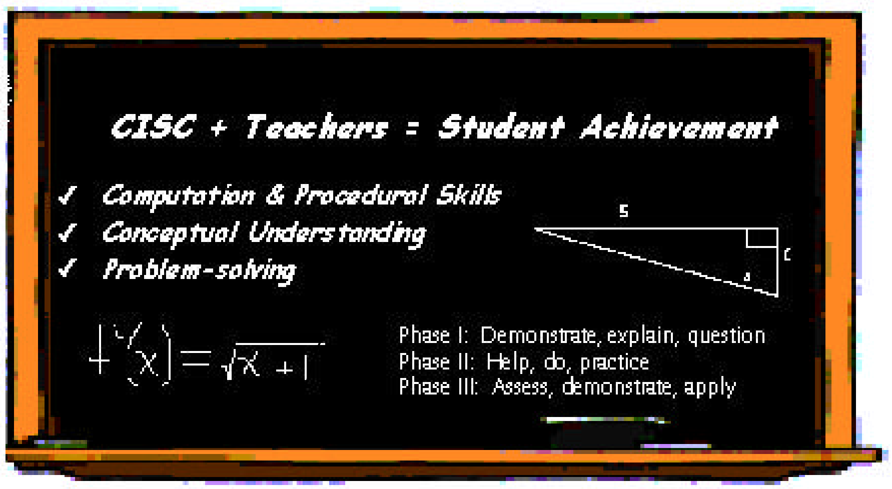
CISC - Curriculum & Instruction Steering Committee

California County Superintendents Educational Services Association

Primary Content Module IX Measurement & Geometry

The Winning EQUATION

A HIGH QUALITY MATHEMATICS PROFESSIONAL DEVELOPMENT PROGRAM FOR TEACHERS IN GRADES 4 THROUGH ALGEBRA II



STRAND: Measurement and Geometry

MODULE TITLE: **PRIMARY CONTENT MODULE IX**

MODULE INTENTION: The intention of this module is to inform and instruct participants in the underlying mathematical content in the areas of meaurement and Geometry.

THIS ENTIRE MODULE MUST BE COVERED IN-DEPTH.

In addition to the underlying mathematical content provided by this module, the facilitator should use the classroom connections provided within this binder and referenced in the facilitator’s notes.

TIME: 2 hours

PARTICIPANT OUTCOMES:

- Demonstrte understanding of "dissection: as the basic idea that permeates geometric measurement.
- Demonstrate knowledge of perimeter and area for polygons.
- Demonstrate how to use dissection to find the area of a triangle.
- Demonstrate how to use dissection to establish the Pythagorean Theorem.
- Demonstrate how to represent the area of a circle using dissection.
- Demonstrate why the sum of the measures of the angles of a triangle is 180^0.

PRE-POST TEST

T-1

PRIMARY CONTENT MODULE IX
Measurement and Geometry

Facilitator's Notes

Ask participants to take the pre-test. Explain the rationale behind the pre and post tests. Explain to the participants the importance of establishing how much we learn as we take part in this unit. Distribute **Pre-Assessment** and tell the participants that this will help them keep track of their learning. Allow 20 minutes for them to complete the test.

Advise participants that the grade 4 – 7 standards involve ideas from three areas of geometry: MENSURATION, SYNTHETIC GEOMETRY, AND ANALYTIC GEOMETRY. (T-1)

Take time to review these terms with the participants. You may want to provide a concrete example for each area.

Mensuration relates to the process of measuring length, area and volume of geometric objects. Here measuring refers to a well defined process for assigning a numerical value-i.e., an integer, rational, or irrational number to a geometric attribute. While some mensuration problems require calculus-like tools (circumference of a circle, area of a circle, or volume of a sphere), others can be dealt with at a pre-calculus level (area of a trapezoid or volume of a prism). Write the word "measurement" on the transparency.

Synthetic Geometry refers to the formal deductive system established by the ancient Greeks and subsequently refined by modern mathematicians. It is based on relationships between/among geometric objects such as points, lines, planes, angles, circles. Rather than dealing with measurement, synthetic geometry focuses on relationships such as intersection, congruence, and similarity. Write the word "relationships" on the transparency.

Analytic Geometry is a 17th century development that bridges the areas of geometry and algebra. It calls for the use of algebraic expressions to define geometric shapes such as points, lines, circles, etc. Write "merging of Algebra and Geometry" on the transparency.

In the grades 4 – 7 standards, analytic geometry tends to be dealt with under the heading of Algebra and Functions. For this reason we will focus on mensuration and synthetic geometry.

Perimeter (Polygons)

T-2

Put on T-2. The basic idea that geometric measurement pervades is dissection, "cutting up and rearranging the pieces." in finding the **perimeter** of a polygon, the idea is so natural that we tend not to mention it explicitly. The perimeter of a polygon is found by laying it's sides end-to-end.

T-3

Put on T-3 Note that the perimeter of a triangle is again found by laying its sides end-to-end. The perimeter is the sum of the lengths of the sides. When the same idea is extended to rectangles, the formula is simplified. Make sure to point out to participants *the use of the Distributive Property*.

T-4/H-4

Put on T-4. Hand out copies or ask participants to copy and do each problem.

1. 8 1/4 units
2. 36 units
3. 12 units
4. 13.2 units
5. 26 units
6. 26 units

Point out to participants that problems 5 and 6 have perimeters that are equal. Are the areas the same?

T-5

Put on T-5. Define and show the formula for circumference of a circle.

T-5A

$$C = \pi \times D, \text{ or equivalently, } C = \pi 2r = 2\pi r$$

Supplemental materials can be used here:
- Radius, Diameter Circumference a Circle (ST-1)
- π Chart (ST-2)
- Circumference Practice Problems (ST-3)
- Scoops and Circle (ST-4)

T-5B/H5B

Answers:
1) 31.4 ft. 2) 44 cm 3) $2\pi D = 40\pi$ D=20 in

Area (Polygons)

T-6

Put on T-6 Explain to participants that dissection is also central to finding the area of polygonal shapes. Here we begin by finding the areas of rectangles with sides of whole number length.

Area is measured in square units, that is, a square where each side is 1 unit long.

T-7

Point out to participants that a rectangle with a base of 5 and a height of 3 can be covered by 15 square units. Put on T-7. It is natural to assign it an area of 3 x 5 = 15 or

$$\mathbf{A = b \times h}$$

Special note to facilitator

The fundamental idea of area is the number of unit squares that fit inside a region. How should area be defined and understood for a rectangle whose sides have fractional lengths? The point is that such a rectangle cannot be partitioned into unit squares.

Transparencies T-8 through T-12 are devoted to resolving this question and explaining why the area of a rectangle, even under this circumstance, is base times height.

T-8 through T-12

Put on and present transparencies T-8 through T-12. Make duplicate transparencies of T-13 through T-16 and T-19 and T-22. Cut apart one transparency to model the dissection method to prove the area formulas and the Pythagorean Theorem.

T-13

Refer the participants to T-13.

T-14
T-15

Having established A = b x h for a parallelogram, we have a technique for finding the area of a triangle. Put on T-14. Dissection works as well for trapezoids. Put on T-15.

Ask participants to think about the area of a circle for a moment. Ask them if they can approximate the area of a circle in terms of sectors that resemble triangles.

T-16

Put on T-16 and discuss that since these sectors approximate half the area of a rectangle with base one half the circumference and height r, the figure on the slide can be used to suggest why the area of a circle of radius r is given by

$$\mathbf{A = (1/2) \times 2\pi r = \pi \times r^2}$$

Perimeter vs Area

T-17

Put up T-17. Discuss that while we may want to start the study of areas by asking students to "cover rectangles with unit squares" (like we did in the previous slide, it is important to note that this is equivalent to "cutting a rectangle in unit squares". The latter interpretation introduces children to the concept of dissection , which is at the heart of finding areas of more complex figures such as triangles, parallelograms and trapezoids. Be sure that participants understand that knowing the perimeter of a rectangle provides only limited information about the rectangle's area. By making b (or h) close to zero while keeping b + h constant, we can make the area b x h arbitrarily small even as the perimeter remains constant.

T-17A/H-17A

Ask participants to work problems and ask participants to present solutions.

1) $A = 36cm^2$ 2) $A = 30cm^2$
3) $h = 4cm$ 4) $A = 24cm^2$
5) $A = 70 + \frac{49}{4}\pi cm^2$ 6) $A = (40 + \frac{25}{2}\pi)cm^2$

Supplemental materials, Grade 6/7:
- Area of Two-Dimensional Shapes (ST-6)
- Practice with Area (ST-7)
- Scale Factor and Area (ST-8)
- Estimating Area (ST-9)
- Surface Area of Cylinders (ST-10)

T-18

Discuss with participants that dissection techniques even enable us to establish the Pythagorean Theorem. The Pythagorean Theorem applies only to right triangles. Show participants T-18. Explain that when you square length b, for example, you can represent this geometrically as a square with sides of length b and area equal to b^2.

T-19

Point out that given a right triangle with legs a and b and hypotenuse c, the challenge is to dissect smaller squares out of area a^2 and b^2 and rearrange the pieces into a square of area c^2 therefore, showing that $a^2 + b^2 = c^2$. This is shown in T-19.

Supplemental materials, Grade 6/7:
- Pythagorean Theorem (ST-11)
- Practice (ST-12)
- Pythagorean Theorem Converse (ST-13)

T-20

Put up T-20

T-21/H-21

T-21 is an algebraic proof of the Pythagorean Theorem. Begin by showing the illustration only and challenging your participants to complete the proof. Depending upon your audience you may need to give them the first line. Be sure to work them through the proof as a class.

Discuss that while dissection applies to solids, the process has limited application at grades 4 - 7. Here the standards call for helping students visualize 3-dimensional shapes and developing formulas for volume and surface areas of special figures such as rectangular prisms, pyramids and cylinders. (This content material is covered within the classroom lesson portion of each grade level 4/5 and/or 6/7 as appropriate and as determined by the content standards.)

Supplemental materials, Grade 6/7:

- Volume of Prism, Pyramid, and Cylinders (ST-14)
- Volume of Cylinders (ST-15)
- Surface Area and Volume Practice Problems (ST-16)

Synthetic Geometry

Discuss with participants that in order to prepare students for its more formal study, the grades 4 - 7 standards call for students to develop the vocabulary of Euclidean geometry and an understanding of the relations and properties central to geometry in the plane. The essential terminology is that of points, lines, angles, triangles, circles, parallelism, bisectors etc. (This vocabulary is found within the 4/5 module.)

Supplemental materials, Grade 6/7:

- Angle (ST-17)
- Types of Angles (ST-18)
- Complementary and Supplementary Angles (ST-19)
- Types of Angles (ST-21)
- Properties of a Triangle (ST-22)
- Making Angles (ST-23)

A Mathematics Content Standard for grade 5 under Measurement and Geometry 2.2 is

> "Know that the sum of the angles of any triangle is 180^0
> and the sum of the angles of any quadriliateral is 360^0
> and use this information to solve problems."

