Low Temperature Smoothness of the Pressure for Antiferromagnets and other Models

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Abstract. We prove under certain conditions on a grandcanonical Hamiltonian that at low temperature the pressure is infinitely differentiable with respect to the inverse temperature and other parameters in the Hamiltonian, when the parameters are chosen so that the number of Gibbs states is at least equal to the number of ground states. Applications are made to Antiferromagnets and hard-core gases.

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1. Introduction

An important task for classical equilibrium statistical mechanics is to find general mathematical criteria for phase transitions and phase coexistence. One approach to this problem is the general theory of infinite volume Gibbs states, probability measures determined by the DLR equations first given in Refs. 1 and 2. The problem in this context is to describe all limit Gibbs measures for a given Hamiltonian. Results of this type include proofs of the uniqueness or nonuniqueness of Gibbs states for various values of the thermodynamic parameters associated with the system of particles under consideration.

Another approach to the study of phase transitions is to try to locate all singularities of an appropriate thermodynamic function, e.g., the pressure. Phase transitions are then associated with these singularities. In the low temperature region, investigations along these lines have established smoothness or analyticity for Ferromagnetic-type systems, as in Basuev[3] and the references contained in Slawny[4]. The related question of the smoothness of the low temperature phase diagram has been analyzed by Zahradnik[5].

In this paper, we establish some connections between these two criteria for phase transitions in the low temperature region. For a fairly general class of Hamiltonians, we give a simple proof that when parameters in the Hamiltonian like external field strengths are chosen so that the number of Gibbs states at low temperature is at least than the number of ground states, then the pressure is infinitely differentiable with respect to inverse temperature and these parameters. Our requirement on the number of Gibbs states is not as restrictive as it may appear. It is frequently possible to fix certain parameters while letting the others vary in such a way that smoothness of the pressure is established in the (low temperature) regions of the phase diagram where the number of phases is constant. This is carried out for the antiferromagnet and hard-core gas.

Although Pirogov-Sinai Theory plays a major role here, the parameters we consider are not necessarily associated with fields which split the degeneracy of our Hamiltonians. Examples are given in Section 4. In Section 3 we state and prove our main result, Theorem 3.1. Section 2 summarizes the parts of Pirogov-Sinai Theory which are needed in the proof of Theorem 3.1. We assume that the reader has some familiarity with the method of cluster expansions (c.f. Ref. 6) which is also used in the proof.

2. Summary of Pirogov-Sinai Theory

In this section we summarize Pirogov-Sinai Theory[7,4,8] in a form which will be useful in Sect. 3.

We consider Hamiltonians formally given by

$$H(x) = \beta \sum_{\Lambda} \varphi(x_{\Lambda}) \tag{2.1}$$

where φ is a finite range periodic interaction potential, x_{Λ} is the restriction of the configuration x to the subset Λ of Z^d, and the sum is over finite subsets of Z^d. The parameter β represents the inverse temperature. For $i \in Z^d$ let x_i, the restriction of x to i, take values in some finite set X. The reference measure on X is the counting measure. The energy of a finite configuration (with empty boundary conditions) is then given by

$$H(x_{\Lambda}) = \beta \sum_{A \subset \Lambda} \varphi(x_A)$$

With only minor modifications in what follows, we may assume that H has a hard-core restriction. In that case, configurations x and y below are assumed to be "allowable" in that they satisfy the hard-core restriction.

It is convenient to use the notion of relative Hamiltonian. Let two configurations x and y be equal at all but a finite number of sites in Z^d. In this case we write x = y (a.s.). Define

$$H(x|y) = \beta \sum_{\Lambda} [\varphi(x_\Lambda) - \varphi(y_\Lambda)] \qquad (2.2)$$

Note that the sum in (2.2) is finite. A configuration x is called a ground state if H(x | y) ≥ 0 for each y such that y = x (a.s.). We consider only periodic ground states and we assume that H has only a finite number of periodic ground states, $x^1, ..., x^n$.

The specific energy e_x (H) of a periodic ground state x is given by

$$e_x(H) = \lim_{\Lambda \uparrow Z^d} \frac{1}{|\Lambda|} \sum_{i \quad \Lambda} U_i(x)$$

where the limit may be taken over an increasing sequence of hypercubes and where

$$U_i(x) = \beta \sum_{A:i \quad A} \frac{1}{|A|} \varphi(x_A)$$

with | | denoting cardinality.

