INTERNATIONAL RESEARCH EXPERIENCE FOR STUDENTS
UNIVERSIDADE ESTADUAL DE CAMPINAS - UNICAMP

ON THE COMPACT SUBMANIFOLDS OF CODIMENSION 3 WITH NONNEGATIVE CURVATURE

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Introduction

In this project, we've studied the holonomy algebra of a manifold immersed in $\mathbb{R}^{n+3}$.

Also, we applied our classification result in a problem related to covering spaces of a compact manifold with non-negative curvature.

Vectors fields

We will denote by $\mathbb{R}^N_p$ the set of vectors that have origin at p.

Definition

A **vector field** *in* $\mathbb{R}^N$ *is a map* $X : \mathbb{R}^N \to \mathbb{R}^N_p$.

Identifying the vectors $\{e_i\}_{i=1}^N$ of the standard basis of $\mathbb{R}^N$ with their translates $\{e_{i_p}\}_{i=1}^N$ at the point p, we write

$$X(q) = \sum_{i=1}^{N} a_i(q) e_i.$$

A vector field X is said to be differentiable if the functions a_i's are differentiable.

Covariant derivatives and connections

Definition

Let X be a differentiable vector field in $\mathbb{R}^N$ and $c : [0,1] \to \mathbb{R}^N$ a differentiable curve in $\mathbb{R}^N$. Setting $X(c(t)) = (x_1(t), \ldots, x_N(t))$, the **covariant derivative** of X over $c(t)$ is given by

$$\frac{DX}{dt} = (x_1'(t), \ldots, x_N'(t)).$$

Definition

Given two vector fields X and Y in $\mathbb{R}^N$, we define the **connection** $\tilde{\nabla}_X Y$ as the vector field that at $p \in \mathbb{R}^N$ is given by the following procedure: Let $c(t)$ be a curve such that $c(0) = p$ and $c'(0) = X(p)$. Consider Y defined over $c(t)$, $Y(c(t)) = (y_1(t), \ldots, y_N(t))$. Then,

$$\tilde{\nabla}_X Y(p) = \frac{DY}{dt}(0) = (y_1'(t), \ldots, y_N'(t)).$$

Manifolds

Definition

Let $\mathcal{M}$ be an topological Hausdorff space with a countable basis. Then $\mathcal{M}$ is said to be a **manifold** if there existes a natural number n and for each point $p \in \mathcal{M}$ an open neighbourhood U of p and a continuous map $\phi : U \to \mathbb{R}^n$ which is a homeomorphism onto its image $\phi(U)$, an open subset of $\mathbb{R}^n$. The pair (U, ϕ) is called a **chart** on $\mathcal{M}$. The natural number N is called the **dimension** of $\mathcal{M}$. To denote that the dimension of $\mathcal{M}$ is n we write $\mathcal{M}^n$.