T-22

Put on T-22. A question a student might ask is "Why is the sum of the measures of the angles of a triangle 180^0? An activity that a person might do to show the students why is as follows:

> With scissors and paper cut out a triangle. It does not matter whether the triangle is scalene (has sides of different lengths), isosceles (at least 2 sides with equal lengths), or equilateral (all three sides are equal in length, a special case of the isosceles). Also, it does not matter whether the triangle is obtuse (has an angle that measures more than 90^0), right (has an angle that measures 90^0), acute (all angles measure less than 90^0), or equiangular (the measures of all angles are equal).
>
> Tear off each angle. Position the three angles so that they have common vertices as shown on the transparency. When placed in these positions, the angles appear to have a sum of 180^0.

An indeed it is! But the question still remains:

> How does one know that the sum of the angles is 180^0 and not 179.9^0 or 180.1^0 , for example?

Through synthetic geometry we can prove that the measures of any triangle's angles have a sum of 180^0.

To do this we must first define some terms, present a postulate, and prove a couple of theorems.

T-23/H-23

Put on T-23. Point out the vertical angles. Walk the participants through the proof.

T-24

Put on T-24. Again, point out the alternate interior angles and corresponding angles.

T-25/H-25

Put on T-25. In this transparency the lines j and k are parallel (they never intersect). Line L is called a transversal. A postulate is an accepted fact.

T-26/H-26

Put on T-26. Using the theorem from PT-22 and the postulate from T-24 it can be proven that the measures of the alternate interior angles are equal.

T-27/H-27

Put on T-27. This transparency requires the participants to agree to the idea (a couple of postulates actually) that a straight angle (an angle that has a measure of 180^0) can be decomposed into three angles whose measures have a sum of 180^0.

Then by substitution the measures of the angles of a triangle can be shown to have a sum of 180^0.

Provide time for participants to ask clarifying questions.

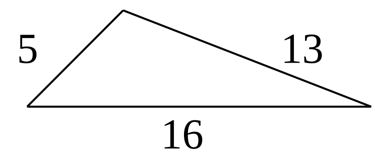
Supplemental materials:

- Properties of Quadrilaterals (ST-24)
- Properties of Quadrilaterals (ST-25)
- Quadrilateral Comparisons (ST-26)
- Congruence (ST-27)

Measurement and Geometry

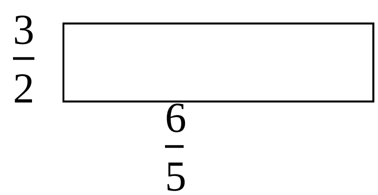
Pre- Post-Test

1. Find the perimeter of the triangle shown below



2. Define the term "dissection" in relation to geometry.

3. Find the area of the rectangle below:



4. Explain and illustrate how you would find the area of a triangle using dissection.

5. Write the Pythagorean Theorem.

6. Prove the Pythagorean Theorem.

7. Describe and illustrate how using dissection enables us to represent the area of a circle in terms of sectors that resemble triangles.

8. Explain why the sum of the measure of the angles of a triangle is 180^0. How do you know that it is not only 179.9^0?

Measurement and Geometry

Pre- Post-Test Answer Key

1. The perimeter of the triangle is 34 units.

2. Dissection, in relation to geometry, refers to the "cutting up and rearranging of pieces".

3. The area of the rectangle is $\frac{9}{5}$ square units.

4. See Facilitator Notes and T-14

5. For any right triangle whose sides have lengths, *a*, *b*, and *c* with the hypothenuse of length *c*, $a^2 + b^2 = c^2$.

6. See Facilitator Notes and T-19 or T-21.

7. See Facilitator Notes and T-16

8. See T-27

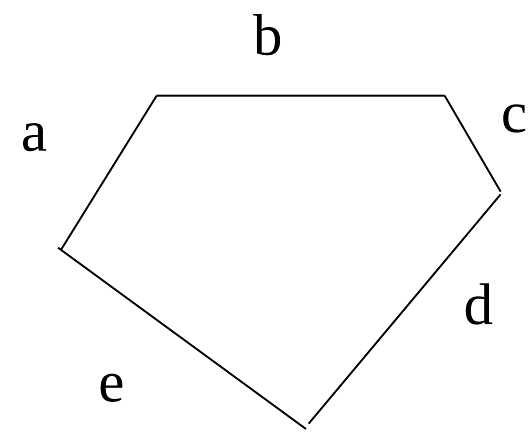
Standards involve ideas from three areas of geometry

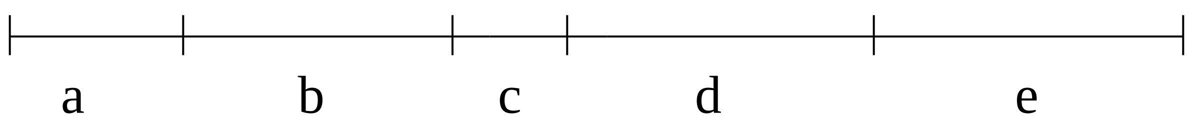
Mensuration

Synthetic Geometry

Analytic Geometry

The basic idea that permeates geometric measurement is that of "cutting up and rearranging pieces."



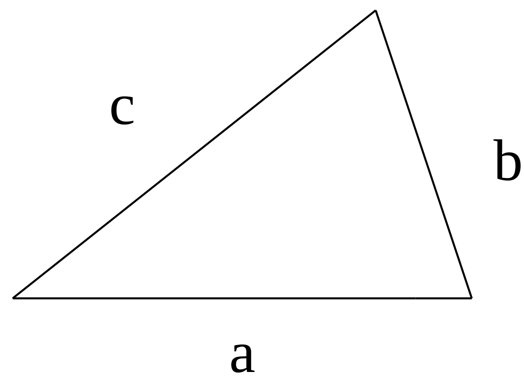


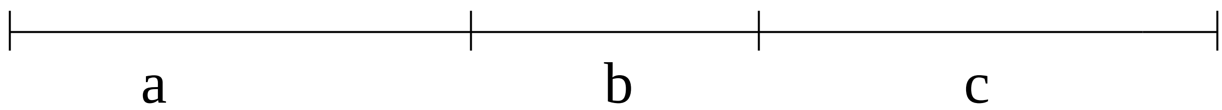
Perimeter = P = a + b + c + d + e

The same concept underlies finding the perimeter of any polygonal shape.

Perimeter of a Triangle and a Rectangle

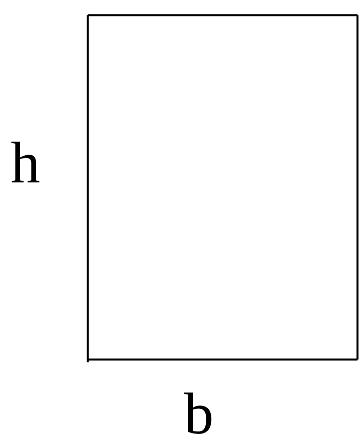
Triangle

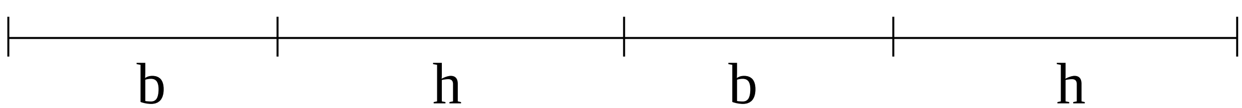




$P = a + b + c$

Rectangle

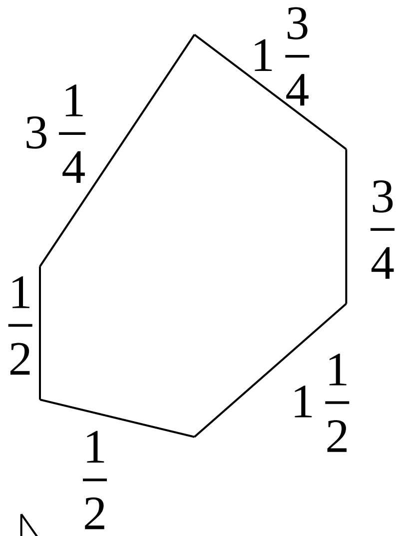




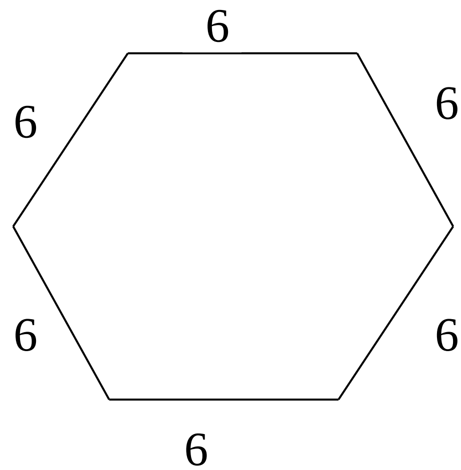
$$\begin{aligned} P &= b + h + b + h \\ &= 2b + 2h \\ &= 2\,(b + h) \end{aligned}$$

Find the perimeter of each figure.

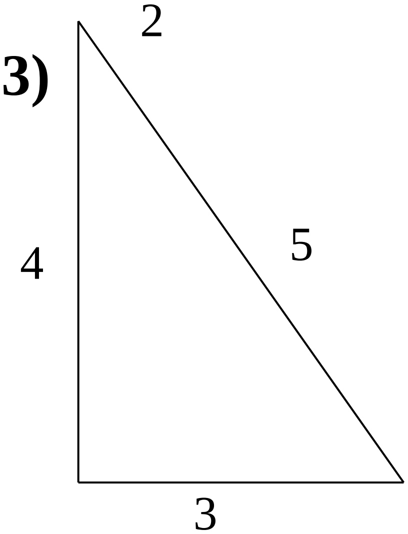
1)



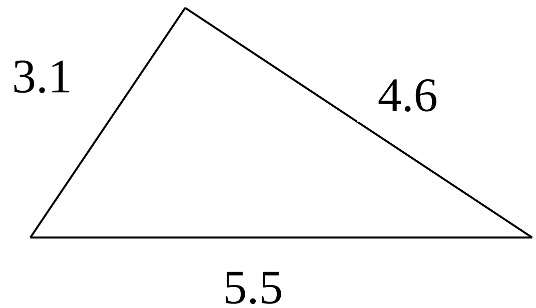
2)



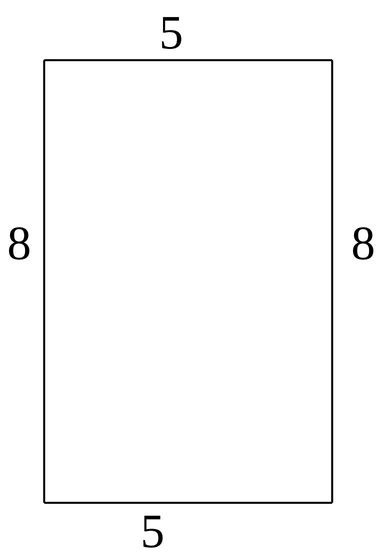
3)



