pm 1, k \pm 1)$
- $r_3(k, k, k \pm 1) = (k, k, k \mp 1)$

With these properties, we can compute the real and imaginary roots which leads to:

$$\mathcal{R}^{re} = \{(k \pm 1, k, k), (k, k \pm 1, k), (k, k, k \pm 1)\}_{k \in \mathbb{Z}}$$

To calculate the imaginary roots, we note that:

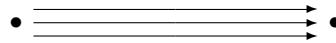
$$\begin{aligned} M &= \{(z_1, z_2, z_3) \mid z_1, z_2, z_3 \in \mathbb{Z}_+, 2z_1 - z_2 - z_3 \leq 0, -z_1 + 2z_2 - z_3 \leq 0, -z_1 - z_2 + 2z_3 \leq 0\} \\ &= \{(z_1, z_2, z_3) \mid z_1, z_2, z_3 \in \mathbb{Z}_+, z_1 = z_2 = z_3\} \\ &= \mathbb{Z}_+(1, 1, 1) \end{aligned}$$

This set is preserved by \mathcal{W} , so we have:

$$\mathcal{R}^{im} = \mathbb{Z}_+(\alpha_1 + \alpha_2 + \alpha_3)$$

5.5 3-Kronecker

The quiver $Q_E =$



gives Cartan matrix $E = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$

$\mathfrak{g}(E)$ is the Lie Algebra generated by $\{e_1, e_2, f_1, f_2, h_1, h_2\}$ given by the relations:

- $[h_1, e_1] = 2e_1, [h_1, e_2] = -3e_2,$
- $[h_1, f_1] = -2f_1, [h_1, f_2] = 3f_2,$
- $[e_1, f_1] = h_1, [e_1, f_2] = 0,$
- $[e_2, f_1] = 0, [e_2, f_2] = h_2,$
- $[e_1, [e_1, [e_1, [e_1, e_2]]]] = 0, [e_2, [e_2, [e_2, [e_2, e_1]]]] = 0,$
- $[f_1, [f_1, [f_1, [f_1, f_2]]]] = 0, [f_2, [f_2, [f_2, [f_2, f_1]]]] = 0,$
- $[h_1, h_2] = 0.$

The simple roots are $\{\alpha_1, \alpha_2\}$ where:

$$\begin{aligned} \alpha_1(h_1) &= \alpha_2(h_2) = 2 \\ \alpha_2(h_1) &= \alpha_1(h_2) = -3 \end{aligned}$$

and any root is of the form $\alpha = \alpha = (z_1, z_2) \in \Gamma$.

The simple reflections are $r_1(z_1, z_2) = (-z_1 + 3z_2, z_2)$ and $r_1(z_1, z_2) = (z_1, 3z_1 - z_2)$.

The Weyl Group $\mathcal{W} = \langle r_1, r_2 \rangle / \langle r_1^2 = 1, r_2^2 = 1 \rangle$ is an infinite dimensional group. Its action on the set of simple roots give rise to a sequence $\{a_n\}$ satisfying $a_n = 3a_{n-1} - a_{n-2}$. After solving the equation for this "Fibonacci" sequence, we find the sequence of roots given by

$$a_n = \left(\frac{3\sqrt{5}}{10} + \frac{1}{2} \right) \left(\frac{3 + \sqrt{5}}{2} \right)^n + \left(-\frac{3\sqrt{5}}{10} + \frac{1}{2} \right) \left(\frac{3 - \sqrt{5}}{2} \right)^n$$

This sequence allows us to explicitly give the properties of the elements of the Weyl Group.

- $(r_2 r_1)^j = \begin{pmatrix} -a_j & a_{j+1} \\ a_{j+1} & a_{j+2} \end{pmatrix}$
- $r_1 (r_2 r_1)^j = \begin{pmatrix} -a_{j+2} & a_{j+3} \\ -a_{j+1} & a_{j+2} \end{pmatrix}$
- $(r_2 r_1)^j r_2 = \begin{pmatrix} -a_{j+1} & -a_j \\ a_{j+2} & -a_{j+1} \end{pmatrix}$
- $(r_1 r_2)^j = \begin{pmatrix} a_{j+2} & -a_{j+1} \\ a_{j+1} & -a_j \end{pmatrix}$

Finally, we have that

$$\mathcal{R}^{re} = \{(\pm a_k, \pm a_{k+1}), (\pm a_{k+1}, \pm a_k), \text{ where } a_k \in \mathbb{Z} \text{ and } a_n \text{ satisfies } a_n = 3a_{n-1} - a_{n-2}\}$$

!!!!!!! We need to formalize what we will say about imaginary roots of the 3-kronecker.... $M = \{(m, n) \mid m, n \in \mathbb{Z}, \frac{2}{3}m \leq n \leq \frac{3}{2}m\}$

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