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An “FKG equality” with applications to random environments

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Abstract

We consider a sequence of independent random variables, $X_1, X_2, X_3, \ldots,$ taking values in $\{1, 2, \ldots, m\}$. We introduce a σ-algebra of “nonpivotal” events and prove the following 0–1 Law: $P(A) = 0$ or 1 if and only if A is nonpivotal. All tail events are nonpivotal. The proof is based on an “FKG equality” which provides exact error terms to the FKG inequality. We give some applications for independent random variables in a random environment in the sense that the probabilities of particular outcomes are random. © 2000 Elsevier Science B.V. All rights reserved

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The FKG inequality has resulted in great advances in probabilty theory, especially in statistical mechanics, percolation theory, and reliability theory. With appropriate restrictions on the probability measure P, the FKG inequality says that the correlation between increasing events, A and B, is always nonnegative, i.e.,

$$P(AB) - P(A)P(B) \geqslant 0.$$

This inequality was first established by Harris (1960) in the case of a product measure P and subsequently generalized by Fortuin et al. (1971), and many others. Considering the importance of the FKG inequality, it is natural to investigate the existence of a more detailed relationship between $P(AB)$ and $P(A)P(B)$ of the form

$$P(AB) = P(A)P(B) + \text{ error terms},$$

where the error terms are nonnegative in the case of increasing events A and B, and arise in a natural way related to the structure of the underlying probability space in general. In this paper we carry out this project for the case of a product measure P associated with finitely-valued random variables. We give an expansion

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formula for $P(AB) - P(A)P(B)$ in terms of the probabilities of "pivotal events" determined by A and B. Precise definitions are given below. Our expansion formula can be viewed as an "FKG equality".

The idea for pivotal events first appeared in Russo (1981) and it plays a prominent role in percolation theory. Our expansion connects this idea of "pivotal event" with the FKG inequality. Our second theorem provides an application of our FKG expansion formula to obtain a zero–one law for product measures. It says that an event A is "nonpivotal" if and only if $P(A) = 0$ or 1. We then apply this new zero–one law to get results for random variables in random environments. An application to Bayesian statistics of our final corollary is also discussed in the context of a simple model for human sex ratios.

Let $S = \{1,2,\dots,m\}$ and $\Lambda = \{1,2,3,\dots\}$. Let $X_1, X_2, X_3, \dots$ be independent random variables with $P(X_i = \alpha) = p_i(\alpha) > 0$ so that $\sum_{\alpha=1}^{m} p_i(\alpha) = 1$ for all $i \in \Lambda$. Let $\Omega = S^\Lambda$.

For $x = (x_1, x_2, \dots) \in \Omega$, $b \in \Lambda$ and $\alpha \in S$, define $x^{\alpha b} \in \Omega$ by

$$x_i^{\alpha b} = \begin{cases} x_i & \text{for } i \neq b, \\ \alpha & \text{for } i = b. \end{cases}$$

Let F denote the σ-algebra of subsets of Ω generated by $\{X_i\colon i \in \Lambda\}$, where $X_i(x) = x_i$. For $A \in F$, define

$$A_b(\alpha,\beta) = \{x \in \Omega\colon x^{\alpha b} \in A \text{ and } x^{\beta b} \notin A\}.$$

We describe $A_b(\alpha,\beta)$ as the event that b is $\alpha\beta$ pivotal for A. Note that $A_b(\alpha,\beta)$ does not depend on x_b, i.e., $A_b(\alpha,\beta) \in \sigma(X_i\colon i \neq b)$, the σ-algebra generated by all X_i except for $i = b$. The indicator function for $A_b(\alpha,\beta)$ satisfies

$$1_{A_b(\alpha,\beta)}(x) = 1_A(x^{\alpha b}) 1_{A^c}(x^{\beta b}), \tag{1}$$

where A^c denotes the complement of A. The notion of "pivotal events" when $m=2$ has appeared in percolation theory, as for example in Russo (1981) and Yang and Zhang (1992).

Consider a duplicate system. Let $\Omega^2 = \Omega \times \Omega$ and $P^2 = P \times P$, where P is the probability measure on Ω generated by the random variables $\{X_i\}$. Define a projection mapping

$$\phi_k : \Omega \times \Omega \to \Omega$$

for each $k = 1,2,3,\dots$ by

$$\phi_k(x,y) = (y_1, y_2, \dots y_{k-1}, x_k, x_{k+1}, \dots).$$

Let $\Gamma_k(A) = \phi_k^{-1}(A) = \{(x,y) \in \Omega \times \Omega\colon (y_1, y_2, \dots, y_{k-1}, x_k, x_{k+1}, \dots) \in A\}$.