Manifolds

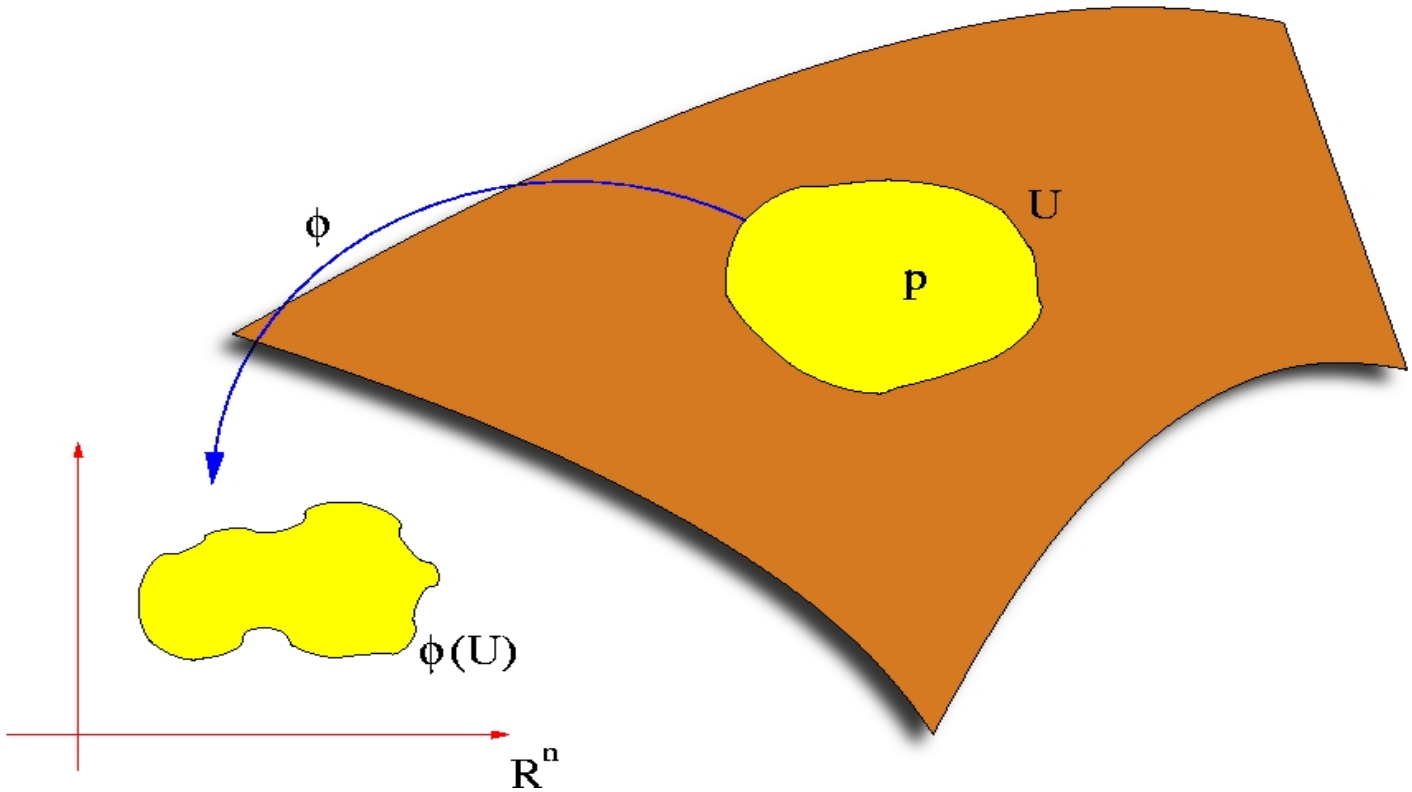


Figure: Manifold: an easier definition

Immersions and tangent space

Definition

We say that a map $f : \mathcal{M}^n \to \mathbb{R}^N$ is an **immersion** if for every $p \in \mathcal{M}$, exists an open set $W \subseteq \mathcal{M}$ $(p \in W)$ and a map $\phi : W \to U \subseteq \mathbb{R}^n$, U open in $\mathbb{R}^n$, such that the jacobian matrix of $f \circ \phi^{-1} : U \to \mathbb{R}^N$ has rank n.

For a differentiable curve $\alpha : (-\epsilon, \epsilon) \to \mathcal{M}$ such that $\alpha(0) = p$, the vector $\alpha'(t)$ is said to be a tangent vector to $\mathcal{M}$ in the point p.

Definition

For a point $p \in \mathcal{M}$, we define the **tangent space** $T_p\mathcal{M}$ as follows:

$$T_p\mathcal{M} = \{\alpha'(0) \mid \alpha : (-\epsilon, \epsilon) \to \mathcal{M},\ \alpha(0) = p\}.$$

The number $N - n$ is called **codimension** of the immersion.