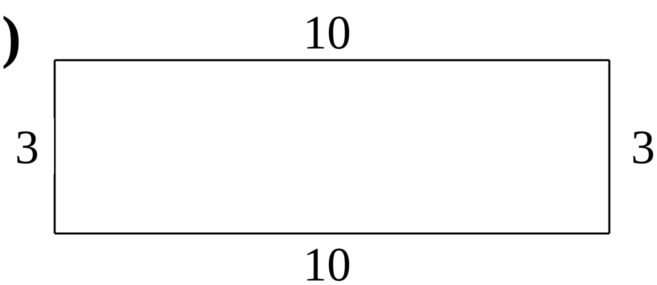
4)



5)



6)



The Number π

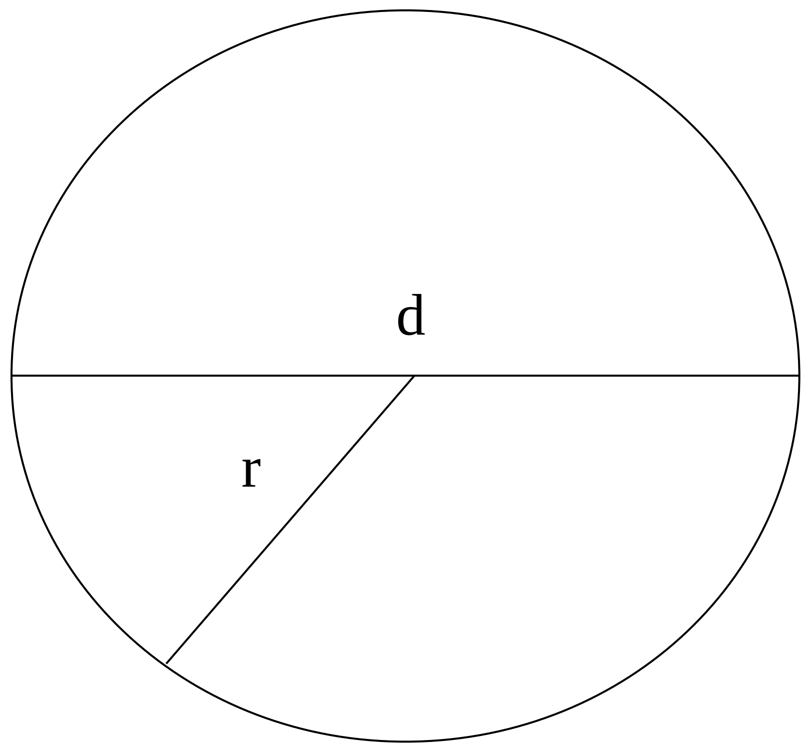
What is the number π? Where does it come from?

<u>Definition</u>: π is the ratio of the circumference of any circle to its diameter.

The ratio $\pi = \frac{C}{d}$ is the same for all circles.
Here C = circumference or distance around the circle, and d = diameter. If we multiply both sides of $\pi = \frac{C}{d}$ by d, we get

$$C = \pi d$$

Circumference of a Circle



$$C = \pi d$$

$$C = \pi \bullet 2r$$

$$C = 2\pi r$$

It can be shown that a π is irrational and that

$$\pi = 3.1415926535....$$

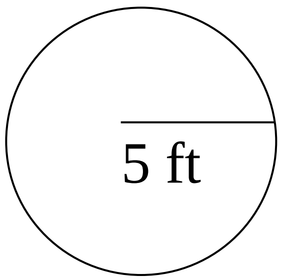
Two commonly used rational approximations are:

$$\pi \approx 3.14$$

$$\pi \approx \frac{22}{7}$$

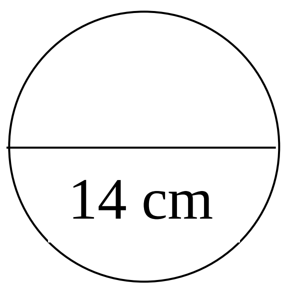
Practice

Approximate the circumference of each circle

1) 

$C \approx$ ______________

Use $\pi \approx 3.14$

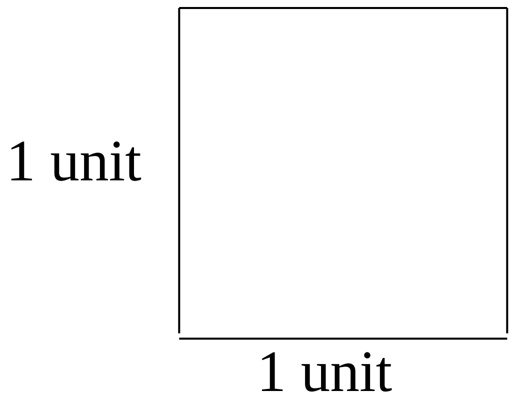
2) 

$C \approx$ ______________

Use $\pi \approx \frac{22}{7}$

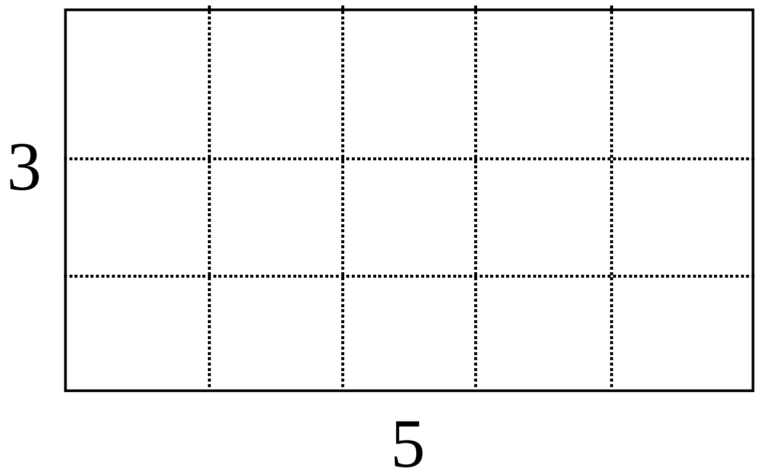
3) A bicycle travels 40π inches in two revolutions. What is the diameter of the wheel of the bicycle? ____________________
(No Calculators)

Area



Some units to measure area are:

square feet	ft^2
square inches	in^2
square yards	yd^2
square miles	mi^2
square centimeters	cm^2
square meters	m^2
square kilometers	km^2



With a base of 5 units and a height of 3 units, it is natural to assign it the area of:

3 units x 5 units = 15 square units

= 15 $(\text{units})^2$

more generally,

A = number of square units

A = (no. of unit squares in each row) x (no. of rows)

A = base x height

A = b x h

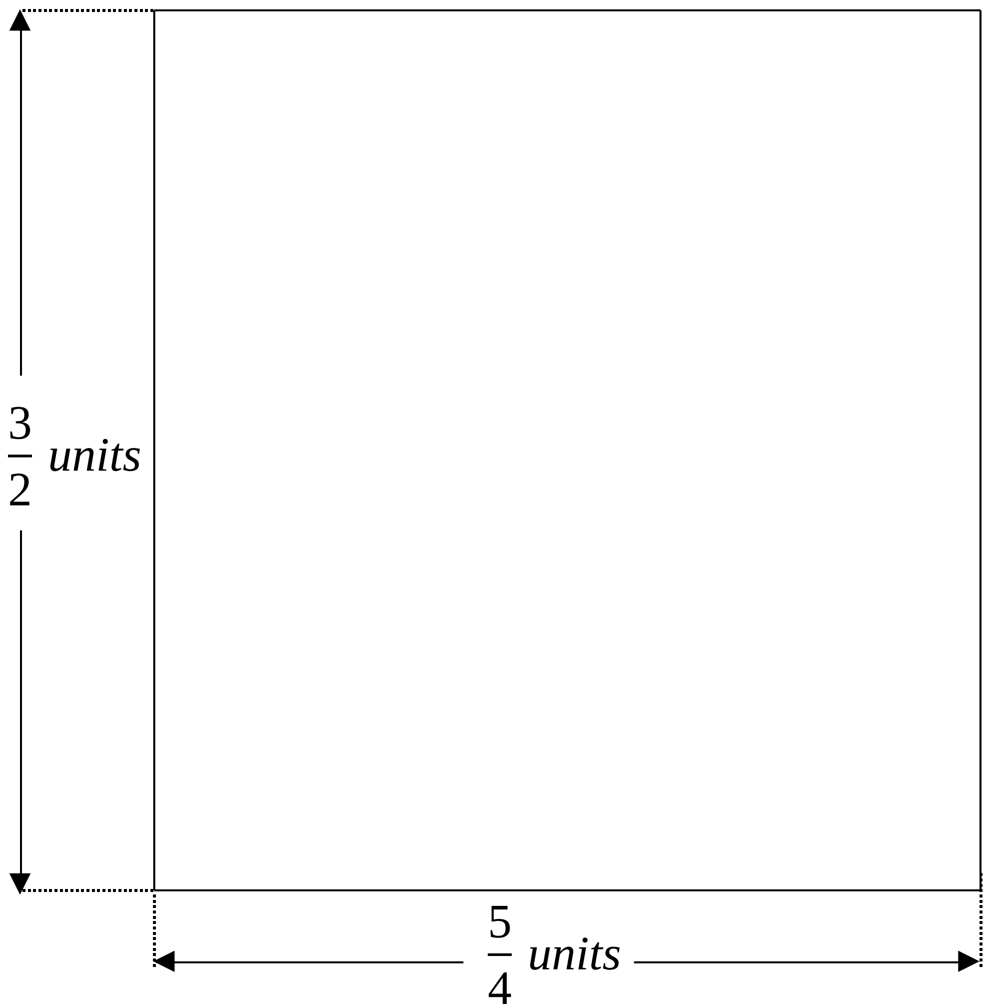
Rectangles Whose Dimensions Are Not Whole Numbers

For a rectangle whose sides have integer lengths, area is the number of unit squares required to fill the interior.

Alternatively, area may be thought of as the number of unit squares which may be "cut" from the interior of the rectangle.

What does area mean if the lengths of the sides of the rectangle involve fractions?