Lemma 1. *Let $A, B \in F$. Then $\lim_{n\to\infty} P^2(\Gamma_1(A)\Gamma_n(B)) = P(A)P(B)$.*

Proof. For any $\varepsilon > 0$, there exists k and $B^k \in \sigma(X_1,\dots,X_k)$ such that $P(B \Delta B^k) < \varepsilon$. This implies $|P(B) - P(B^k)| < \varepsilon$. Then

$$\begin{aligned} |P^2(\Gamma_1(A)\Gamma_n(B)) - P(A)P(B)| &\leqslant |P^2(\Gamma_1(A)\Gamma_n(B)) - P^2(\Gamma_1(A)\Gamma_n(B^k))| \\ &\quad + |P^2(\Gamma_1(A)\Gamma_n(B^k)) - P(A)P(B^k)| + |P(A)P(B^k) - P(A)P(B)|. \end{aligned} \tag{2}$$

The second term on the right-hand side of (2) equals zero if $n \geqslant k+1$. The third term is bounded by $P(A)\varepsilon \leqslant \varepsilon$. The first term equals

$|E^2(1_{\Gamma_1(A)} 1_{\Gamma_n(B)}) - E^2(1_{\Gamma_1(A)} 1_{\Gamma_n(B^k)})| \leqslant E^2|1_{\Gamma_n(B)} - 1_{\Gamma_n(B^k)}| = P(B \Delta B^k) < \varepsilon$ because $P^2(\Gamma_n(C)) = P(C)$ for any $C \in F$. $\square$

Theorem 1. *For any* $A, B \in F$,

(a) $$P^2(\Gamma_1(A)\Gamma_k(B)) = P^2(\Gamma_1(A)\Gamma_r(B)) + \sum_{b=k}^{r-1} \sum_{\substack{1\leqslant \alpha,\beta\leqslant m \\ \alpha\neq\beta}} p_b(\alpha)p_b(\beta)[P^2(\Gamma_1(A_b(\alpha,\beta))\Gamma_b(B_b(\alpha,\beta))) - P^2(\Gamma_1(A_b(\alpha,\beta))\Gamma_b(B_b(\beta,\alpha)))] \quad \text{for all } r>k.$$

(b) $$P^2(\Gamma_1(A)\Gamma_k(B)) = P(A)P(B) + \sum_{b=k}^{\infty} \sum_{\substack{1\leqslant \alpha,\beta\leqslant m \\ \alpha\neq\beta}} p_b(\alpha)p_b(\beta)[P^2(\Gamma_1(A_b(\alpha,\beta))\Gamma_b(B_b(\alpha,\beta))) - P^2(\Gamma_1(A_b(\alpha,\beta))\Gamma_b(B_b(\beta,\alpha)))].$$

(c) *In particular*,

$$P(AB) = P(A)P(B) + \sum_{b=1}^{\infty} \sum_{\substack{1\leqslant \alpha,\beta\leqslant m \\ \alpha\neq\beta}} p_b(\alpha)p_b(\beta)[P^2(\Gamma_1(A_b(\alpha,\beta))\Gamma_b(B_b(\alpha,\beta))) - P^2(\Gamma_1(A_b(\alpha,\beta))\Gamma_b(B_b(\beta,\alpha)))].$$

Remark 1. If A or B is in the σ-algebra, σ-$(X_1, X_2, \ldots, X_n)$ for some n, then the sums in parts (b) and (c) of Theorem 1 involve only finitely many terms; the upper limit ∞ may be replaced by n.

Proof. Part (b) follows from Lemma 1 and part a by letting $r\to\infty$. Part (c) follows from part (b) by choosing $k=1$. To prove part (a), write

$$\begin{aligned} P^2(\Gamma_1(A)\Gamma_k(B)) - P^2(\Gamma_1(A)\Gamma_r(B)) &= \sum_{b=k}^{r-1} [P^2(\Gamma_1(A)\Gamma_b(B)) - P^2(\Gamma_1(A)\Gamma_{b+1}(B))] \\ &= \frac{1}{2}\sum_{b=k}^{r-1} [E^2(1_{A'}1_{B'}) + E^2(1_{A''}1_{B''}) - E^2(1_{A'}1_{B''}) - E^2(1_{A''}1_{B'})], \end{aligned} \quad (3)$$