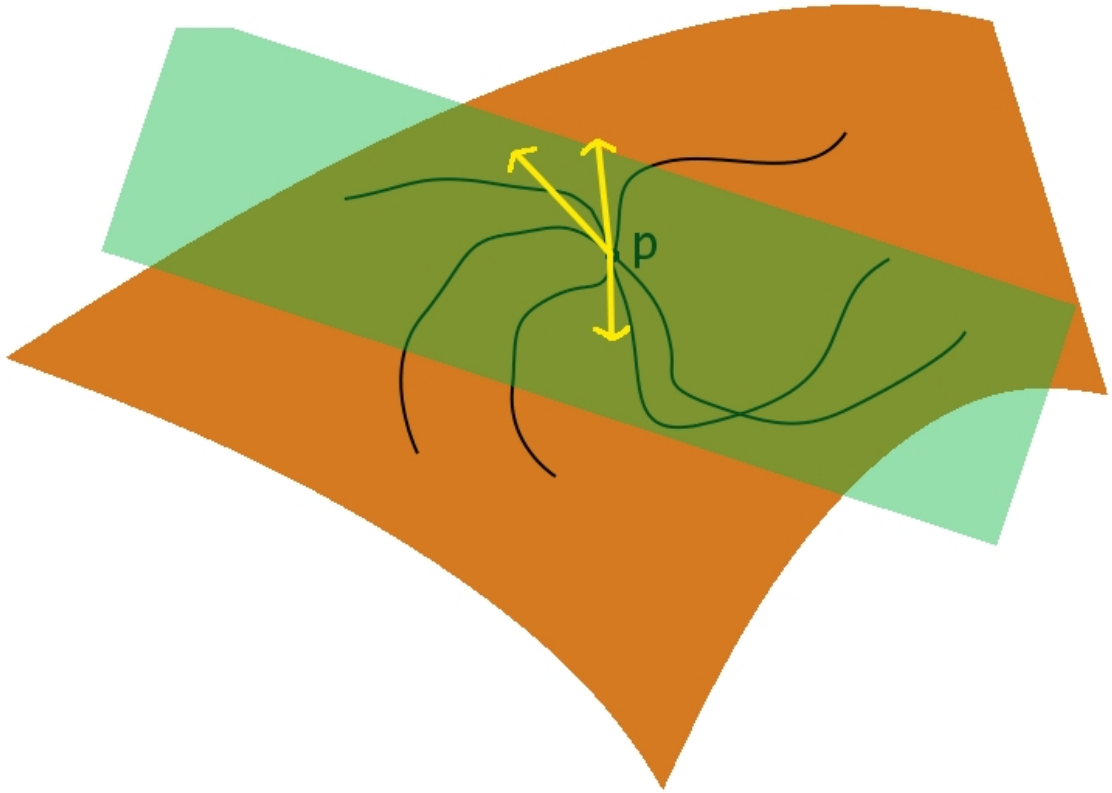


Figure: The (green) tangent space of the (orange) manifold is the set of (yellow) tangent vectors of (black) curves in the point p.

Definition

A **(tangent) vector field** in $\mathcal{M}$ is a map X that associates to each point $p \in \mathcal{M}$ a vector $X(p) \in T_p\mathcal{M}$.

Given two vector fields on $\mathcal{M}$, X and Y, we can extend them to $\tilde{X}$ and $\tilde{Y}$ vector fields in $\mathbb{R}^N$, then define a connection in $\mathcal{M}$ as

$$\nabla_X Y(p) = (\tilde{\nabla}_{\tilde{X}} \tilde{Y}(p))^T,$$

where $(\cdot)^T$ denotes the orthogonal projection onto $T_p\mathcal{M}$. So, we have:

$$\tilde{\nabla}_{\tilde{X}} Y = \nabla_X Y + (\tilde{\nabla}_{\tilde{X}} \tilde{Y})^N,$$

with $(\cdot)^N$ the orthogonal projection onto $T_p^{\perp}\mathcal{M}$.

We denote $(\tilde{\nabla}_{\tilde{X}} \tilde{Y})^N$ by $\alpha(X, Y)$ and call it the **second fundamental form** of the immersion. Note that α is a symmetric bilinear form.

Definition

A **normal vector field** in $\mathcal{M}$ is a map ξ that associates to each point $p \in \mathcal{M}$ a vector $\xi(p) \in T_p^{\perp}\mathcal{M}$.

Given a tangent vector field in $\mathcal{M}$ and a normal vector field ξ, we define the **normal connection** as

$$\nabla_X^{\perp}\xi = (\tilde{\nabla}_X\xi)^N.$$

The tangent component of $\tilde{\nabla}_x\xi$ defines an operator

$$A_\xi : T_p\mathcal{M} \to T_p\mathcal{M},$$

called **shape operator** (or **Weingarten operator**), given by

$$A_\xi(X) = -(\tilde{\nabla}_X\xi)^T.$$

Some identities

The shape operator is symmetric. Now we'll present some important identities.