We note that for any two periodic ground states x^1 and x^2 of H,

$$e_{x^1}(H) = e_{x^2}(H) \qquad (2.3)$$

This is proved in Lemma 2.1 of Ref.4.

For i Z^d, let

W(i) = {j Z^d : || i - j || ≤ r}

where || || denotes the L_∞ --norm of $\mathbf{R}^d$ restricted to Z^d and r is a fixed number greater than the radius of interaction of φ and large enough so that all periodic ground states are uniquely determined by their restrictions to W(i).

A hypercube W(i) is said to be an irregular cube of the configuration x if x restricted to W(i) is not equal to any ground state restricted to W(i). The boundary B(x) of x is the union of all irregular cubes of x.

A subset Λ of Z^d is connected if Λ cannot be written as $\Lambda_1 \quad \Lambda_2$ with $d(\Lambda_1, \Lambda_2) > 1$. Here the distance is determined by the L_∞ norm.

Let $y = x^q$ (a.s.) for some $q = 1,\ldots, n$ and let M be a connected component of the boundary B(y). The pair (M, y_M) is a contour of y. A pair (M, x_M) is called a contour if it is the contour of some configuration. M is called the support of Γ and we write $|\Gamma| = |M|$.

Let $\Gamma = (M, x_M)$ be a contour. Then there exists a unique configuration x_Γ such that $x_\Gamma = x_M$ on M and $M = B(x)$. Also since $|M| < \infty$, there exists a unique infinite connected component of M^c which is called the exterior of Γ, ext Γ, and the rest of M^c is called the interior of Γ, Int Γ. On each connected component of M^c, x_Γ is equal to one of the ground states. We will write $\Gamma = \Gamma^q$ if $x_\Gamma = x^q$ on ext (Γ). The m-interior, $\text{Int}_m\Gamma$, of Γ is the union of components of the interior of Γ on which $x_\Gamma = x^m$. The contour Γ is called an exterior contour if it is not contained in the interior of any other contour.

Let Λ be a finite subset of Z^d. Let $R_q(\Lambda)$ be the set of all configurations x such that $x_i = x_i^q$ for all $i \quad \Lambda^c$, $d(B(x), \Lambda^c) > 1$, and for any contour Γ of x, $\text{Int}(\Gamma) \quad \Lambda$. The rarified partition function in volume Λ with boundary conditions x^q for the Hamiltionian H is defined by

$$Z^q(\Lambda|H) = \sum_{x \quad R_q(\Lambda)} \exp[-H(x|x^q)]$$

Let Z^q_Λ denote the usual partition function, i.e.,

$$Z^q_\Lambda = \sum_{x \quad \Lambda} \exp[-H^q(x)]$$

where the sum is over (allowable) configurations in Λ and

$$H^q(x) = \sum_{A \cap \Lambda \neq} \varphi(y_A)$$

and $y = x_\Lambda$ on Λ and $y = x^q$ on Λ^c. Observe that given a finite set $\Lambda \quad Z^d$, there is a unique subset $\Lambda' \quad \Lambda$ such that

$$Z^q(\Lambda|H) = Z^q_{\Lambda'} \exp[-H(x^q_{\Lambda'})] \qquad (2.4)$$

As in Ref. 8, we write $\Gamma^q \quad \Lambda$ if supp $\Gamma^q \quad \Lambda$, dist (supp $\Gamma^q, \Lambda^c) > 1$, and Int $\Gamma^q \quad \Lambda$. Two subsets of Z^d are said to be far apart if the L_∞ distance between them is greater than one. Let $C_q(\Lambda)$ be the ensemble whose elements are finite sets of contours $\{\Gamma^q_1,\ldots, \Gamma^q_m\}$, Γ^q_i Λ with supports pairwise far apart. These contours $\{\Gamma^q_1,\ldots, \Gamma^q_m\}$ in $C_q(\Lambda)$ are in general not contours of a configuration x. Each contour has boundary condition x^q.

Let F_q be a nonnegative functional defined on $C_q(\Lambda)$. The contour rarified partition function in volume Λ is defined by

$$Z(\Lambda|F_q) = \sum \exp[-\sum_i F_q(\Gamma_i^q)]$$

where the summation is over all possible $\{\Gamma_1^q,..., \Gamma_m^q\}$ in $C_q(\Lambda)$.