where $A' = \Gamma_1(A)$, $B' = \Gamma_b(B)$, $A'' = \{(x,y)\colon (x_1, x_2, \ldots, x_{b-1}, y_b, x_{b+1}, \ldots) \in A\}$, and $B'' = \Gamma_{b+1}(B) = \{(x,y)\colon (y_1, y_2, \ldots, y_b, x_{b+1}, x_{b+2}, \ldots) \in B\}$. The last equality follows from the symmetry of x_b and y_b. Thus,

$$E^2(1_{A''}1_{B''}) = E^2(1_{A'}1_{B'}) = P^2(\Gamma_1(A)\Gamma_b(B))$$

and

$$E^2(1_{A''}1_{B'}) = E^2(1_{A'}1_{B''}) = P^2(\Gamma_1(A)\Gamma_{b+1}(B)).$$

The right-hand side of (3) equals

$$\frac{1}{2}\sum_{b=k}^{r-1} E^2[(1_{A'} - 1_{A''})(1_{B'} - 1_{B''})]. \quad (4)$$

To evaluate (4), we consider the following four disjoint sets:

$$C_{ij} = \{1_{A'} - 1_{A''} = i \text{ and } 1_{B'} - 1_{B''} = j\},$$

where i and $j=\pm 1$. Then (4) may be rewritten as

$$\frac{1}{2}\sum_{b=k}^{r-1}[P^2(C_{11})+P^2(C_{-1-1})-P^2(C_{1-1})-P^2(C_{-11})]. \tag{5}$$

Let

$$C_{ij}^{\alpha\beta}=C_{ij}\cap\{(x,y)\colon x_b=\alpha, y_b=\beta\}.$$

Then,

$$C_{ij}=\bigcup_{\substack{1\leqslant\alpha,\beta\leqslant m\\ \alpha\neq\beta}} C_{ij}^{\alpha\beta},$$

is a disjoint union. Observe that $(x,y)\in C_{11}^{\alpha\beta}$ if and only if $[x\in A_b(\alpha,\beta)$ and $x_b=\alpha,\ y_b=\beta]$ and $[(y_1,y_2,\dots,y_{b-1},x_b,x_{b+1},\dots)\in B_b(\alpha,\beta)$ and $x_b=\alpha, y_b=\beta]$. In other words,

$$C_{11}^{\alpha\beta}=\Gamma_1(A_b(\alpha,\beta))\cap\Gamma_b(B_b(\alpha,\beta))\cap\{x_b=\alpha,y_b=\beta\}.$$

Thus,

$$P^2(C_{11})=\sum_{\substack{1\leqslant\alpha,\beta\leqslant m\\ \alpha\neq\beta}} p_b(\alpha)p_b(\beta)P^2(\Gamma_1(A_b(\alpha,\beta))\Gamma_b(B_b(\alpha,\beta))),$$

where we have used the fact, for example, that $\Gamma_1(A_b(\alpha,\beta))\cap\Gamma_b(B_b(\alpha,\beta))$ does not depend on x_b and y_b. The same argument shows that

$$P^2(C_{-1-1}^{\alpha\beta})=p_b(\alpha)p_b(\beta)P^2(\Gamma_1(A_b(\beta,\alpha))\Gamma_b(B_b(\beta,\alpha))).$$

Summing on α and β yields $P^2(C_{-1-1})=P^2(C_{11})$. A similar argument shows that

$$P^2(C_{-11})=P^2(C_{1-1})=\sum_{\substack{1\leqslant\alpha,\beta\leqslant m\\ \alpha\neq\beta}} p_b(\alpha)p_b(\beta)P^2(\Gamma_1(A_b(\alpha,\beta))\Gamma_b(B_b(\beta,\alpha))).$$

Substituting these last expressions into (5) yields the desired result. □

Remark 2. Statement (c) is what we refer to as an expansion of correlations. Note that if A is an increasing event, then $A_b(\alpha,\beta)=\varnothing$ when $\beta>\alpha$ Therefore (c) immediately implies the following FKG inequality for finitely-valued, independent random variables.