The Gauss formula:

$$\tilde{\nabla}_X Y = \nabla_X Y + \alpha(X, Y) \tag{1}$$

$$\langle A_\xi X, Y \rangle = \langle X, A_\xi Y \rangle = \langle \xi, \alpha(X, Y) \rangle \tag{2}$$

Curvature tensor

Definition

Let $\chi(\mathcal{M})$ denote the set of vectors fields in $\mathcal{M}$. For $X, Y, Z \in \chi(\mathcal{M})$ we define the **curvature tensor** as

$$\mathcal{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z,$$

and the **sectional curvature** of the plane spanned by X and Y as

$$k(X,Y) = \frac{\langle \mathcal{R}(X,Y)Y, X \rangle}{||X \wedge Y||^2}.$$

Curvature tensor II

Using the equations 1 and 2, we can express the curvature tensor and the sectional curvature as

$$\mathcal{R}(X,Y)Z = A_{\alpha(Y,Z)}X - A_{\alpha(X,Z)}Y \tag{3}$$

$$k(X,Y) = \langle \alpha(X,X), \alpha(Y,Y) \rangle - \langle \alpha(X,Y), \alpha(X,Y) \rangle^2 \tag{4}$$

The equation 3 is know as Gauss Equation, and it implies the equation 4.

Fundamental theorem of submanifolds

Let

$$(D_X\alpha)(Y,Z) = \nabla^{\perp}_X \alpha(Y,Z) - \alpha(\nabla_X Y, Z) - \alpha(Y, \nabla_X Z)$$

$$(D_Y\alpha)(X,Z) = \nabla^{\perp}_Y \alpha(X,Y) - \alpha(\nabla_Y X, Z) - \alpha(X, \nabla_Y Z).$$

Codazzi Equation:

$$(D_X\alpha)(Y,Z) = (D_Y\alpha)(X,Z) \tag{5}$$

Ricci Equation:

$$\langle \mathcal{R}^{\perp}(X,Y)\xi_1, \xi_2\rangle = \langle [A_{\xi_1}, A_{\xi_2}](X), Y\rangle \tag{6}$$

Theorem

If $\mathcal{M}^d$ is a d-dimensional manifold, then $\mathcal{M}$ may be isometrically and locally embedded in a D-dimensional space $\mathcal{M}^D$ if and only if its metric, twisting vector and the extrinsic curvature satisfy the Gauss, Codazzi and Ricci equations.

Exterior algebra

Let $\mathcal{V}$ be an $\mathbb{F}$-vector space. Given two vectors $u, v \in \mathcal{V}$, we define $u \wedge v : \mathcal{V} \times \mathcal{V} \to \mathbb{F}$ as

$$(u \wedge v)(a, b) = u^*(a)v^*(b) - u^*(b)v^*(a),$$

for $a, b \in \mathcal{V}$, where u^* and v^* are the functionals associated with u and v.

Definition

The **exterior algebra** of $\mathcal{V}$, denoted by $\bigwedge^2(\mathcal{V})$, is the set of all possible wedge products $u \wedge v$, for $u, v \in \mathcal{V}$.

Exterior algebra and $o(\mathcal{V})$

The exterior algebra is a $\mathbb{F}$-vector space, with the usual sum and scalar product. Moreover, $\bigwedge^2(\mathcal{V})$ is a Lie algebra, with the bracket defined by

$$[x \wedge y, z \wedge w] = \langle x, z\rangle y \wedge w + \langle y, w\rangle x \wedge z - \langle x, w\rangle y \wedge z - \langle y, z\rangle x \wedge w,$$

for $x \wedge y,\ z \wedge w \in \bigwedge^2(\mathcal{V})$

We can identify $\bigwedge^2(\mathcal{V})$ with $o(\mathcal{V})$, the algebra of skew-symmetric operators on $\mathcal{V}$ in this way: for $x \wedge y \in \bigwedge^2(\mathcal{V})$ we associate the operator $(x \wedge y) : \mathcal{V} \to \mathcal{V}$ such that, for $z \in \mathcal{V}$, $(x \wedge y)(z) = \langle x, z\rangle y - \langle y, z\rangle x$.

Curvature operator

We'll start the transition from geometry to algebra.