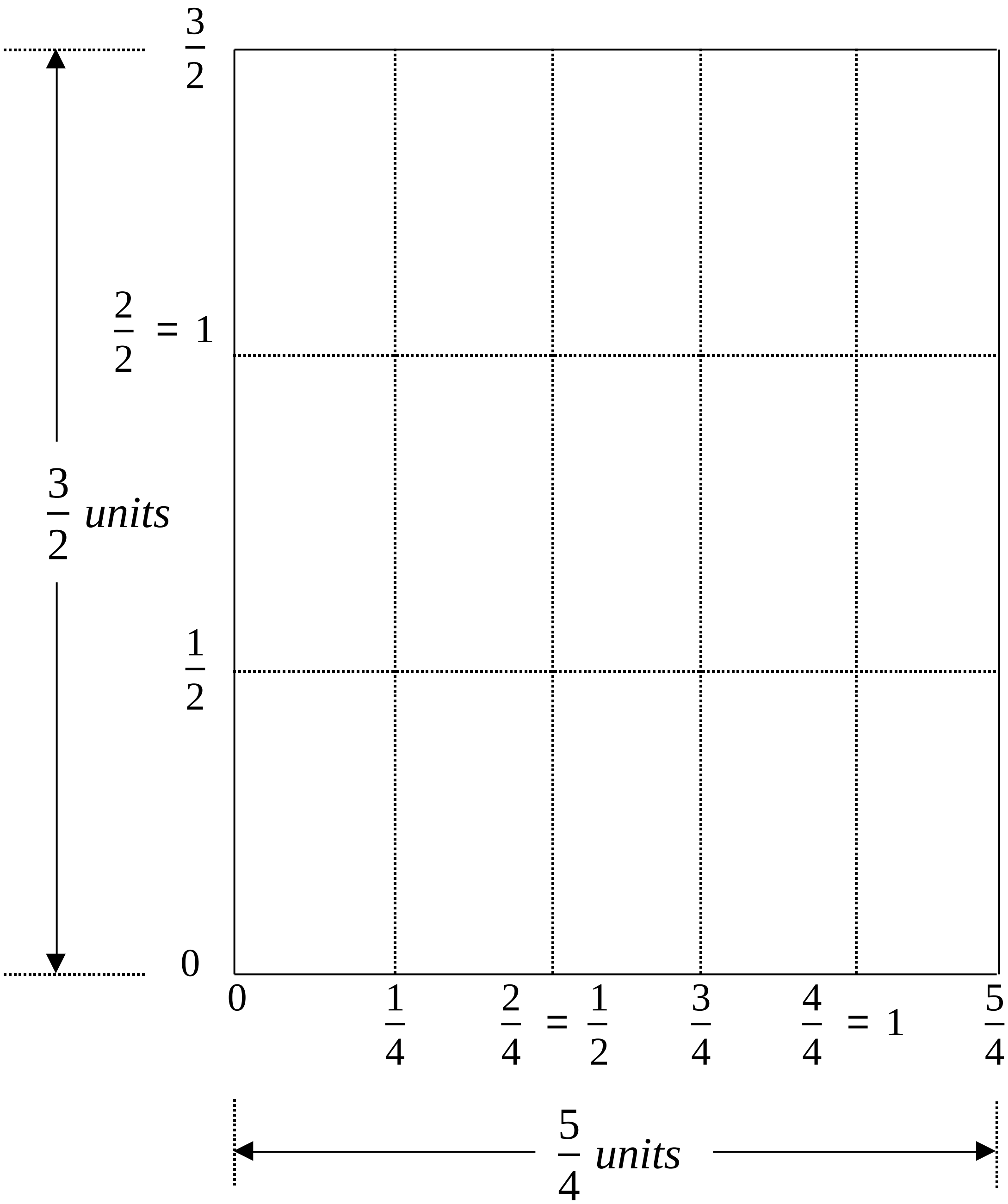
Example



What do we mean by the area of this rectangle?

It is not possible to "fit in" a whole number of unit squares, but area <u>means</u> the number of unit squares that "fit inside".

Give Up Unit Squares

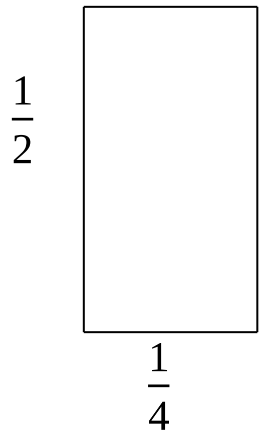


(5 rectangles in each row) x (3 rows)

15 rectangles

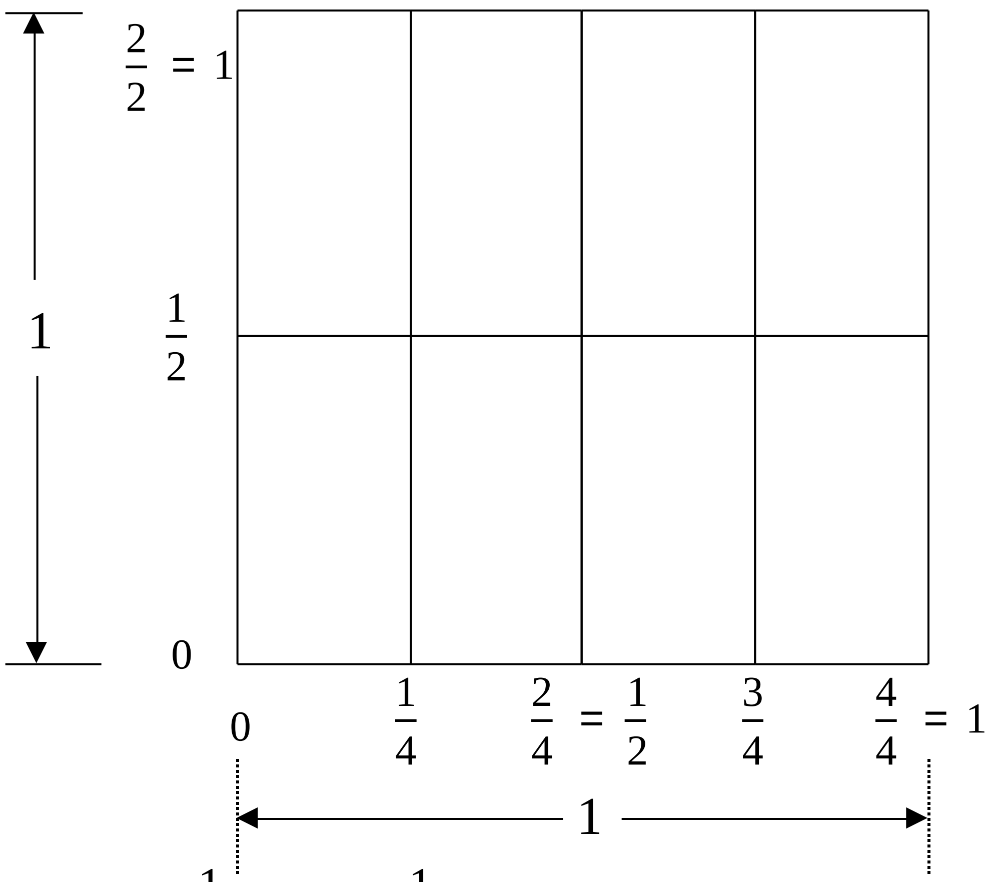
BUT: What is the area of each of the smaller rectangles?

What is the area A of:

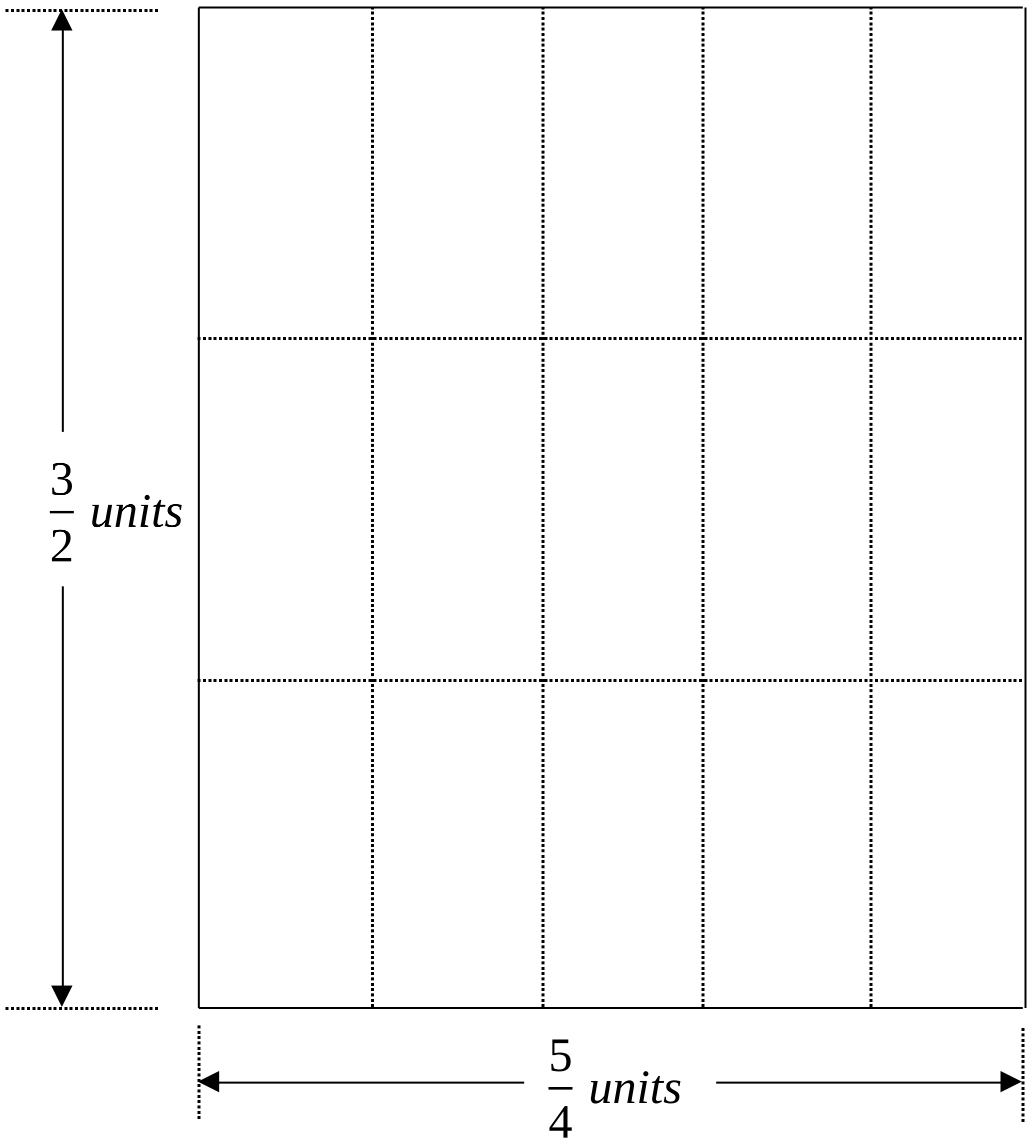


$\frac{1}{2}$ unit by $\frac{1}{4}$ unit rectangle

Back to the one-by-one square



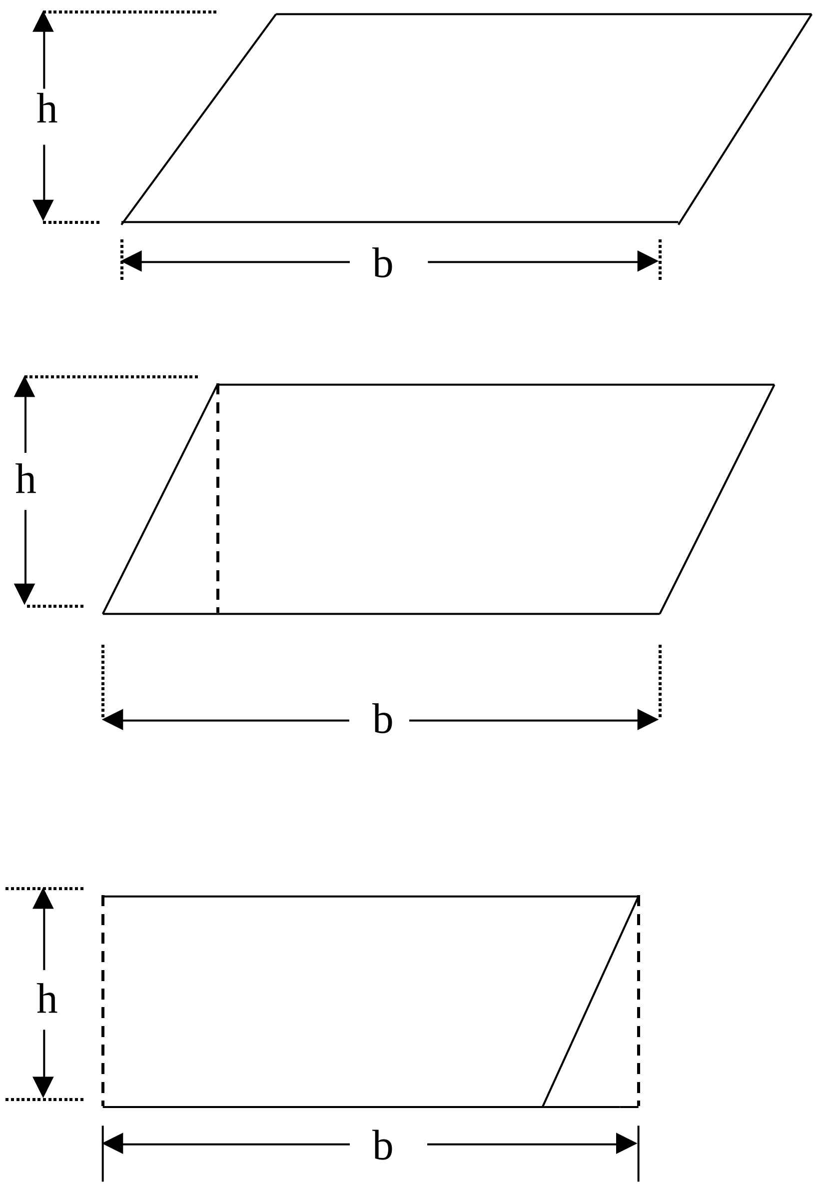
8 of the $\frac{1}{2}$ unit by $\frac{1}{4}$ unit rectangles fit in a unit square. Since all 8 rectangles are congruent, each has an area of $\frac{1}{8}$ of a unit square. $\mathbf{A} = \frac{1}{8}$



$$A = 15 \bullet \frac{1}{8}$$
$$= \frac{5}{4} \bullet \frac{3}{2}$$
$$= b \bullet h$$

Area of a rectangle is the length of the base times length of the height works for fractions.

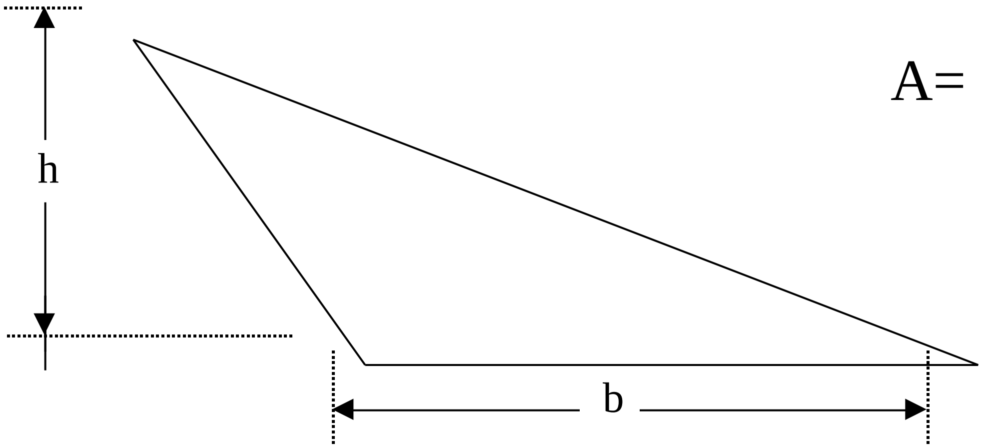
Using dissection to determine the area of a parallelogram with base b and altitude h.



Area of a Parallelogram

$$A = b \cdot h$$

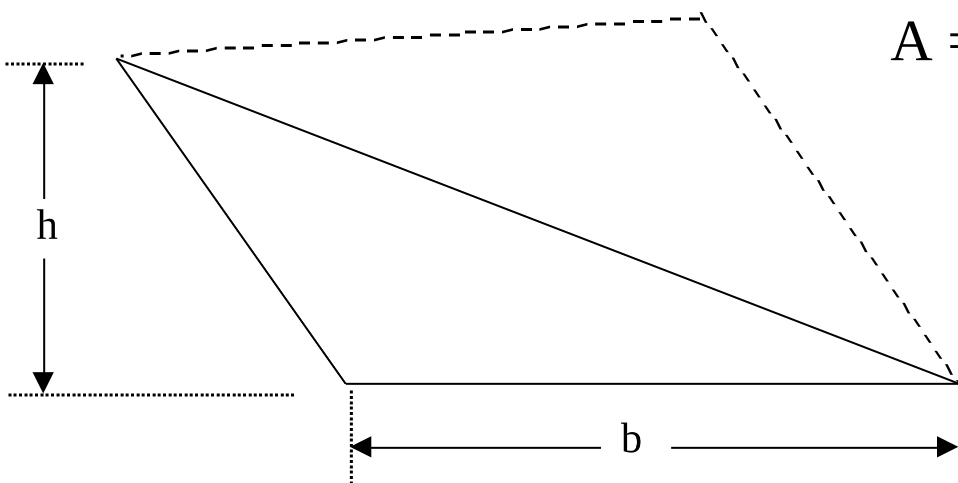
Finding the Area of a Triangle



$A = ?$

$2A = b \bullet h$

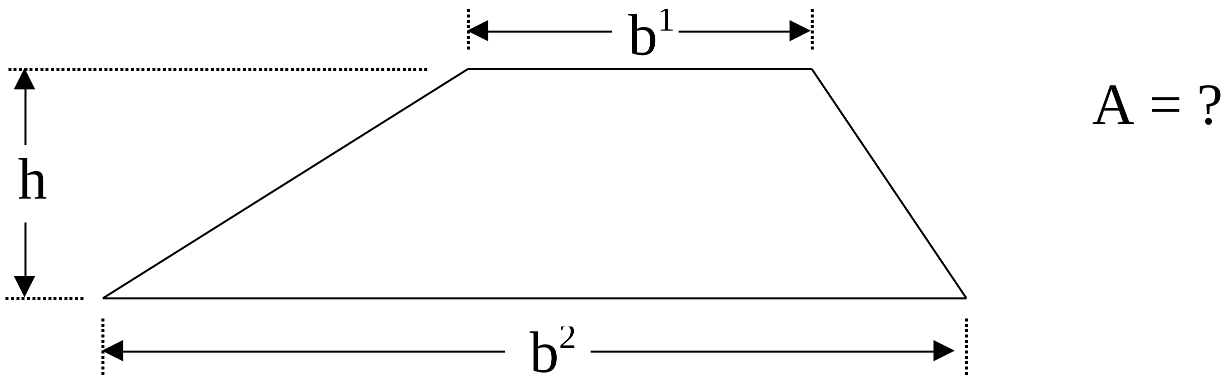
$A = \frac{1}{2} b \bullet h$

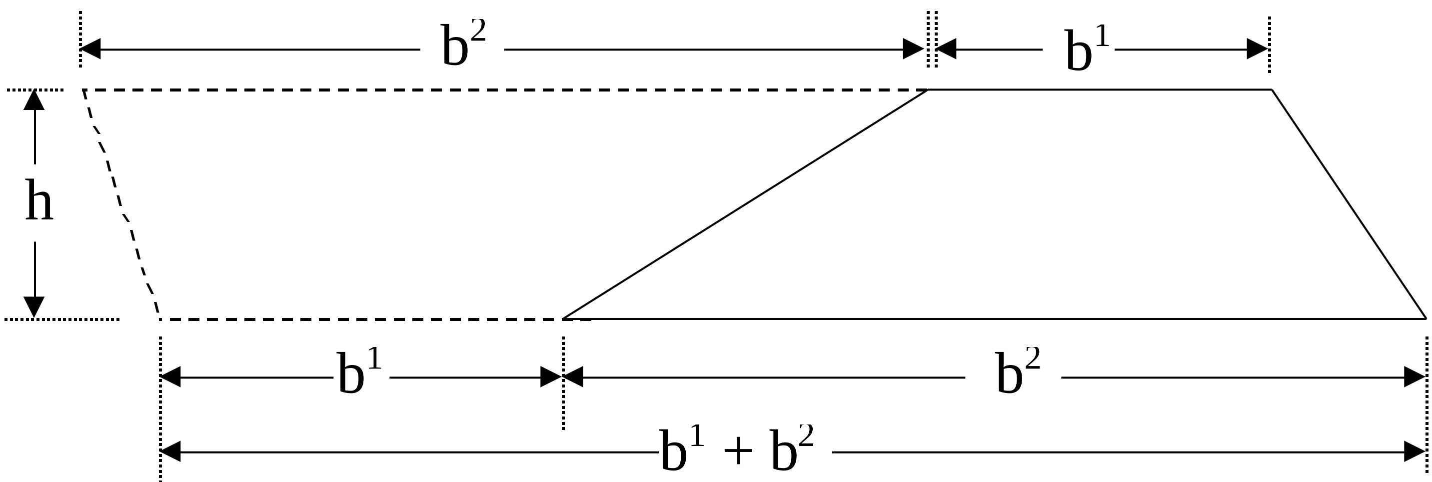


Area of a Triangle

$$A = \frac{1}{2}\, b \bullet h$$

Finding the Area of a Trapezoid



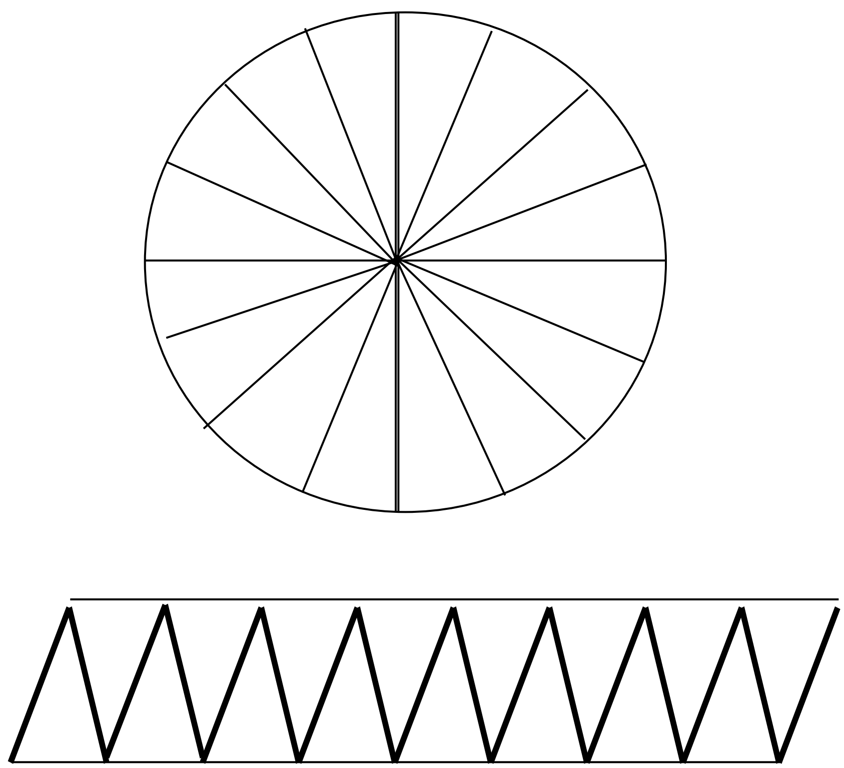


Area of a Trapezoid

$$\mathbf{2A} = (b_1 + b_2)h$$

$$\mathbf{A} = \frac{1}{2}h(b_1 + b_2)$$

Dissection enables us to represent the area of a circle in terms of sectors that resemble triangles