Corollary 1. *Let $A,B\in F$. If A and B are increasing events, then $P(AB)\geqslant P(A)P(B)$.*

Statement (c) of Theorem 1 also gives a characterization for A and B to be independent, viz., A and $B\in F$ are independent if and only if

$$\sum_{b=1}^{\infty}\sum_{\substack{1\leqslant\alpha,\beta\leqslant m\\ \alpha\neq\beta}} p_b(\alpha)p_b(\beta)\,[P^2(\Gamma_1(A_b(\alpha,\beta))\Gamma_b(B_b(\alpha,\beta)))-\ P^2(\Gamma_1(A_b(\alpha,\beta))\Gamma_b(B_b(\beta,\alpha)))]=0.$$

Definition. An event $A\in F$ is said to be nonpivotal if $P(A_b(\alpha,\beta))=0$ for all $b\in\Lambda$ and all $\alpha,\beta\in S$. Let U be the class of all nonpivotal events.

Lemma 2. *U is a σ-algebra and the tail field $T\subset U$.*

Proof. Since Ω_b and $\emptyset_b$, are both empty, Ω and $\emptyset$ are elements of U. Let b be fixed. We write x^α for $x^{\alpha b}$ and x^β for $x^{\beta b}$. By (1) $A \in U$ if and only if for P-a.e. x,

$$1_{A_b(\alpha,\beta)}(x) = 1_A(x^\alpha)1_{A^c}(x^\beta) = 0. \tag{6}$$

Now suppose $A, B \in U$. Then

$$\begin{aligned}1_{(A\cup B)_b(\alpha,\beta)}(x) &= 1_{A\cup B}(x^\alpha)1_{(A\cup B)^c}(x^\beta)\\ &= [1_A(x^\alpha) + 1_B(x^\alpha) - 1_A(x^\alpha)1_B(x^\alpha)]1_{A^c}(x^\beta)1_{B^c}(x^\beta) = 0\end{aligned}$$

P-a.s., by (6). Therefore $A, B \in U$ implies $A \cup B \in U$. Since $(A^c)_b(\alpha,\beta) = (A)_b(\beta,\alpha)$, we have $A^c \in U$ if and only if $A \in U$. Therefore U is an algebra. It now suffices to show that U is a monotone class. Let $A_n \in U$, $A_n \subset A_{n+1}$. Then

$$\begin{aligned}1_{(\bigcup_{n=1}^{\infty} A_n)_b(\alpha,\beta)}(x) &= 1_{\bigcup_{n=1}^{\infty} A_n}(x^\alpha)1_{\bigcap_{n=1}^{\infty} A_n^c}(x^\beta)\\ &= \lim_{N\to\infty}[1_{\bigcup_{n=1}^{N} A_n}(x^\alpha)1_{\bigcap_{n=1}^{N} A_n^c}(x^\beta)]\\ &= \lim_{N\to\infty} 1_{(\bigcup_{n=1}^{N} A_n)_b(\alpha,\beta)}(x) = 0,\end{aligned}$$

P-a.e., where we have used the result that U is an algebra. By the Monotone Class Theorem, U is a σ-algebra.

Let $A \in T$. Then A does not depend on x_b for any $b \in \Lambda$. Therefore $A_b(\alpha,\beta) = \emptyset$ for all α, β. Hence $T \subset U$. □

Theorem 2. *$A \in U$ if and only if $P(A) = 0$ or 1.*

Proof. Assume that $A \in U$. Then $P^2(\Gamma_1(A_b(\alpha,\beta))\Gamma_b(B_b(\alpha,\beta))) \leqslant P^2(\Gamma_1(A_b(\alpha,\beta))) = P(A_b(\alpha,\beta)) = 0$. By Theorem 1(c), with $B = A$, $P(A) = P(A)^2$.

Conversely, assume $P(A) = 0$ or 1. If $P(A) = 1$, then $P(A^c) = 0$, and if $A^c \in U$, then $A \in U$ by Lemma 2. It therefore suffices to consider only the case $P(A) = 0$. By definition, $A_b(\alpha,\beta) \cap \{x_b = \alpha\} \subset A$. Therefore,

$$\begin{aligned}P(A_b(\alpha,\beta)) &= P(A_b(\alpha,\beta) \cap \{x_b = \alpha\}) + P(A_b(\alpha,\beta) \cap \{x_b \neq \alpha\})\\ &= P(A_b(\alpha,\beta) \cap \{x_b \neq \alpha\}) = P(A_b(\alpha,\beta))P(\{x_b \neq \alpha\}).\end{aligned}$$