A functional F_q is called a τ-funtional if

$$F_q(\Gamma^q) \geq \tau \mid \Gamma^q \mid.$$

In order to apply Pirogov-Sinai theory, we need to check Peierls' condition for H, namely, there exists a positive constant ρ such that for any periodic ground state x^q, and any x such that $x = x^q$ (a.s.)

$$H(x \mid x^q) \geq \rho \, |B(x)| \qquad (2.5)$$

The following Lemma is a consequence of Pirogov-Sinai Theory.

Lemma 2.1 *Let H have n distinct periodic ground states. Assume that for all* β *sufficiently large, H has n distinct Gibbs states which are limits of finite volume Gibbs measures with boundary conditions equal respectively to the periodic ground states. Each ground state is then associated with one Gibbs state. Assume that there are external fields* $H_1, ..., H_{n-1}$ *which completely split the n-fold degeneracy of H in the sense of Pirogov-Sinai. Then for* β *sufficiently large there exists a number* τ *and* τ*-functionals* F_q*, q= 1,..., n such that*

$$Z^q(\Lambda|H) = Z(\Lambda|F_q) \qquad (2.6)$$

for all q and Λ*. Moreover each* F_q *satisfies*

$$F_q(\Gamma^q) = H(\Gamma^q) + \sum_{k=1}^{n} \log Z(\,\mathrm{Int}_k\, \Gamma^q \mid F_q) - \log Z(\,\mathrm{Int}_k\, \Gamma^q \mid F_k) \qquad (2.7)$$

where $H(\Gamma^q) = H(x_{\Gamma^q} | x^q)$.

The proof of Lemma 2.1 follows easily from the Main Theorem B and the proof of Proposition 2.6 in Ref. 8. We note that τ may be chosen arbitrarily large by choosing β sufficiently large.

Remark 2.1 The existence of the external fields $H_1, ..., H_{n-1}$ is a technical requirement which allows us to apply Pirogov-Sinai Theory. We are only interested in the case $\mu_i \equiv 0$ for the Hamiltonian $H' = H + \mu_1 H_1 + \cdots + \mu_{n-1} H_{n-1}$. With the hypotheses of Lemma 2.1, $\mu_i \equiv 0$ corresponds to the case of n (the maximal number) of coexisting phases for H' at low temperature and small values of $\{\mu_i\}$.

3. Differentiability of the Pressure

Assume that H depends on parameters $\lambda_1,...,\lambda_m$, where $\lambda_1 = \beta$, the inverse temperature. The remaining parameters might correspond to external field strengths or to variables which parameterize a hypersurface in the phase diagram (c.f. Sect. 4). In this section we use Lemma 2.1 to prove with certain hypotheses that the pressure corresponding to H is a C^∞ function of these parameters.

With the notation of Sect. 2, the pressure P for H is defined by

$$P(\lambda_1,..., \lambda_m) = \lim_{\Lambda \uparrow Z^d} \frac{1}{|\Lambda|} Z_\Lambda \qquad (3.1)$$

where indicates empty boundary conditions. Under general conditions (see, for example, Preston[8]) which are satisfied here, may be replaced by an arbitrary boundary condition in (3.1). We exploit this fact in the proof of Theorem 3.1 below.

In what follows we use the notation $\langle \ \rangle_\Lambda^q$ for expectation with respect to the finite volume Gibbs state for H in volume Λ with boundary configuration x^q.

Theorem 3.1 *Let H be a finite range periodic Hamiltonian depending on parameters* $\lambda_1,..., \lambda_m$ *as described above. Let intervals* $I_1, ..., I_m$ *be given where* $I_1 = (\beta_1, \infty)$. *For* $\lambda_k, k = 1,..., m$, *in any finite open subinterval of* I_k *assume that H satisfies the following conditions:*

a) H satisfies Peierls' condition

b) H has exactly n periodic ground states $x^1,..., x^n$

c) H has n extremal Gibbs states which are limits of finite volume Gibbs measures with boundary configurations given respectively by $x^1, \ldots, x^n$

d) $\max_{x_\Lambda} \left| \frac{\partial}{\partial \lambda_k} H(x_\Lambda) \right| \le c_1 |\Lambda|$ *for some constant* c_1 *and all finite* $\Lambda \quad Z^d$.

e) For any ground state configuration x^q *and any N, there exists a constant c(N) such that*

$$\left| \frac{\partial^N}{\partial \lambda_k^N} H(\Gamma^q) \right| = \left| \frac{\partial^N}{\partial \lambda_k^N} H(x_{\Gamma^q} | x^q) \right| \le c(N) |\Gamma^q|^N \text{ and } \left| \frac{\partial^N}{\partial \lambda_k^N} H(x_\Lambda^q) \right| \le c(N) |\Lambda| \text{ for any finite } \Lambda.$$

Then for any k = 1,..., m there exists β_0 *such that* $\frac{\partial^N P}{\partial \lambda_k^N}$ *exists for all N = 1, 2, ..., when* $\lambda_k \in I_k$ *provided* $\beta \equiv \lambda_1 > \beta_0$.