Definition

Let $\mathcal{M}$ be a manifold and $p \in \mathcal{M}$. We define the **curvature operator** as the operator $R : \bigwedge^2(T_p\mathcal{M}) \to \bigwedge^2(T_p\mathcal{M})$ implicit given by

$$\langle \mathcal{R}(X,Y)Y, X\rangle = \langle R(X \wedge Y), X \wedge Y\rangle.$$

The curvature operator in p may be also given as

$$R(X \wedge Y) = \sum_{\xi} A_\xi(X) \wedge A_\xi(Y), \tag{7}$$

for ξ the normal fields in p (in other words, A_ξ are the components of 2nd fundamental form). As our spaces are finite-dimensional, we adopt the notation $\{S_i\}_i$ for $\{A_{\xi_i}\}_i$.

Some definitions

Given a symmetric operator S we denote by $D(S)$ the subspace which is simultaneously the range of S, the orthogonal complement of the kernel of S, and the span of the nonnullity eigenvectors of S.

For a immersed manifold $\mathcal{M}$, if p is its codimension, the **relative curvature subspace** at m is , $k(m) = D(S_1) + \ldots + D(S_p)$.

The Lie group consisting of parallel translations of $\mathcal{M}$ around loops based at m is called the **holonomy group** at m. The holonomy groups at different points are isomorphic. We shall denote its Lie algebra h_m. In fact, h_m is spanned by the parallel translates to m of the curvature transformations at all of the points of $\mathcal{M}$.

Denote by $r(m)$ the Lie algebra generated by the curvature transformations at m; thus $r(m)$ is a subalgebra of h_m and it is clear that $r(m) \subseteq \bigwedge^2(k(m))$ for every m.

We will present a (partial) classification theorem for $r(m)$ of immersed manifolds with codimension 3.

Just to fix our notations, let S_1, S_2, S_3 be the components of the 2nd fundamental form of $\mathcal{M}$ at the point m and:

$$R(x \wedge y) = S_1(x) \wedge S_1(y) + S_2(x) \wedge S_2(y) + S_3(x) \wedge S_3(y),$$

$$\mathcal{V}_1 = D(S_1),\ \mathcal{V}_2 = D(S_2),\ \mathcal{V}_3 = D(S_3),$$

$$k(m) = \mathcal{V} = \mathcal{V}_1 + \mathcal{V}_2 + \mathcal{V}_3,$$

$r(m)$=algebra generated by $Im(R)$.

Equation $\star$

Our manifolds satisfies the following relations: if Z_i are the sets defined by

$$Z_i = \{v \in \mathcal{V} \mid v \in \mathcal{V}_j,\ j \neq i\},$$

then we assume that

$$Z_i \neq 0,\ \ i = 1,\ 2,\ 3. \tag{8}$$

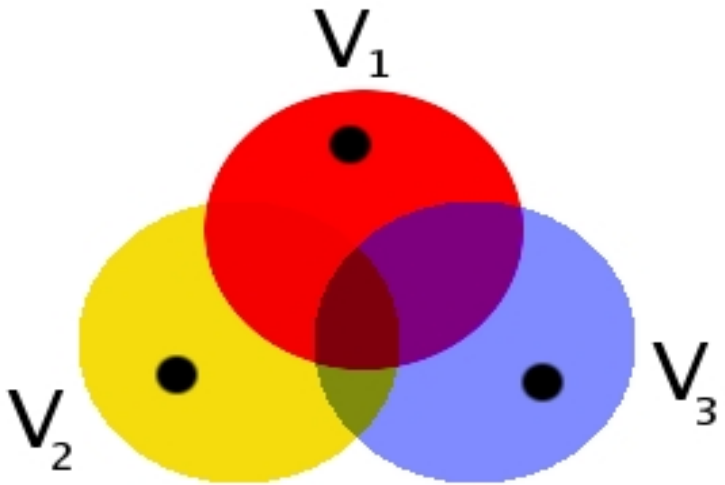


Figure: We need one element in V_i which is not on any other V_j.

Some lemmas

The classification for $r(m)$ will be given after some lemmas. We will work with the cases $\mathcal{V} \neq \mathcal{V}_1$ and S_1 has nonzero signature.