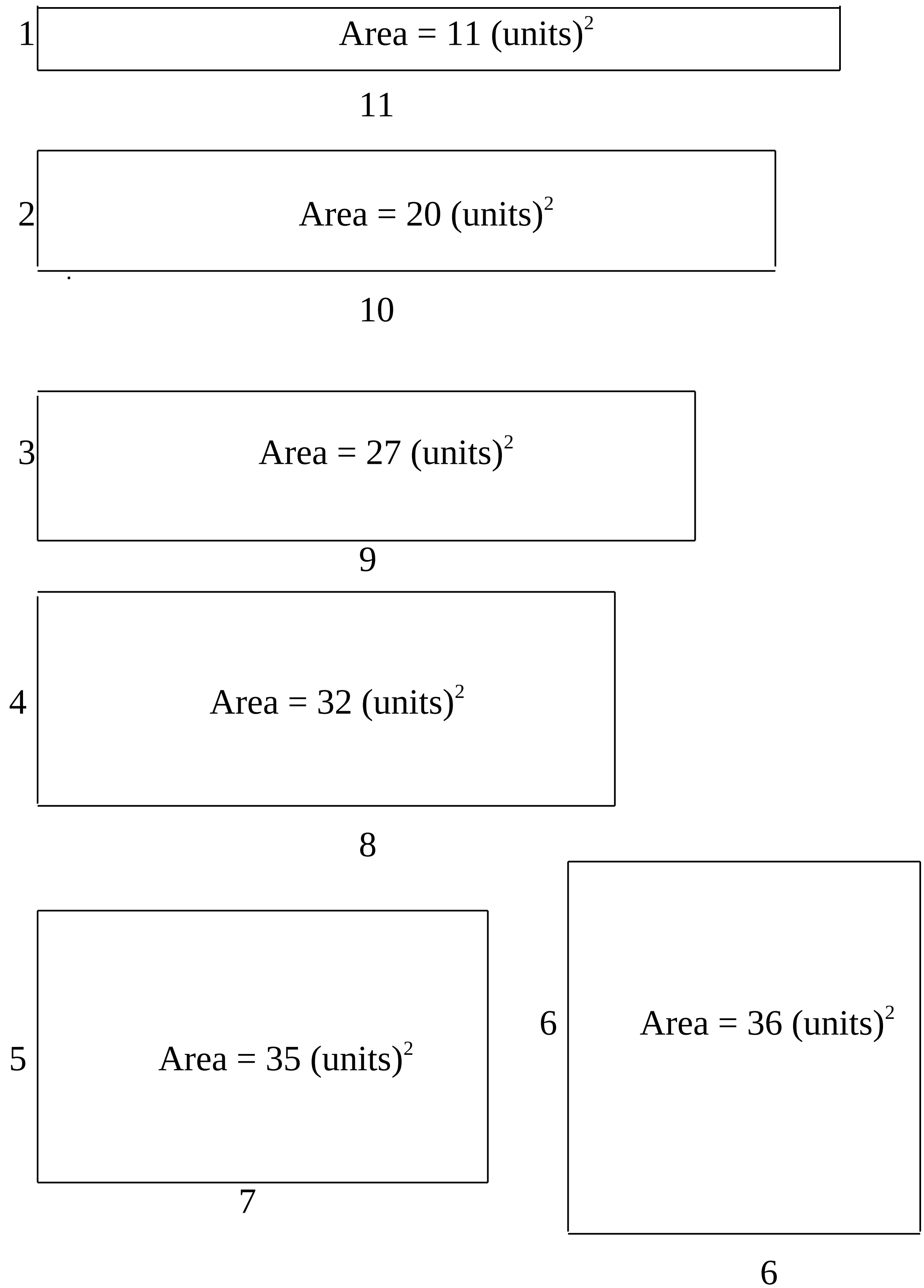


Recall: $C = \pi d = 2\pi r$

Area of a rectangle: $\left(\frac{1}{2}C\right) \cdot r = \frac{1}{2}(2\pi r) \cdot r$

Area of a circle: $\mathbf{A = \pi r^2}$

Rectangles with the same perimeter, but different areas.

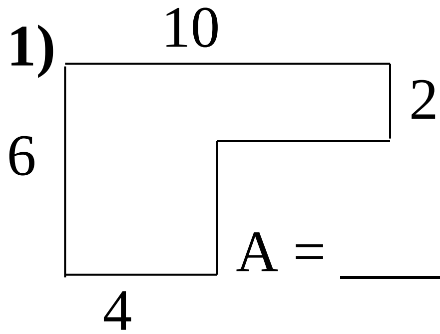


Practice

Find the missing measure of each figure.

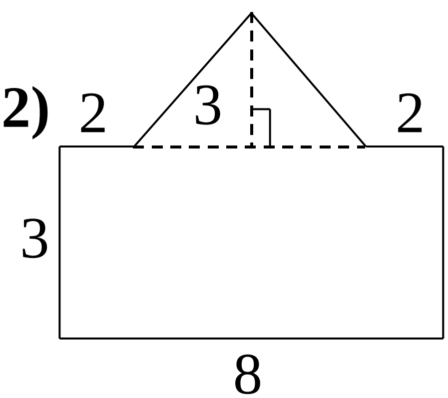
Leave answers in terms of π. All measures in cm.

1)



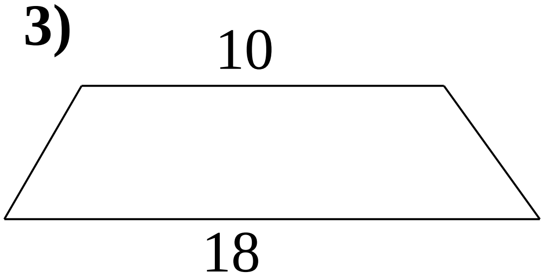
A = _________

2)



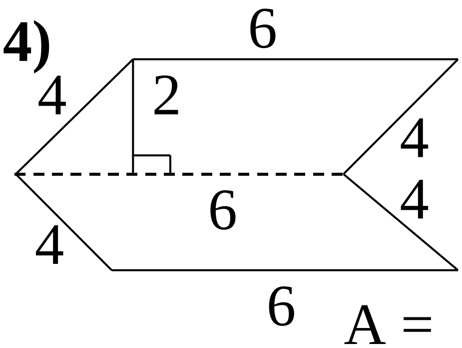
A = _________

3)



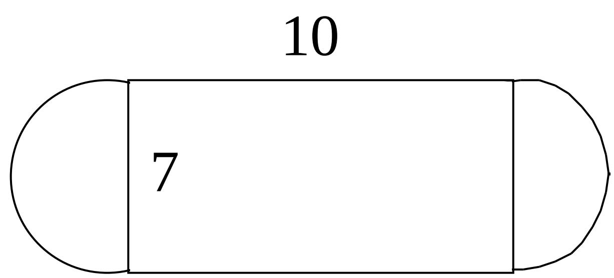
If A = 56cm^2
then h = __________

4)



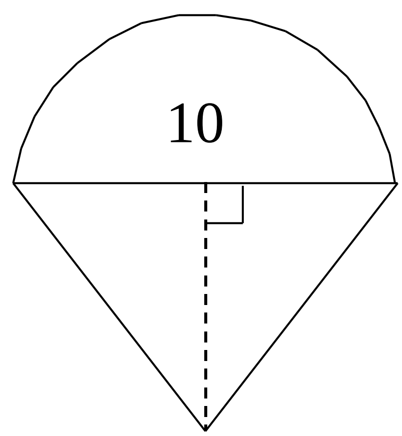
A = _________

5)



A = _________

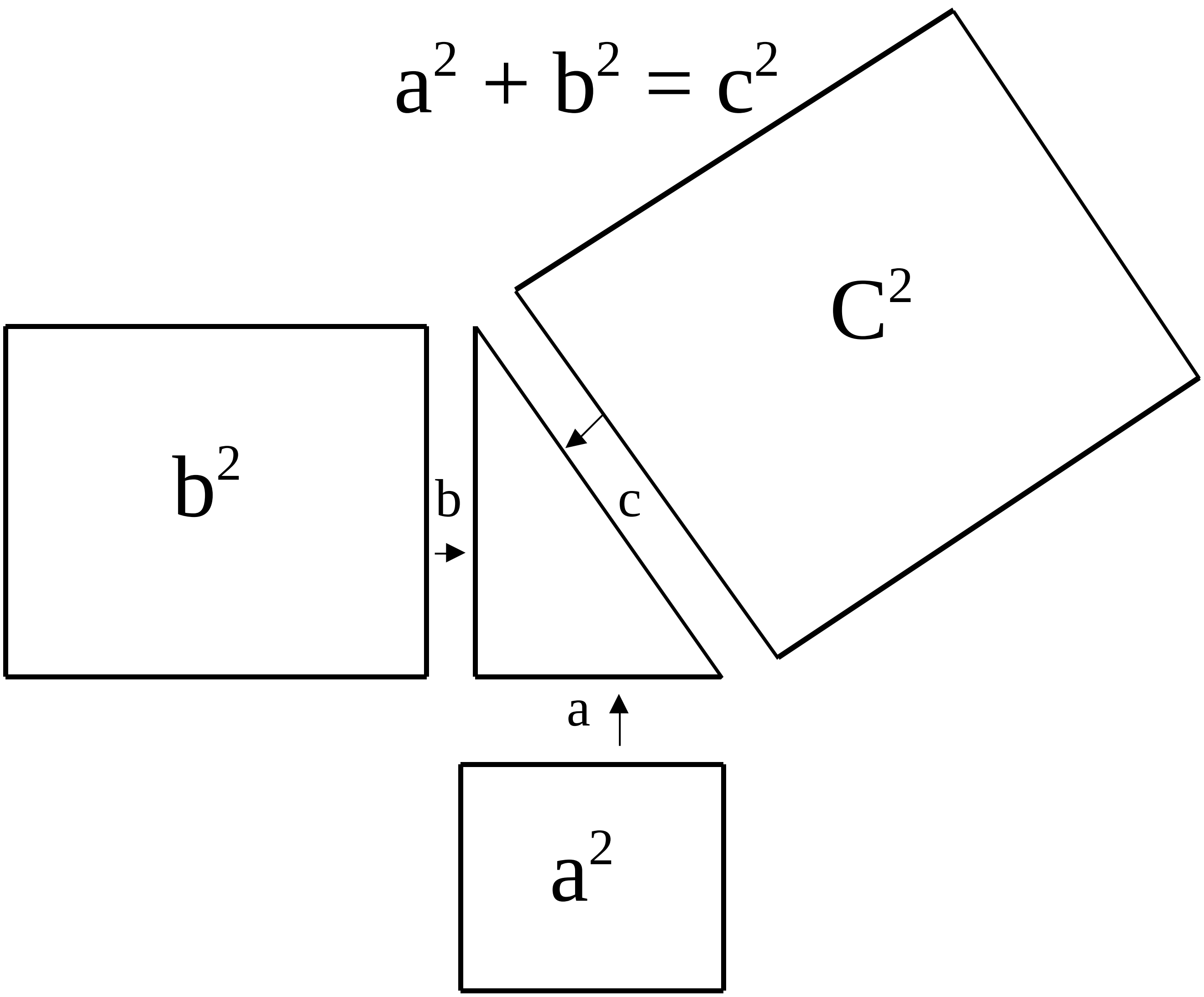
6)