Remark 3.1 We note that condition c) of Theorem 3.1 does not require the number of Gibbs states to equal the number of ground states; the number of Gibbs states may exceed the number of ground states. Thus, for example, the theorem applies to Hamiltonians with the parameters chosen so that there is only one ground state, but possibly more than one Gibbs state. However, examples exist where a) and b) of Theorem 3.1 are satisfied but not c) as shown in the introduction of Ref. 4. In an interesting example in the introduction of Ref. 4, a Hamiltonian and its low temperature phase diagram are shown. There are points on the phase diagram corresponding to i) three ground states and one Gibbs state ii) one ground state and two Gibbs states iii) two ground states and three Gibbs states, etc.

Proof. Let $\{\Lambda_n\}$ be a sequence of volumes converging to Z^d and for a fixed value of q let $f_n = \frac{1}{|\Lambda|} \log Z_{\Lambda_n}^q$. If $\left\{ \frac{\partial f_n}{\partial \lambda_k} \right\}$ is a family of equicontinuous functions in λ_k which are uniformly bounded for all λ_k in any open subinterval of I_k then by the Arzela-Ascolli Theorem, there exists a subsequence $\left\{ \frac{\partial f_{n_m}}{\partial \lambda_k} \right\}$ uniformly convergent in any open subinterval of I_k. This implies that P is differentiable with respect to λ_k on I_k and

$$\frac{\partial P}{\partial \lambda_k} = \lim_{m \to \infty} \frac{\partial f_{n_m}}{\partial \lambda_k}$$

Similarly, by induction, if $\left\{ \frac{\partial^N f_n}{\partial \lambda_k^N} \right\}$ is uniformly bounded in any open subinterval of I_k for all $N = 1, 2, \ldots$, then it follows that P is infinitely differentiable in λ_k.

Thus it suffices to show that for β in a finite open interval with $\beta > \beta_0$ (for some β_0) and λ_k in an open subinterval of I_k

$$\frac{1}{|\Lambda'|} \left| \frac{\partial^N \log Z^q_{\Lambda'}}{\partial \lambda_k^N} \right| \leq M'_N \qquad (3.2)$$

for all hypercubes Λ' with side length sufficiently large and some constants M'_N which may depend on the subintervals of I_1 and I_k.

We next modify (3.2) by using equation (2.4). By condition e) of the Theorem,

$$\frac{1}{|\Lambda'|} \frac{\partial^N}{\partial \lambda_k^N} \log \exp\{H(x^q_{\Lambda'})\} = \frac{1}{|\Lambda'|} \frac{\partial^N}{\partial \lambda_k^N} H(x^q_{\Lambda'})$$

is bounded for each N and all Λ'. Also $\frac{|\Lambda'_i|}{|\Lambda_i|} \to 1$ as $i \to \infty$ for any sequence $\Lambda_i \uparrow Z^d$. To prove the Theorem it therefore suffices to prove that there exists a β_0 such that for β in any open subinterval of (β_0, ∞),

$$\frac{1}{|\Lambda|} \left| \frac{\partial^N \log Z^q(\Lambda|H)}{\partial \lambda_k^N} \right| \leq M_N \qquad (3.3)$$

for all λ_k in a finite open subinterval of I_k, all Λ, and some constants M_N which may depend on the subintervals, and all $N = 1, 2, \ldots$.