The last equality follows because $A_b(\alpha,\beta)$ does not depend on x_b. Since $P(\{x_b = \alpha\}) > 0$,

$$P(A_b(\alpha,\beta)) = 0.$$

Thus, $A \in U$. □

We next consider a sequence of independent random variables $X_1, X_2, X_3, \ldots$, where $\{P(X_i = \alpha)\} = \{p_i(\alpha)\}$ is also a sequence of independent random variables. Let μ_i be a probability measure on the portion Q of the hyperplane in $\mathbf{R}^m$ determined by $\sum_{\alpha=1}^m p_i(\alpha) = 1$ with each $p_i(\alpha) > 0$. Define $I = Q^\Lambda$ and let $\mu(\mathrm{d}p) = \prod_{i\in\Lambda}\mu(\mathrm{d}\boldsymbol{p}_i)$ where $\boldsymbol{p}_i = (p_i(1), \ldots, p_i(m))$. For each fixed $p = (\boldsymbol{p}_i\colon i \in \Lambda) \in I$, let P_p be the product measure on Ω for the independent random variables $X_1, X_2, X_3, \ldots$ satisfying $\{P_p(X_i = \alpha)\} = p_i(\alpha)$.

Let $\tilde{\Omega} = \Omega \times I$ and let $\tilde{P}$ be the probability measure on $\tilde{\Omega}$ defined by $\tilde{P}(\mathrm{d}(x,p)) = \mu(\mathrm{d}p)\, P_p(\mathrm{d}x)$. Note that under $\tilde{P}$, $\{X_i\}$ is a sequence of independent random variables with $\tilde{P}(X_i = \alpha) = \int \boldsymbol{p}_i(\alpha)\,\mathrm{d}\mu_i$, and Theorem 2 applies to any event $\tilde{A} \subseteq \tilde{\Omega}$ with $\tilde{A} \in \sigma(X_i, i \in \Lambda)$.

Corollary 2. *Let $A \subseteq \Omega$ and $A \in \sigma(X_i,\ i \in \Lambda)$. If for μ almost every p, $P_p(A) = 0$ or 1, then either $P_p(A) = 0$ for μ almost every p, or $P_p(A) = 1$ for μ almost every p.*

Proof. If for μ almost every p, $P_p(A) = 0$ or 1, then for all b, $P_p(A_b(\alpha, \beta)) = 0$, by Theorem 2. For any set $B \in \sigma(X_i,\ i \in \Lambda)$, define $\tilde{B} = \{(x, p)\colon x \in B\}$. Let

$$\bar{p} = \left(\int \boldsymbol{p}_1 \, \mathrm{d}\mu_1, \int \boldsymbol{p}_2 \, \mathrm{d}\mu_2, \int \boldsymbol{p}_3 \, \mathrm{d}\mu_3, \dots \right).$$

Then

$$P_{\bar{p}}(A_b(\alpha, \beta)) = \tilde{P}(\tilde{A}_b(\alpha, \beta)) = \int \mathrm{d}\mu(p) P_p(A_b(\alpha, \beta)) = 0.$$

From Theorem 2, it follows that $P_{\bar{p}}(A) = \tilde{P}(\tilde{A}) = 0$ or 1.

If $\tilde{P}(\tilde{A}) = \int \mathrm{d}\mu(p) P_p(A) = 0$, then $P_p(A) = 0$ for μ almost every p. Similarly, if $\tilde{P}(\tilde{A}) = 1$, then $P_p(A) = 1$ for μ almost every p. □

Consider a function f which is σ-$(X_1, X_2, \dots)$ measurable.

Let $\mathrm{Var}_p(f) = E_p(f^2) - (E_p(f))^2$ where E_p denotes expectation with respect to P_p. It is easily shown that $\mathrm{Var}(f) = E(\mathrm{Var}_p(f)) + \mathrm{Var}(E_p(f))$. Therefore, $0 \leqslant E(\mathrm{Var}_p(f)) \leqslant \mathrm{Var}(f)$.

Corollary 3. *With the notation above,* $E(\mathrm{Var}_p(f)) = 0$ *if and only if* $\mathrm{Var}(f) = 0$.