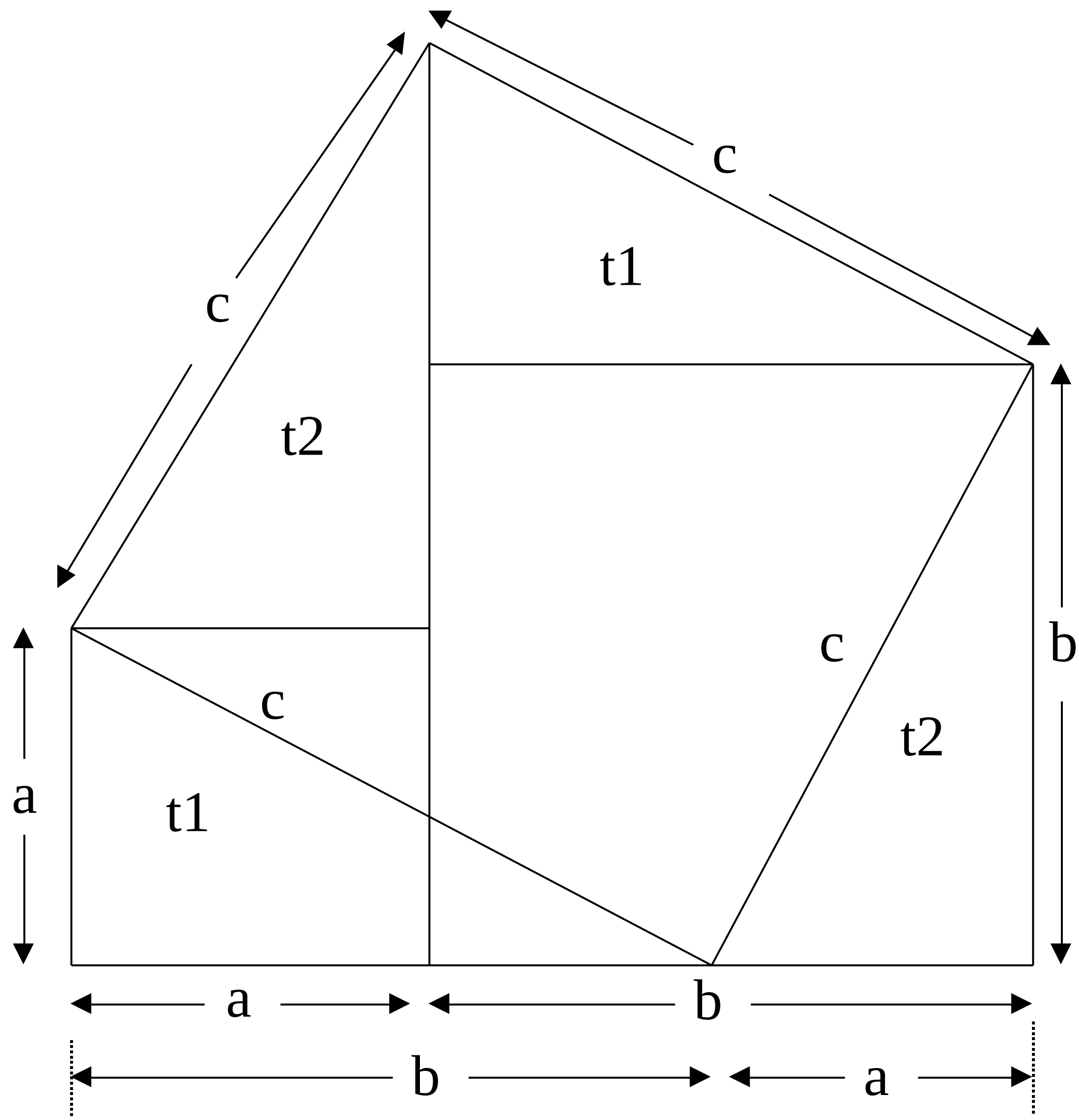
A = _________

Dissection techniques enable us to establish the Pythagorean Theorem

$$a^2 + b^2 = c^2$$



Rearrange the pieces into a square with an area of c^2



Prelude to 2nd Proof

$$(a + b)^2 = ?$$

Distributive Property

$$a(b + c) = ab + ac$$

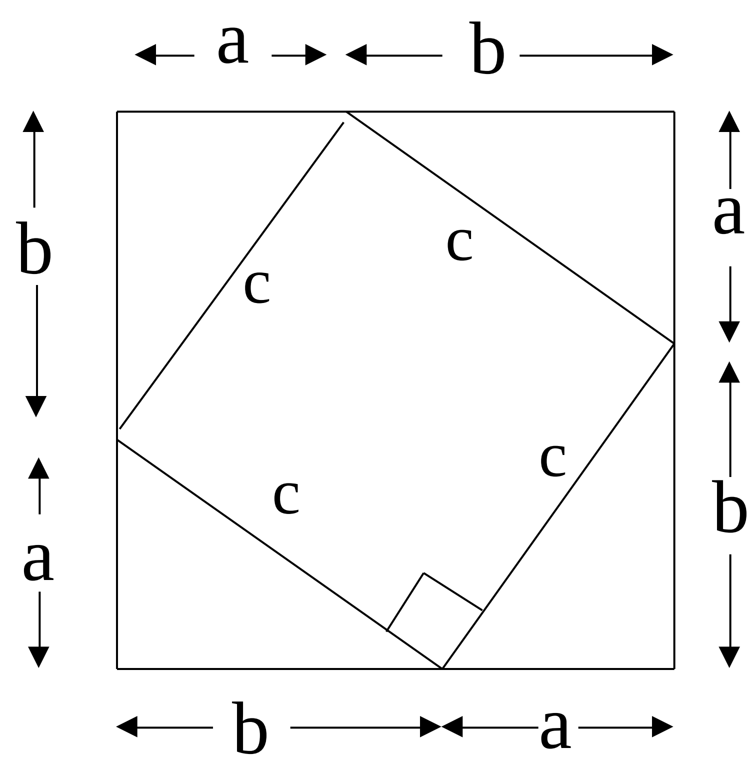
$$A(a + b) = Aa + Ab$$

Let $A = (a + b)$

$$\begin{aligned}(a + b)^2 &= (a + b)(a + b) \\ &= A(a + b) \\ &= Aa + Ab \\ &= (a + b)a + (a + b)b \\ &= a^2 + ba + ab + b^2\end{aligned}$$

$$\mathbf{(a + b)^2 = a^2 + 2ab + b^2}$$

Another way to establish the Pythagorean Theorem:



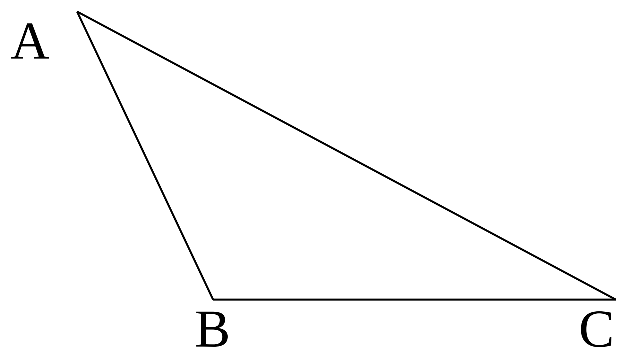
$$A = (a + b)^2 \text{ but also } A = 4\,(1/2ab) + c^2$$

$$(a + b)^2 = 4\,(1/2ab) + c^2$$

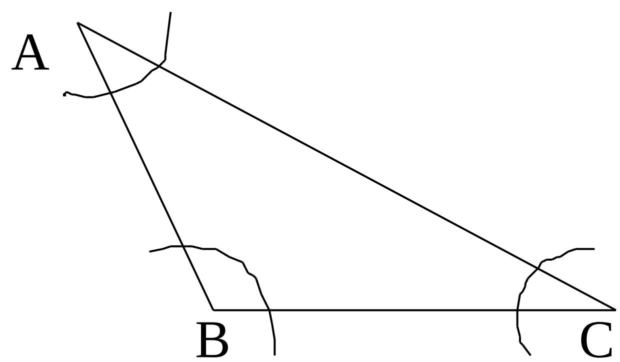
$$a^2 + 2ab + b^2 = 2ab + c^2$$

$$a^2 + b^2 = c^2$$

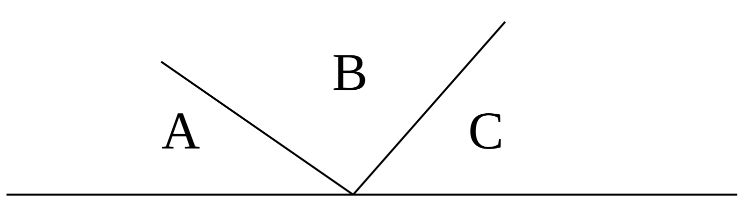
What is the sum of the measures of the angles of a triangle?