By (2.6) there exists β_0 such that if $\beta > \beta_0$ then

$$\begin{aligned} Z^q(\Lambda|H) &= Z(\Lambda|F_q) \\ &= \sum_{\{\Gamma_1^q,\ldots,\Gamma_m^q\} \, C(\Lambda)} \exp[-\sum_{i=1}^m F_q(\Gamma_i^q)] \\ &= \sum_{m=0}^{\infty} \frac{1}{m!} \sum_{\Gamma_1^q,\ldots,\Gamma_m^q \, \Lambda} \psi(1,\ldots,m) \exp[-\sum_{i=1}^m F_q(\Gamma_i^q)] \qquad (3.4) \end{aligned}$$

where

$$\psi(1,\ldots,m) = \exp[-\sum_{1 \leq i < j \leq m} V(i,j)]$$

and V(i,j) = ∞ if $\Gamma_i{}^q$ and $\Gamma_j{}^q$ are not far apart and V(i,j) = 0 if $\Gamma_i{}^q$ and $\Gamma_j{}^q$ are far apart. As in Mayer's expansion[6], let

$$\psi_c(1,\ldots,m) = \sum_{\gamma} \prod_{(i,j)\in\gamma} \{\exp[-V(i,j)]-1\}$$

where γ is a connected graph[6] on {1, ..., m}.

Then

$$\log Z(\Lambda|F_q) = \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{\Gamma_1^q,\ldots,\Gamma_m^q \subset \Lambda} \psi_c(1,\ldots,m) \exp[-\sum_{i=1}^{m} F_q(\Gamma_i^q)] \qquad (3.5)$$

and

$$\frac{1}{|\Lambda|}\frac{\partial^N}{\partial\lambda_k^N} \log Z(\Lambda|F_q) = \frac{1}{|\Lambda|} \sum_{m=1}^{\infty} \frac{1}{m!} \sum_{\Gamma_1^q,\ldots,\Gamma_m^q \subset \Lambda} \psi_c(1,\ldots,m) \frac{\partial^N}{\partial\lambda_k^N} \exp[-\sum_{i=1}^{m} F_q(\Gamma_i^q)] \qquad (3.6)$$

Let $f = -\sum_{i=1}^{m} F_q(\Gamma_i^q)$. Then $\frac{\partial^N}{\partial\lambda_k^N}\exp f$ is a sum of at most N! terms and each term is of the form $e^f \frac{\partial^{i_1} f}{\partial\lambda_k^{i_1}} \cdots \frac{\partial^{i_L} f}{\partial\lambda_k^{i_L}}$ with $i_1+\cdots+i_L = N$, $i_j \geq 1$ for all j = 1,..., L. We use formula (2.7) to compute $\frac{\partial^\ell}{\partial\lambda_k^\ell} F_q(\Gamma^q)$. It is necessary to first estimate

$$\frac{\partial^\ell}{\partial\lambda_k^\ell} \log Z(\Lambda|F_q) = \frac{\partial^\ell}{\partial\lambda_k^\ell} \log\left[Z_{\Lambda'}^q \exp\{-H(x_{\Lambda'}^q)\right] \qquad (3.7)$$

For $\ell \geq 1$ the right side of (3.7) is a sum of at most $2^\ell\, \ell\,!$ terms of the form

$$\pm \left\langle \left(\frac{\partial}{\partial\lambda_k} H\right)^{k_1} \right\rangle_{\Lambda'}^q \left\langle \left(\frac{\partial}{\partial\lambda_k} H\right)^{k_2} \right\rangle_{\Lambda'}^q \cdots \left\langle \left(\frac{\partial}{\partial\lambda_k} H\right)^{k_t} \right\rangle_{\Lambda'}^q \qquad (3.8)$$

where $k_1+\cdots+k_t = \ell$, $i_j \geq 1$ for j = 1,..., t.

By condition d),

$$\left|\frac{\partial^\ell}{\partial\lambda_k^\ell} \log Z(\Lambda|F_q)\right| \leq 2^\ell \ell\,! c_1^\ell |\Lambda|^\ell \equiv c_2 |\Lambda|^\ell \qquad (3.9)$$

