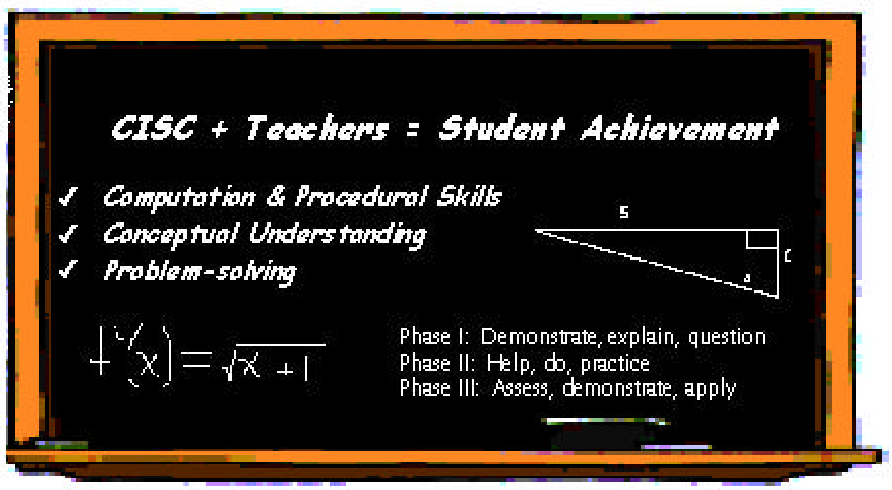
CISC - Curriculum & Instruction Steering Committee

California County Superintendents Educational Services Association

Primary Content Module IV NUMBER SENSE: Factors of Whole Numbers

The Winning EQUATION

A HIGH QUALITY MATHEMATICS PROFESSIONAL DEVELOPMENT PROGRAM FOR TEACHERS IN GRADES 4 THROUGH ALGEBRA II



STRAND: NUMBER SENSE: Factors of Whole Numbers

MODULE TITLE: **PRIMARY CONTENT MODULE IV**

MODULE INTENTION: The intention of this module is to inform and instruct participants in the underlying mathematical content in the area of factoring whole numbers.

THIS ENTIRE MODULE MUST BE COVERED IN-DEPTH.

The presentation of these Primary Content Modules is a departure from past professional development models. The content here, is presented for individual teacher's depth of content in mathematics. Presentation to students would, in most cases, not address the general case or proof, but focus on presentation with numerical examples.

TIME: 2 hours

PARTICIPANT OUTCOMES:

- •Demonstrate understanding of factors of whole numbers.
- •Demonstrate understanding of the principles of prime and composite numbers.
- •Demonstrate how to determine the greatest common factor and the least common multiple when relating two whole numbers.

PRIMARY CONTENT MODULE IV
NUMBER SENSE: Factors of Whole Numbers

Facilitator's Notes

Pre-Post Test

Ask participants to take the pre-test. After reviewing the results of the pre-test proceed with the following lesson on factors of whole numbers.

T-1

Divisability: While expressions of the form $a = b \cdot q + r$ enable us to divide any whole number a by any counting number b, assuming $a > b$, special importance is attached to the case when $r = 0$. A number $b > 0$ is called a **divisor** or **factor** of a if there exists a whole number q so that $a = b \cdot q$.

Since $0 = b \cdot 0$, 0 is divisible by any $b > 0$.

Whenever $a > 0$ and $a = b \cdot q$, there is a rectangular array of $b \cdot q$ dots that corresponds to the factorization of a by b.

T-2

Zero and One: Note that,

$a = 1 \cdot a$ or $a \div 1 = a$

and

$0 = a \cdot 0$ or $0 \div a = 0$

leads to the conclusion that 0 is divisible by any number $a \neq 0$ and any number a is divisible by 1.

T-3

Caution: Division by zero is not allowed.

Suppose $a \div 0 = q$ Then $a = 0 \cdot q$.

This is impossible if $a \neq 0$. If $a = 0$, any value of q works, but for division there can only be one answer.

T-4

An example is $a = b \cdot q$, assuming $a > b$,

$24 = 4 \cdot 6$
a factors as $b \cdot q$
24 factors as $4 \cdot 6$

Have participants think about how to represent 24 as factors another way, and how to represent factors of 7.

Factoring is an important skill for later applications (e.g., it arises in the addition of fractions) and a concept of interest in its own right. An engaging way to introduce students to this topic is to relate it to the study of prime numbers.

T-5

A whole number is said to be **prime** if it has exactly two factors: one and itself. This definition keeps 1 from being a prime. A number > 1 that is not prime is called a **composite**. The first of these definitions leads to the following list of primes:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37,…

Primes have been studied for thousands of years, and many of their important properties were established by the ancient Greeks. Among these is the fact that the above list goes on indefinitely; i.e., **there is no largest prime**.

T-6

Show participants short cuts for testing if whole numbers are divisible by other whole numbers.

T-6A
T-6B/H-6
T-6C

Number	Shortcut
2	ones digit is 0 or even
3	sum of digits is divisible by 3
4	last two digits are divisible by 4
5	ones digit is 0 or 5
6	rules for 2 and 3 both work
8	last 3 digits are divisible by 8
9	sum of digits are divisible by 9
10	ones digit is 0

[Optional:
T-20 through T-23]

Use appendix section on justification for divisibility rules at this point, if desired.