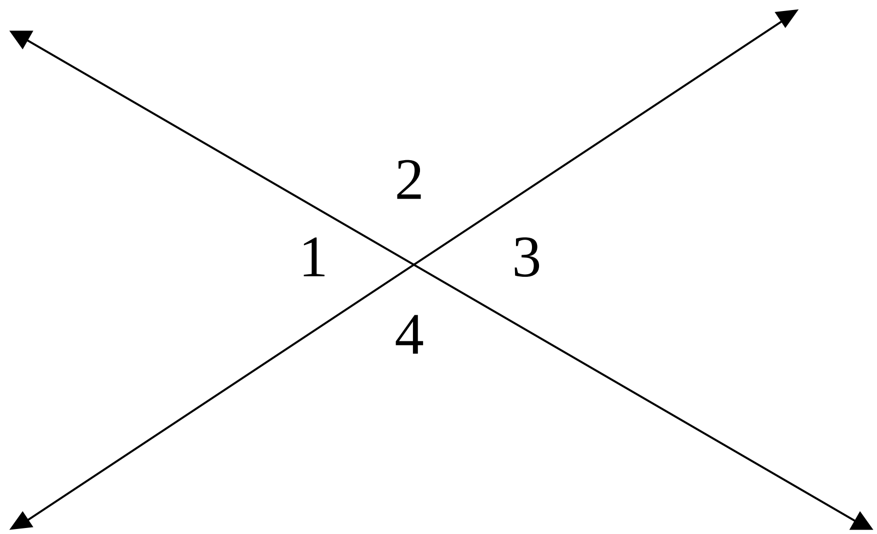
Tear off the angles



Position the three angles so their vertices share the same point.



Types of Angles



Vertical Angles

1 and 3

2 and 4

Note: m 1 + m 2 = 180 = m 2 + m 3

m 1 + m 2 = m 2 + m 3

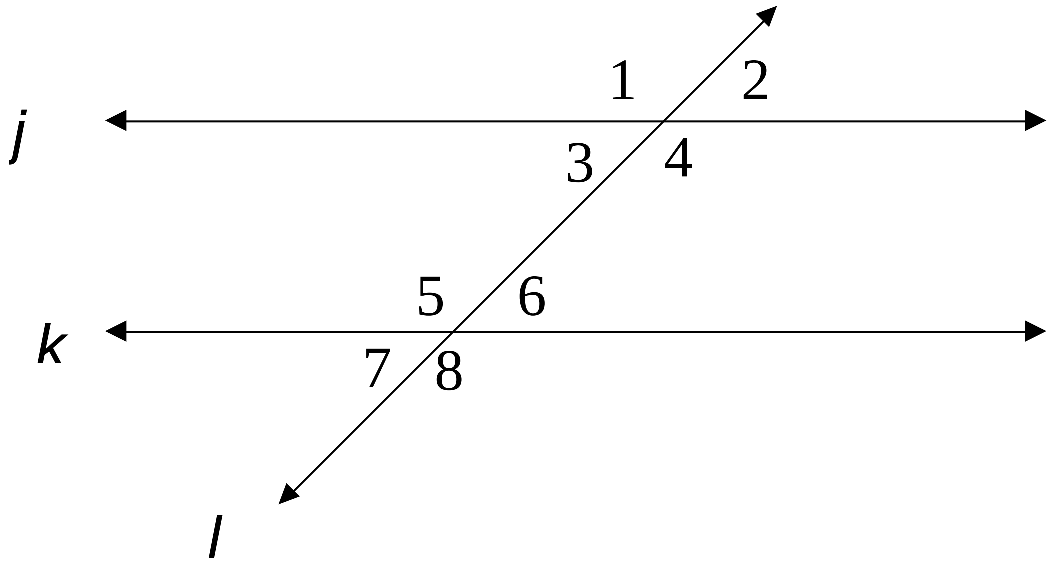
m 1 = m 3

A similar procedure can be done for 2 and 4

Theorem:

The measures of vertical angles are equal.

More Types of Angles



Alternate Interior Angles

3 and 6

4 and 5

Corresponding Angles

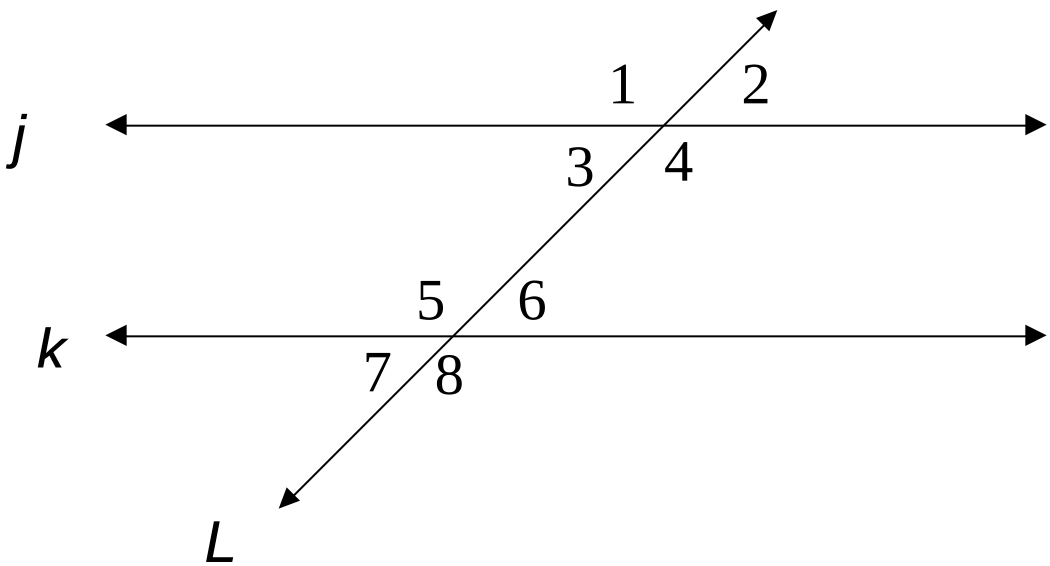
1 and 5

2 and 6

3 and 7

4 and 8

Parallel Lines and Corresponding Angles



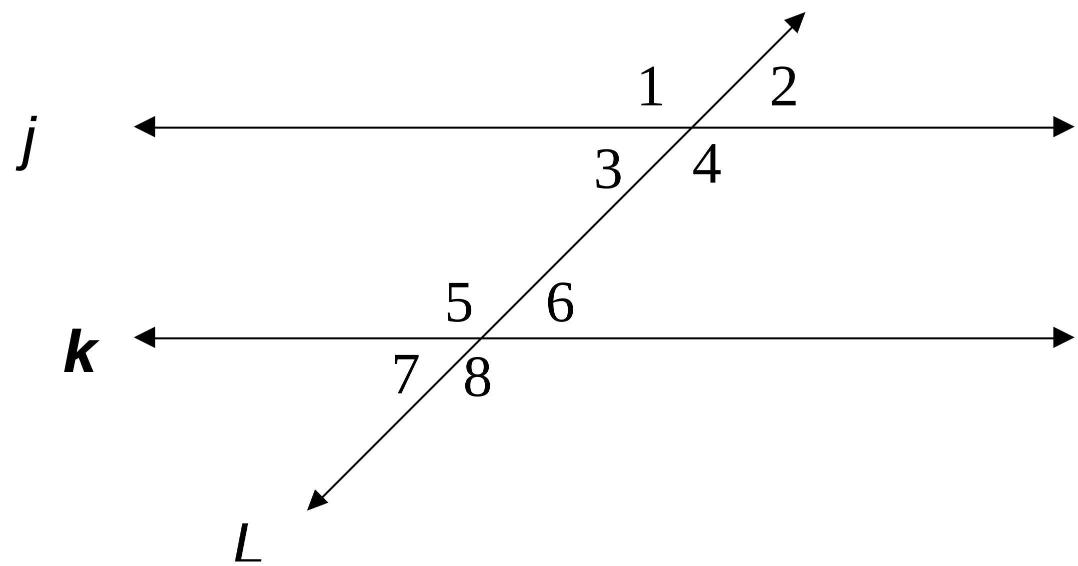
Lines j and k are parallel.

Postulate: If two parallel lines are cut by a transversal, then measures of the corresponding angles are equal.

$m \;\; 1 = m \;\; 5$ $m \;\; 3 = m \;\; 7$

$m \;\; 2 = m \;\; 6$ $m \;\; 4 = m \;\; 8$

Parallel Lines and Alternate Interior Angles



Lines j and k are parallel.

Note: m 1 = m 5

m 4 = m 1

m 4 = m 5

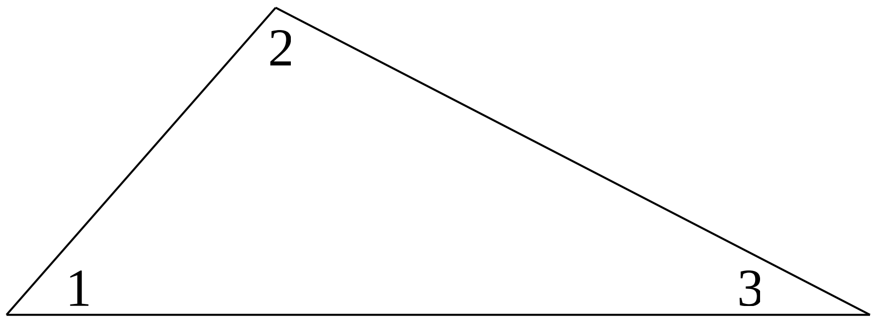
The alternate interior angles 4 and 5 have equal measures.

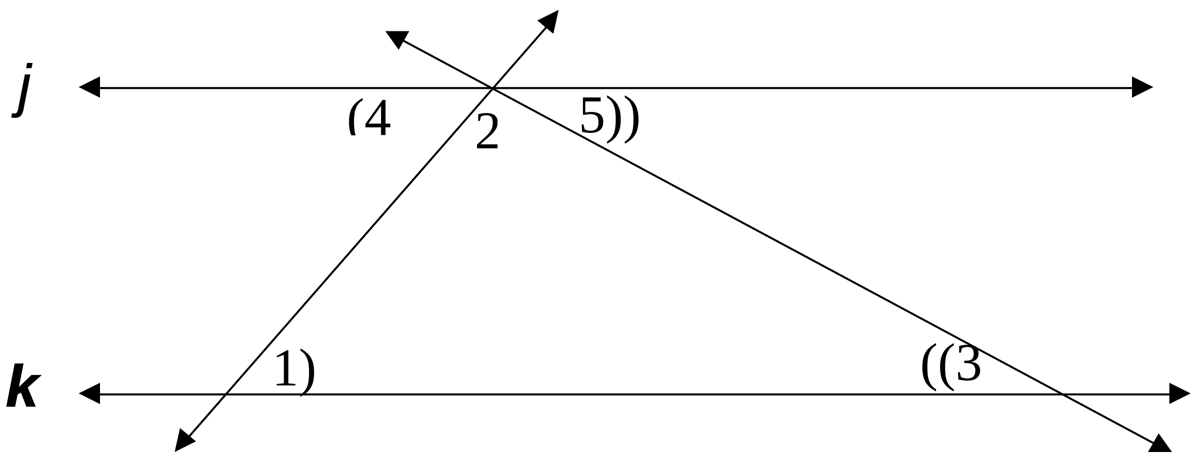
A similar procedure can be done for 3 and 6.

Theorem:

The measures of alternate interior angles are equal.

The Sum of the Measures of the Angles of a Triangle





Lines j and k are parallel.

$m\ 4 + m\ 2 + m\ 5 = 180$

$m\ 1 = m\ 4$ $\quad$ $m\ 3 = m\ 5$

$m\ 1 + m\ 2 + m\ 3 = 180^0$

This procedure can be done on any triangle .

Theorem: The sum of the measures of the angles of a triangle is 180^0