Combining (3.9) and (2.7) with condition e) gives

$$\left|\frac{\partial^\ell}{\partial\lambda_k^\ell} F_q(\Gamma^q)\right| \leq c(\ell)\left|\Gamma^q\right|^\ell + nc_2\left|\mathrm{Int}\,\Gamma^q\right|^\ell \leq c(\ell)\left|\Gamma^q\right|^\ell + c_3\left|\Gamma^q\right|^{d\ell} \leq c_4\left|\Gamma^q\right|^{d\ell} \qquad (3.10)$$

for some geometric constants c_3 and c_4. Hence

$$\left|\frac{\partial^\ell f}{\partial\lambda_k^\ell}\right| \leq c_4 \sum_{i=1}^{m} \left|\Gamma_i^q\right|^{d\ell} \leq c_4 \left(\sum_{i=1}^{m} \left|\Gamma_i^q\right|\right)^{d\ell} \qquad (3.11)$$

and

$$\left|\frac{\partial^N e^f}{\partial \lambda_k^N}\right| \le e^f \; N!\, c_4 \left(\sum_{i=1}^{m} \left|\Gamma_i^q\right|\right)^{dN}$$

$$\le c_4\, e^f\, d!(N!)^2 \exp\left(\sum_{i=1}^{m} \left|\Gamma_i^q\right|\right) \equiv c_5\; e^f \exp\left(\sum_{i=1}^{m} \left|\Gamma_i^q\right|\right) \quad (3.12)$$

Combining (3.6) , (3.12), and using the fact that F_q is a τ-functional gives

$$\frac{1}{|\Lambda|}\left|\frac{\partial^N}{\partial \lambda_k^N} \log Z(\Lambda|F_q)\right| \le \frac{c_5}{|\Lambda|}\sum_{m=1}^{\infty}\frac{1}{m!}\sum_{\Gamma_1^q,\ldots,\Gamma_m^q \subset \Lambda} |\psi_c(1,\ldots,m)| \exp[(1-\tau)\sum_{i=1}^{m}|\Gamma_i^q|] \quad (3.13)$$

Now using Penrose tree graph bounds and standard arguments[6], it follows that (3.13) is bounded by a constant M_N if

$$\sum_{\Gamma^q : 0 \in \Gamma^q} e^{-(\tau-2)|\Gamma^q|} \; < \; 1 \qquad (3.14)$$

Since τ may be chosen arbitrarily large by choosing β sufficiently large, (3.14) holds if $\beta \ge \beta_0$ for some $\beta_0 > 0$. The proof is completed by combining (3.13) with (3.3).

The following is a corollary to the proof of Theorem 3.1.

Corollary 3.1 *Assume all of the hypotheses of Theorem 3.1 except that condition e) holds only for N = 1, 2, ..., M+1. Then the pressure is M times differentiable with respect to* λ_k *for* $\beta > \beta_0$.

4. Examples

In this section we give some examples and applications of Theorem 3.1.

A) Ferromagnetic Ising Model with $d \ge 2$

The purpose of this example is just to illustrate Theorem 3.1. We note that low temperature analyticity in β is well-known; references are given in Slawny[4] (see also Ref. 3). Let

$$H(x) = \beta J \sum_{|i-j|=1} x_i\, x_j - \beta h \sum_i x_i \qquad (4.1)$$

with $J < 0$ and $x_i = \pm 1$ and where $|\cdot|$ denotes the Euclidean norm. Then H satisfies Peierls' condition for all $\beta = \lambda_1$, and $h = \lambda_2$. The Hamiltonian $H_1 = \beta \sum_i x_i$ completely splits the degeneracy of H. If h = 0, H has two periodic ground states and otherwise one. The remaining hypotheses of Theorem 3.1 are easily checked. The following Corollary holds.

Corollary 4.1 *There exists* β_0 *(h) depending on h such that if* $\beta > \beta_0$ *(h) and*

a) h = 0, then the pressure is infinitely differentiable with respect to β.

b) h ≠0, then the pressure is infinitely differentiable with respect to β *and with respect to h.*

We note that Lebowitz and Martin-Lof[10] proved that the pressure is not differentiable with respect to h at h = 0 for all values of the temperature below the critical temperature.

B) Antiferromagnet, d ≥ 2

Let

$$H(x) = \beta \sum_{|i-j|=1} x_i x_j - \beta h \sum_i x_i \qquad (4.2)$$

with $x_i = \pm 1$. Note that the term $\beta h \sum_i x_i$ in (4.2) does not split the degeneracy of the Hamiltonian consisting of the first term alone. H(x) has two ground states if |h| < 2d and one ground state if |h| > 2d. If |h| < 2d, a perturbation Hamiltonian and coupling constant $\beta\mu$ which split the degeneracy of (4.2) is $\beta\mu \sum_{i \text{ even}} x_i$, where the sum is over sites in $\mathbf{Z}^d$, the sum of whose components is even. The coefficient μ is of little physical interest and we consider only the case $\mu = 0$. It can be shown that (for $\mu = 0$) H(x) has exactly two ground states and at least two Gibbs states for all |h| < 2d and for $\beta > \beta_0$(h) where β_0(h) may be chosen to depend only on min (|h - 2d|, |h + 2d|). For a discussion and further references on this we refer the reader to Ref. 11. Analysis of the case |h| > 2d is straightforward. As to the differentiability of the pressure with respect to β and h we have the following corollary to Theorem 3.1.

