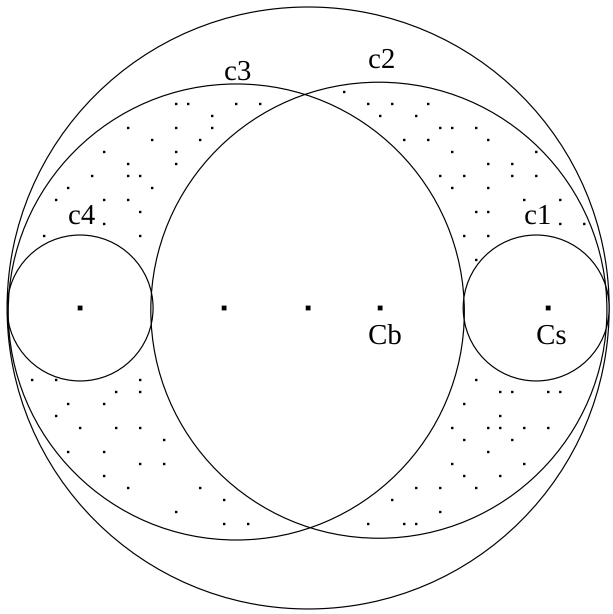
1 Introduction

1.1 The Problem

Consider a spherical surface with radius R. Suppose that the surface has been cut by four (4) right circular cylinders with parallel axes. Consider a plane that passes by the center of the spherical surface and is orthogonal to the axes of the cylinders. The plane section of the spherical surface when it is cut by this plane is illustrated in the following picture:



Let $c_1, c_2, c_3 and c_4$ be the circles that represent the plane section of the cylinders by this plane. Observe that c_1 is tangent to c_3 and c_2 is tangent to c_4. Furthermore, c_1 and c_4 share the same radius (r_s) and c_2 and c_3 share the same radius (r_b). Observe that $c_1, c_2, c_3 and c_4$ are tangent to the plane section of the sphere with radius R.

Calculate the radii of cylinders that maximize the surface area on the sphere that is projected on the pigmented region of the above figure.

1.2 Historical Perspective

The above problem was first explored by Japanese Mathematicians during the period in which Japan was isolated from the western culture (1603-1867). It is important to observe that during this era, the Japanese didnt have access to the progress that was being made in the calculus field. However, they were still able to work on geometry problems using their own techniques of Calculus. In this problem we found that even by using advanced calculus techniques, it was still difficult to solve by hand. Probably because of this difficulty we never found a satisfactory answer in the source literature.

2 Solving the Problem

2.1 First Contacts, Parametrization and Integrals

Our first contact with the problem was to get familiar with the concept and understand thoroughly the question. We did this by making sure we undestood all the concepts invloved and by visualizing the problem using "Geometry Sketchpad." In "Geometry Sketchpad," we were able to see all the symmetries invloved. This helped us to:

1. We set up the coordinate system by considering a few things: First, by the definition of tangency, we know that if two circles are internally tangent then their centers and the point of contact are collinear. We already observed that c_1 is tangent to c_3 and c_2 is tangent to c_4 and also $c_1, c_2, c_3 and c_4$ are tangent to the plane section of the sphere. Therefore, the circles in the above figure have collinear centers. Because of this we decided to fix the x-axis such that it contained all the circle centers of the bisector plane. Secondly, we let the z-axis be parallel to the cylinders axes and through the center of the sphere. It follows that the y-axis will be the line through the center of the sphere and perpendicular to the x-z plane. Therefore, the coordinate system that we set up has symmetry with respect to the x and y axis. This symmetry property allows us to work only in the first quadrant.

2. Label the main points: Considering only the circles c_1 and c_3, we have that the sum of the diameters equals the sphere's diameter. So then we can write:

$$2r_s + 2r_b = 2R \Rightarrow R = r_b + r_s$$

   Using the coordinate system, we have that:

$$C_s = (r_b, 0)$$
$$C_b = (r_s, 0)$$

3. Visualizing the limitations: The radius of the small circles cannot surpass the radius of the bigger circles. In fact, $r_s \leq R/2$. We noticed that if we have $r_s = R/2$, then the surface area will be zero because the small circle will coincide with the big circle.