Proof. If $E(\mathrm{Var}_p(f)) = \int \mathrm{d}\mu(p) \mathrm{Var}_p(f) = 0$, then $\mathrm{Var}_p(f) = 0$ for μ almost every p. This implies that $f(x) = c(p)$, a constant for μ almost every p. Therefore, $P_p\{x\colon f(x) > \alpha\} = 0$ or 1 for P_p almost every x and all real α. As in the proof of Corollary 2, it follows that $P_{\bar{p}}\{x\colon f(x) > \alpha\} = 0$ or 1. Thus, $f(x)$ equals a constant $c = E_{\bar{p}}(f(x))$ almost surely, $P_{\bar{p}}$. Since $\int \mathrm{d}\mu(p) E_p(|f(x) - c|^2) = 0$, $\mathrm{Var}(f) = 0$. The other direction is clear. □

Corollary 4. *Let $S_n = X_1 + \cdots + X_n$. If for μ almost every p, $\lim_{n\to\infty} S_n/n = c(p)$, a constant P_p-a.e., then $\lim_{n\to\infty} S_n/n$ is constant $\tilde{P}$ almost everywhere.*

Proof. Let $f(x) = \lim_{n\to\infty} S_n/n$. By assumption, for μ almost every p, $f(x) = c(p)$, a constant depending on p, for P_p almost every x. By Corollary 3, $f(x)$ is constant $\tilde{P}$ almost everywhere. □

Remark 3. It is possible to prove Corollary 4 using conditioning arguments and Kolmogorov's criterion for the convergence of averaged sums of independent random variables with appropriate restrictions on the variances.

We give an application of Corollary 4 relating to human sex ratios at birth, and Bayesian statistics. It is fairly well established that the ratio of males to all infants at birth is approximately 0.51. This ratio appears to be stable with respect to locality, ethnicity, and time for sufficiently large populations (US Census Bureau, 1998; US National Center for Health Statistics, 1998). In spite of the global stability of this ratio, there is evidence that the probability p for a male infant varies among individual couples. Evidence exists (James, 1990) that p may range as widely as 0.31–0.83.

How might this phenomenon be explained? Let us assume a population of n births and let X_i be a random variable which takes the value 1 if the ith birth is male and 0 if it is female. We will assume that the collection $\{X_i\}$ is independent. As above, let $\{P(X_i = 1)\} = \{p_i\}$ also be a collection of random variables. Assume the p_i's are equal to the common random variable P, whose prior distribution we denote by $\pi(p)$.

Conditioning on the event $\{P = p\}$, for a fixed p, and letting n go to infinity gives almost surely,

$$\frac{X_1 + \cdots + X_n}{n} \to p$$

and, in general, without conditioning,

$$\frac{X_1 + \cdots + X_n}{n} \to P.$$

But assuming the validity of the demographic and medical references cited above, we are forced to conclude that this approach does not provide a good model for sex ratios, in the sense that large sample means do not approximate the limiting ratio (unless P were constant, which is not supported by the evidence).

A better model results from the assumption that $p_1, p_2, p_3, \ldots$ are i.i.d. random variables with distribution $\pi(p)$. In this case, intuitively, the sample mean of the collection $\{X_i\}$ should tend to $\hat{p}$, the mean of $\pi(p)$, for almost every given $p_1, p_2, p_3, \ldots$. By Corollary 4 this indeed follows; the sample means approximate for almost all p_i's the limiting global constant ratio of males to all infants. Thus our Corollary 4 provides an explanation for the individual variability of the probabilities for gender as well as the global stability of the sex ratio.

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References

Fortuin, C.M., Kasteleyn, P.W., Ginibre, J., 1971. Correlation inequalities on some partially ordered sets. Commun. Math. Phys. 22, 89–103.

Harris, T.E., 1960. A lower bound for the critical probability in a certain percolation process. Proceedings of the Cambridge Philosophical Society, vol. 56, pp. 12–20.

James, W.H., 1990. Variations of the human sex ratio at birth. Fertil Steril. 54 (5), 956–957.

Russo, L., 1981. On the critical percolation probabilities. Probab. Theory Related Fields 56, 229–237.

U.S. Census Bureau, 1998. World wide statistics for 0 to 4 year olds. http://www.census.gov/ftp/pub/ipc/www/idbsum.html.

U.S. National Center for Health Statistics, 1998. Private communication. ("Per your request, NCHS reports (for the United States) that in 1995, there were 1,996,355 male live births compared with 1,903,234 female live births. These numbers yielded a sex ratio of 1049 male per 1000 female live birth, similar to the sex ratio reported in 1994 and for the last 50 years".)

Yang, W.S., Zhang, Y., 1992. A note on differentiability of the cluster density for independent percolation in high dimensions. J. Statist. Phys. 66, 1123–1138.