Corollary 4.2 *Let P be the pressure for (4.2). Then if $|h| \neq 2d$, there exists $\beta_0(h)$ which depends only on min ($|h - 2d|$, $|h + 2d|$) such that P is infinitely differentiable with respect to h for $\beta > \beta_0(h)$ and infinitely differentiable with respect to β for $\beta > \beta_0(h)$.*

C) Hard-Core Gas, $d \geq 2$

In this example,

$$H(x) = \beta \sum_i x_i \qquad (4.3)$$

where $x_i = 0$ or 1 and H is restricted to allowable configurations. A configuration x is allowable if $x_i\, x_j = 0$ whenever the Euclidean distance between i and j equals one.

Corollary 4.3 *Let P be the pressure for (4.3). Then there exists $\beta_0 > 0$ such if $\beta > \beta_0$, then P is infinitely differentiable with respect to β.*

D) Fisher Antiferromagnet, d = 2

Let

$$H(x) = \beta \sum_{(i,j)} J_{ij}\, x_i\, x_j - \beta h \sum_i x_i \qquad (4.4)$$

where the sum is over pairs (i, j), $x_i = \pm 1$, and J_{ij} equals 1 for nearest neighbor pairs (i,j), J_{ij} is a negative constant when $|i - j| = \sqrt{2}$, and $J_{ij} = 0$ otherwise. This Hamiltonian was analyzed by Pirogov and Sinai[7]. In order to formulate Corollary 4.4 we repeat some of their analysis. For simplicity, we consider only $h \geq 0$. The case $h \leq 0$ is similar.

Consider the Hamiltonian

$$H^1(x) = \beta \sum_{(i,j)} J_{ij}\, x_i\, x_j - 4\beta \sum_i x_i \qquad (4.5)$$

$H^1(x)$ satisfies Peierls' condition and has three ground states

$$x^1_i = (-1)^{|i|}$$

$$x^2_i = (-1)^{|i|+1}$$

$$x^3_i = 1$$

where $|i| = |(i_1, i_2)| = |i_1| + |i_2|$. Let H_1 and H_2 be given by

$$H_1 = \beta \sum_{i:|i| odd} x_i \qquad H_2 = \beta \sum_{i:|i| even} x_i \qquad (4.6)$$

Then clearly $\mu_1 H_1 + \mu_2 H_2$ completely splits the degeneracy of H^1.

It follows from Pirogov-Sinai Theory that there exist $\mu_1(\beta)$ and $\mu_2(\beta)$ such that

$$H^1 + \mu_1(\beta)\, H_1 + \mu_2(\beta)\, H_2$$

has three Gibbs measures, no two of which are a convex combination of the third, when β is sufficiently large. Symmetry arguments show that $\mu_1(\beta) = \mu_2(\beta)$. Referring to the common value of

$\mu_1(\beta)$ - 4 and $\mu_2(\beta)$ - 4 as $h(\beta)$, define

$$H^2(x) = \beta \sum_{(i,j)} J_{ij}\, x_i\, x_j - h(\beta) \sum_i x_i \qquad (4.7)$$

The following Corollary follows from Corollary 3.1.

Corollary 4.4 *Let P be the pressure for H^2 given by (4.7). Suppose that $h(\beta)$ is n times continuously differentiable with respect to β for all β sufficiently large (possibly depending on n). Then there exists a β_0 such that if $\beta > \beta_0$, P is n times differentiable with respect to β.*

Remark 4.1 Applying the results of Zahradnik on the analyticity of the phase diagram[5], it follows that $h(\beta)$ is an analytic function. Thus the hypothesis that $h(\beta)$ is n times differentiable may be removed in Corollary 4.4.

Referring back to H defined by (4.4), it is possible to show that the pressure for H is infinitely differentiable with respect to h and β, for large β, at values of h such that $h > \max[h(\beta), 4]$ or $0 \le h < \min[h(\beta), 4]$. This follows from Theorem 3.1 and an analysis similar to the one given above.

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