After familiarizing ourselves with the problem a little more using Geometry Sketchpad, we can begin solving the problem. We start with the equation of the sphere:

$$R^2 = x^2 + y^2 + z^2$$

By solving this equation for z, we can rewrite this function as $z = \pm\sqrt{R^2 - x^2 - y^2}$. Due to the symmetry of the sphere in relation to the xy-plane, we will only work with the positive case and write $f(x, y) = +\sqrt{R^2 - x^2 - y^2}$ now compute the surface area, we need to use the following formula:

$$\int\int_D \sqrt{1+\left(\frac{\partial f}{\partial x}\right)^2+\left(\frac{\partial f}{\partial y}\right)^2} dxdy$$

where D is the domain of the surface area which we are trying to maximize.

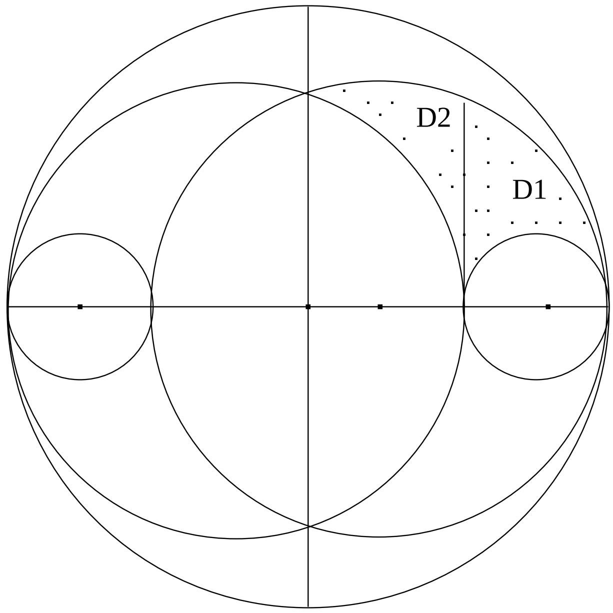
The following is an overview of how we got the above formula:

1. Divide the domain into small rectangles. For each rectangle, choose one of its vertices.
2. Consider the tangent planes to the surface through these vertices, and take the parallelograms bounded by the rectangles in the domain.
3. The summation of the areas of the parallelograms approaches the surface area when we make the rectangles smaller and smaller.

Since the surface area formula involves partial derivatives, it was necessary for us to calculate the partial derivatives of the function $f(x, y) = +\sqrt{R^2 - x^2 - y^2}$ in terms of x and y. Once we calculated the partial derivatives, we then replaced the results in the surface area formula, and got the following:

$$\int\int_D \frac{R}{\sqrt{R^2 - x^2 - y^2}} dxdy$$

Recall that due to the symmetry, we are only working in the first quadrant. Therefore, we can consider D in the integral above as the domain in only the first quadrant. Since the domain is bounded by three different functions, we needed to split it in two different regions named D_1 and D_2, as the following picture:



The next step was to find the equations that describe the circles, using r_s, r_b, C_s and C_b:

$$\begin{aligned}
c_1 &: (x - r_b)^2 + y^2 = r_s^2 \\
c_2 &: (x - r_s)^2 + y^2 = r_b^2 \\
c_3 &: (x + r_s)^2 + y^2 = r_b^2 \\
c_4 &: (x + r_b)^2 + y^2 = r_s^2
\end{aligned}$$

Then we set up the integral for each domain D_1 and D_2 and added them together in order to get the entire integral. But before we obtained the entire integral, we first calculated the partial derivatives and inputed it in the surface area formula, giving us:

$$\int\int_{D_1} \frac{R}{\sqrt{R^2 - x^2 - y^2}} dxdy + \int\int_{D_2} \frac{R}{\sqrt{R^2 - x^2 - y^2}} dxdy$$

$$\int_0^{R-2r_s} \int_{\sqrt{-(x+r_s)^2+r_b^2}}^{\sqrt{-(x-r_s)^2+r_b^2}} \frac{R}{\sqrt{R^2 - x^2 - y^2}} dydx + \int_{R-2r_s}^{R} \int_{\sqrt{-(x-r_b)^2+r_s^2}}^{\sqrt{-(x-r_s)^2+r_b^2}} \frac{R}{\sqrt{R^2 - x^2 - y^2}} dydx$$

In attempting to solve this integral we used differents methods, like polar coordinates, substitutions and integration by parts. However, none of the techniques we used worked. So we decided to use a computer algebra system.