Lemma

$\bigwedge^2(\mathcal{V})$ is generated by $\mathcal{V}_1 \wedge \mathcal{V}_2 + \mathcal{V}_1 \wedge \mathcal{V}_3$.

Lemma

If $v \neq 0$ is a vector in $\mathcal{V}$, then $v \wedge \mathcal{V}$ generates $\bigwedge^2(\mathcal{V})$.

Lemma

If $v_1 \in \mathcal{V}_1$, $v_2 \in \mathcal{V}_2$ and $v_3 \in \mathcal{V}_3$, all of them nonzero with $\mathcal{V}_1 \cap \mathcal{V}_2 \neq 0$ and $\mathcal{V}_1 \cap \mathcal{V}_3 \neq 0$, then $\bigwedge^2(\mathcal{V})$ is generated by $v_1 \wedge \mathcal{V}_1 + v_2 \wedge \mathcal{V}_2 + v_3 \wedge \mathcal{V}_3$.

Some lemmas

Lemma

If $v_i \in \mathcal{V}_i$, $i = 1, 2, 3$ are vectors such that $v_1 \wedge (v_2 + v_3) \neq 0$ then $\bigwedge^2(\mathcal{V})$ is generated by $\bigwedge^2(\mathcal{V}_1) + \bigwedge^2(\mathcal{V}_2) + \bigwedge^2(\mathcal{V}_3) + v_1 \wedge v_2 + v_1 \wedge v_3$.

Lemma

If $\mathcal{V}_i \perp \mathcal{V}_j$ for every $i, j = 1, 2, 3$, if $\dim \mathcal{V}_1$ or $\dim (\mathcal{V}_2 + \mathcal{V}_3)$ do not equals 2, and if $\dim \mathcal{V}_2$ or $\dim \mathcal{V}_3$ do not equals 2, then $X = \bigwedge^2(\mathcal{V}_1) + \bigwedge^2(\mathcal{V}_2) + \bigwedge^2(\mathcal{V}_3)$ is a proper subalgebra of $\bigwedge^2(\mathcal{V})$.

Lemma

If $\mathcal{V}_1 \cap (\mathcal{V}_2 + \mathcal{V}_3) = 0$, $\mathcal{V}_2 \cap \mathcal{V}_3 = 0$, $\mathcal{V}_1 \neq (\mathcal{V}_2 + \mathcal{V}_3)^\perp$, $\mathcal{V}_2 \nsubseteq \mathcal{V}_3^\perp$ and $\dim \mathcal{V}_1 > 2$, $\dim \mathcal{V}_2 > 2$, $\dim \mathcal{V}_3 > 1$, then $\bigwedge^2(\mathcal{V})$ is generated by $\bigwedge^2(\mathcal{V}_1) + \bigwedge^2(\mathcal{V}_2) + \bigwedge^2(\mathcal{V}_3)$.

Somme lemmas

Lemma

The range of R contains

$$S_1(\mathcal{V}_2+\mathcal{V}_3)^{\perp}\wedge\mathcal{V}_1+S_2(\mathcal{V}_1+\mathcal{V}_3)^{\perp}\wedge\mathcal{V}_2+S_3(\mathcal{V}_1+\mathcal{V}_2)^{\perp}\wedge\mathcal{V}_3.$$

If $\mathcal{V}_i\bigcap(\mathcal{V}_j+\mathcal{V}_k)$ for $i,k\neq i$, then the range of R equals

$$\bigwedge^2(\mathcal{V}_1)+\bigwedge^2(\mathcal{V}_2)+\bigwedge^2(\mathcal{V}_3).$$

Lemma

The subalgebra generated by $\bigwedge^2(\mathcal{V}_1)+\bigwedge^2(\mathcal{V}_2)+\bigwedge^2(\mathcal{V}_3)$ is $r(m)$.