2.2 Using Mathematica

The standard package of Mathematica for solving integrals didn't work because the integral is probably not analytic. At this point, we realized that we could not solve it directly through analytic means. So we tried to solve some specific numerical cases in an attempt to generalize a solution.

So, we looked for other ways to solve it in Mathematica and we found a package that solves integrals using complex variables and residuals, called Calculus.

However, for this package to work correctly we needed to calculate the integral under the limits from $-\infty$ to $+\infty$. Therefore, we had to use other strategies to define our domain.

To do this we used the function `Boole`, which returns 0 if a logic sequence is false or 1 if the sequence is true. For example, if a point is outside the domain, the function `Boole` returns 0. However, if the point is within the domain, the function `Boole` returns 1. So the integral under the limits from $-\infty$ to $+\infty$ is nulled outside the domain

Once we did that, we were able to use the following integral to solve our integral:

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{R}{\sqrt{R^2-x^2-y^2}}.Boole[D_1(x,y)]dxdy+$$
$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{R}{\sqrt{R^2-x^2-y^2}}.Boole[D_2(x,y)]dxdy$$

To define D_1 and D_2 we used the following sentences in the function `Boole`:

$$D_1: \ [x<R-2r_s\Lambda x>0\Lambda(x+r_s^2)+y^2>(r_b)^2\Lambda(x-r_s^2)+y^2<(r_b)^2]$$

$$D_2: \ [x<R\Lambda x>R-2r_s\Lambda(x-r_b^2)+y^2>(r_s)^2\Lambda(x-r_s^2)+y^2<(r_b)^2]$$

Since this package only allowed us to calculate numerical integrals, we needed to take a value to R and a set of values to r_s and compute the integrals for these values to find an approximation to the maximum surface area.

In a primary analysis, we used $R=4$ and $r_s=0,0.25,0.5,0.75,1,1.25,1.5,1.75,2$ (remember that $r_s\leq R/2$). We obtained these values for the surface area [1]:

[1]the commands used to do this are in the Appendix

Small radius	Surface area
0	0
0.25	11.95
0.5	14.96
0.75	15.84
1	15.24
1.25	13.41
1.5	10.37
1.75	6.06
2	0

Observe that in this table the function reaches the maximum when the small radius equals 0.75.

Now, we need to refine these results by taking points closer to 0.75. Since the function seemed to have a unique maximum at $r_s = 0.75$, we took points between 0.5 and 1: $r_s = 0.60, 0.65, 0.70, 0.75, 0.80, 0.85, 0.90$ and obtained the following values:

Small radius	Surface area
0.6	15.52
0.65	15.69
0.7	15.80
0.75	15.84
0.8	18.83
0.85	15.76
0.9	15.64

Observe that again, the function reaches the maximum when $r_s = 0.75$ in this table. To finish, we did one more refinement by taking the values between 0.7 and 0.8:

Small radius	Surface area
0.71	15.810
0.72	15.821
0.73	15.830
0.74	15.837
0.75	15.841
0.76	15.843
0.77	15.842
0.78	15.840
0.79	15.836

After the refinements we have a new maximum at 0.76.

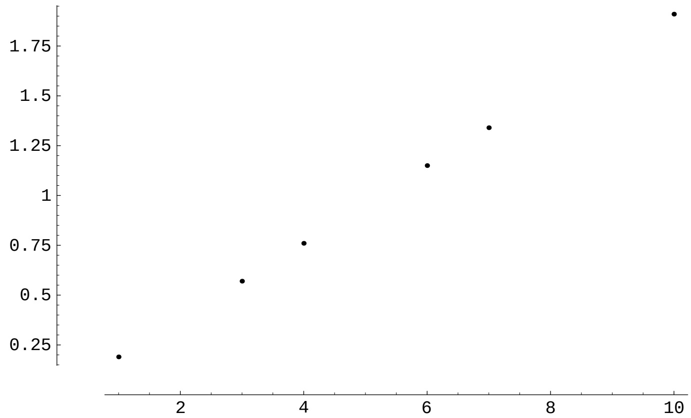
This procedure can be repeated as many times as needed if one wants to increase the precision of the answer. However, we stopped here.

Afterwards, we started to do the same procedure using $R = 1, 3, 6, 7, 10$ [2].

Denoting φ as the value of r_s which maximize the Surface Area, we have the following values to φ:

Sphere radius (R)	φ
1	0.19
3	0.57
4	0.76
6	1.15
7	1.34
10	1.91

Now we can plot these points in a Cartesian System:



Observe that the points are almost lined up and pass through the the Origin (because when $R = 0$ the surface area will be 0), this suggest to us that the function is linear. So we can look for the line that minimizes the distance between the points and the line. This line approaches the linear function which describes the behavior of φ in terms of R.

We found the slope (a) of the line and the approximation error using a Scientific Calculator:

$$a = 0.191 \pm 0.002$$

Observe that the error is very small. So this confirms the function is linear. Now we can write a function of R to φ, using the value of a:

$$\varphi(R) = 0.191R$$

[2] You can find the results in Appendix

2.3 Conclusion

Therefore, we found a relationship between the radius of the sphere (R) and the value of the small radius which maximizes the proposed surface area (φ). Although we calculated the results through a numerical method, they are only an approximation.

3 This Problem as a Activity for students

3.1 Pedagogical Aspects

This problem is a good example of how a teacher can improve his/her class by involving different mathematical contents. Contents such as Geometry, Calculus and technology. By incorporating all these subjects together, a teacher can get his/her class more involved and more willing to learn the material.

However, it is important to note that the material being exposed to the students, should be in such a way that it depends on the grade level and on their mathematical background. For example:

1. Middle School: The material invloved in this grade level should incorporate the geometry aspects of this problem in a two dimensional system , i.e., the tangency properties, symmetry, and the coordinate systemIt should also include the technological aspects, such as a dynamic geometry software.

2. High School: The material involved in this grade level should incorporate the aspects of geometry, calculus, and some introduction to computer algebra systems.

3. College: This problem can be incorporate into a research project. It can also be used to explore: advanced computer systems, advanced calculus, and other numerical methods, such as linear fits.

3.2 Using Computers

As mentioned earlier, this problem which was created by the Japanese Mathematicians before the complete development of calculus, remained unsolved because the involved calculations were far too complicated. And although, now in days we are far more advanced in the calculus field, this is still not enough to solve this problem. In fact, in order to solve this problem, we need the help of a computer.

The use of computers is very important. For example, with the help of a computer, we are able to simulate and visualize some aspects of the problem.

In our case, we used computers to make some simulations which would be harder to do by hands, after, we need the computer to solve the numericals integrals, but in both cases the interpretation of answers must be done by us.

This is an important aspect about the use of computers. These are just tools to help us in some works, like complicated calculations, drawings and simulations. However, it is our responsability to know how to use these tools and to understand its answers to better facilitate learning.

For example, on this problem, the first week we thought all we needed to do was find the domain and solve a few integrals by hand. By the second week we realized that the integral was to complicated to solve. So we figured a simple computer program that would be able to solve what we could not by hand. During our attempts to solve it by computer we ran into a few problems. We came to the conclusion that we needed to change our methods to solve the original problem. The computer showed us that our approach was wrong, but it was still our responsibility to understand our mistakes and to develope a new approach. The main conclusion was that we could not solve the problem analytically, so we developed a method to solve it numerically. This was our research and mathematical skill that allowed us to obtain numerical results in the original problem. Even though, the computer helped us in our calculations along the way, it could not produce the final solution.

That is why it is importat for teachers to know and to stress to their students the fact that there are limitaions to technology. This is why it is up to us to do the hard work and to understand the mathematics behind the computers answers.