The theorem

Theorem

Let $\mathcal{M}$ be a manifold, $m \in \mathcal{M}$ and $r(m)$ the Lie algebra generated by the image of curvature operator $R : T_m\mathcal{M} \to T_m\mathcal{M}$, as defined in 7. There are only the following options for $r(m)$:

$$r(m) = \begin{cases} o(n) \\ o(k) \oplus u(2) \\ o(k) \oplus o(n-k) \\ o(k_1) \oplus o(k_2) \oplus o(n-k_1-k_2) \end{cases}$$

Covering spaces

Definition

Let $\mathcal{M}$ be a manifold. We say that a pair (E, p), $p : E \to \mathcal{M}$, is a **covering space** of $\mathcal{M}$ if for every $m \in \mathcal{M}$ has an open set $m \in U \subseteq \mathcal{M}$ such that $p^{-1}(U)$ is a disjoint union of open sets ω_i in E, each of which is mapped homeomorphically onto U by p.

Definition

We say that a manifold $\mathcal{M}$ has **nonnegative sectional curvature** if for every point $p \in \mathcal{M}$ and every plane $\{X, Y\}$, where X and Y are vector fields in $T_p\mathcal{M}$, we have $k(X, Y) \geq 0$.

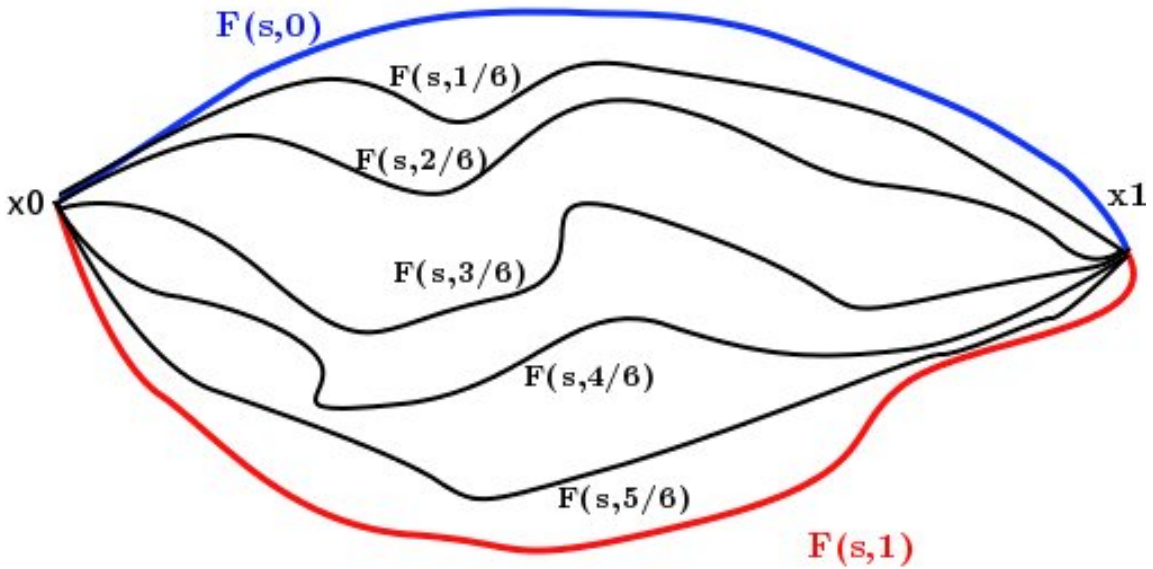
Homotopy

Let $\sigma : [0,1] \to X$ and $\tau : [0,1] \to X$ be paths in a topological space X with the same endpoints, $\sigma(0) = \tau(0) = x_0$, $\sigma(1) = \tau(1) = x_1$.

We say that σ and τ are **homotopic** if there is a continuous map $F : [0,1] \times [0,1] \to X$ such that

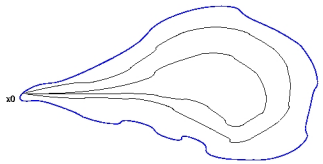
i) $F(s,0) = \sigma(s)$ ii) $F(s,1) = \tau(s)$

iii) $F(0,t) = x_0$ iv) $F(1,t) = x_1$



Fundamental group

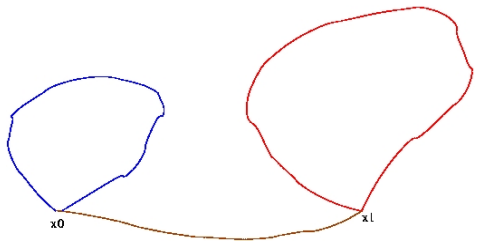
Consider now a topological space X and $x_0 \in X$. If c is a loop at x_0, let $[c]$ denote the equivalence class of all loops at x_0 that are homotopic to c.



Let $\Pi_1(X, x_0)$ be the group of all homotopy classes of loops at x_0, with the composition operation.

Theorem

If X is path connected, then for all $x, y \in X$, $\Pi_1(X, x) \cong \Pi_1(X, y)$.



Homotopy

Thanks to uniqueness (up to isomorphism), we can define the **fundamental group of** X and denote it by $\Pi_1(X)$.

Definition

We say that a space X is **simply connected** if $\Pi_1(X) = \{1\}$, that is, every loop is homotopically trivial.

Theorem

Every connected manifold has a covering that is simply connected. Such space is unique (up to diffeomorphisms) and is called **universal covering**.

Theorem

Let $\mathcal{M}$ be a compact manifold with a compact universal covering. Then $\Pi_1(\mathcal{M})$ is finite.

Cheeger-Gronoll

Theorem (Cheeger-Gronoll)

Let $\mathcal{M}$ be a compact manifold such that $k \geq 0$. Then the universal covering $\tilde{\mathcal{M}}$ is given by

$$\tilde{\mathcal{M}} = \bar{\mathcal{M}} \times \mathbb{R}^l,$$

where $\bar{\mathcal{M}}$ is a compact manifold.

Let $\mathcal{M}$ be a compact manifold, $k \geq 0$, $(\tilde{\mathcal{M}}, p)$ a covering space and f an immersion.

$$\tilde{\mathcal{M}} \to^{p} \mathcal{M}^n \to^{f} \mathbb{R}^{n+3}$$

Then $\tilde{f} = f \circ p : \tilde{\mathcal{M}} \to \mathbb{R}^{n+3}$ is a immersion.

Theorem

Exists $x \in \mathcal{M}$ *such that dim* $N(x) = 0$, *where*

$$N(x) = \{X \in T_p\mathcal{M} \mid \alpha(X, Y) = 0, \ \forall \ Y \in T_p\mathcal{M}\}.$$

So, for $\tilde{f}$, there exists $\tilde{x} \in \tilde{\mathcal{M}}$ such that *dim* $N(\tilde{x}) = 0$ or, in other words, *dim* $\mathcal{V} = n$.

In this case, we can apply our classification theorem.

As we have proved, the options for $r(\tilde{x})$ are

$$r(\tilde{x}) = \begin{cases} o(n) \\ o(k) \oplus u(2) \\ o(k) \oplus o(n-k) \\ o(k_1) \oplus o(k_2) \oplus o(n-k_1-k_2) \end{cases}$$

But by Cheeger-Gronoll theorem we know that $r(\tilde{x}) \subseteq o(n-l) \oplus o(l)$. Let us analyse the options for l.

- If $l = 0$, then
$$\tilde{\mathcal{M}} = \bar{\mathcal{M}} \times \mathbb{R}^0.$$
So $\mathcal{M}$ and $\tilde{\mathcal{M}}$ are compact and $\Pi(\mathcal{M})$ is finite.

- If $l = 1$ then we have the following options:
  - if $n = 5$, then $o(n-l)$ becames $o(4)$, and $u(2) \subseteq o(4)$. So, $r(\tilde{x})$ may be $u(2)$.
  - the other options are
    - $o(1) \oplus o(n-1)$
    - $o(1) \oplus o(k) \oplus o(n-1-k)$

- If $l = 2$ we have $r(\tilde{x}) \subseteq o(n-2)$. The only options where $o(n-2)$ appears is when $r(\tilde{x}) = o(1) \oplus o(1) \oplus o(n-2)$.

For $l \geq 3$, our classification theorem says that we can't have nothing newer. So, we have the following:

Theorem

Let $\mathcal{M}^n$ a compact manifold immersed in $\mathbb{R}^{n+3}$, with $k \geq 0$ and $\tilde{\mathcal{M}}$ as a covering space. Then, either $\Pi_1(\mathcal{M})$ is finite or

$$\tilde{\mathcal{M}} = \bar{\mathcal{M}}^{n-l} \times \mathbb{R}^l,$$

for $l \leq 2$.

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