Every whole number > 1 can be written as the product of prime factors.

T-7

For example: "Chipaway" technique

$$24 = 2 \bullet 12$$
$$= 2 \bullet 2 \bullet 6$$
$$= 2 \bullet 2 \bullet 2 \bullet 3$$
$$= 2^3 \bullet 3$$

"Split Asunder" Technique

$$24 = 4 \bullet 6$$
$$= 2 \bullet 2 \bullet 2 \bullet 3$$
$$= 2^3 \bullet 3$$

"Chipaway" technique

$$1200 = 2 \bullet 600$$

$= 2 \cdot 2 \cdot 300$
$= 2 \cdot 2 \cdot 150 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 75$
$= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 25$
$= 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 \cdot 5 \cdot 5$
$= 2^4 \cdot 3 \cdot 5^2$

"Split Asunder" Technique

$1200 = 30 \cdot 40$
$= 5 \cdot 6 \cdot 5 \cdot 8$
$= 5 \cdot 2 \cdot 3 \cdot 5 \cdot 2 \cdot 2 \cdot 2$
$= 2^4 \cdot 3 \cdot 5^2$

T-8

These techniques may be accomplished with the process of factor trees. Transparency T-8 demonstrates this process.

T-8A/H-8

Have participants use both techniques on 60 and 500 and draw the corresponding factor tree.

T-9

The **Fundamental Theorem of Arithmetic** asserts that except for order, you will obtain the same list of primes regardless of the method used to arrive at a prime factorization.

It states that every composite number greater than one can be expressed as a product of prime numbers. Except for the order in which the prime numbers are written, this can only be done in one way.

T-10A

Factors and GCFs: Aside from being able to factor whole numbers into products of primes, it is also important to be able to develop lists containing **all** factors (prime and composite) of a particular number.

For example,

T-10B

The factors of 24 are: 1, 2, 3, 4, 6, 8, 12, 24.

The factors of 37 are: 1, 37.

The factors of 64 are: 1, 2, 4, 8, 16, 32, 64.

Except in the case of perfect squares, factors appear in pairs whose product is the number being factored.

T-11/H-11

Have participants find the factors of 32 and of 80 using H-11. Have participants discuss the difference between "finding all factors" and "finding the prime factorization." Example: A list of all factors of 80 is 1, 2, 4, 5, 8, 10, 16, 20, 40, 80 because 80 is divisible by all of these. The prime factorization of 80 is $2 \cdot 2 \cdot 2 \cdot 2 \cdot 5 = 2^4 \cdot 5$ because 2 and 5 are both primes.

T-12

Given two numbers such as 24 and 64, we can form a list of **common factors**. Referring to the above lists of factors of 24 and 64, we conclude that

The common factors of 24 and 64 are 1, 2, 4, and 8.

The last list leads to the important concept of **greatest common factor (GCF)**:

The greatest common factor of 24 and 64 is 8.

This is sometimes written as GCF(24, 64) = 8.

A) List of factors

30: 1, 2, 3, 5, 6, 10, 15, 30
96: 1, 2, 3, 4, 6, 8, 12, 16, 24, 32, 48, 96

Common factors of 30 and 96 are 1, 2, 3, 6
GCF(30, 96) is 6.

B) List of factors

90: 1, 2, 3, 5, 6, 9, 10, 15, 18, 30, 45, 90
75: 1, 3, 5, 15, 25, 75

Common factors of 90 and 75 are 1, 3, 5, 15
GCF(90, 75) is 15.

T-13A
T-13B

It is possible to determine GCF(*a*,*b*) from the prime factorizations of *a* and *b*. Follow the examples on the slides.

T-13C/H-13

A) $30 = 2 \cdot 3 \cdot 5 = 2 \cdot 3 \cdot 5$
$96 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 = 2^5 \cdot 3$

$GCF(30, 96) = 2 \cdot 3 = 6$

B) $90 = 2 \cdot 3 \cdot 3 \cdot 5 = 2 \cdot 3^2 \cdot 5$
$75 = 3 \cdot 5 \cdot 5 = 3 \cdot 5^2$

$GCF(90, 75) = 3 \cdot 5 = 15$

C) $4500 = 2^2 \cdot 3^2 \cdot 5^3$
$4050 = 2^1 \cdot 3^4 \cdot 5^2$

$\text{GCF}(4500, 4050) = 2^1 \cdot 3^2 \cdot 5^2 = 450$

T-14A

Closely related to GCF is the concept of **least common multiple (LCM)**. A somewhat awkward way of finding LCM(30, 96) is to write lists of multiples of 30 and of 96. Surely 30 • 96 is on both of these lists. We are, however, looking for the **smallest** number common to both lists.

Here we find:

Multiples of 30:
30, 60, 90, 120, 150,…, 450, 480, 510,…, 2880, 2910,…
Multiples of 96:
96, 192, 288, 384, 480, 776,…, 2880, 2976,…

so that LCM(30, 96) = 480.

T-14B

However, LCM(30, 96) can also be found as the smallest product of primes that contains the prime factorizations of both 30 and 96. Recalling that $30 = 2 \cdot 3 \cdot 5$ and that $96 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 3 = 2^5 \cdot 3$, we find that

$\text{LCM}(30, 96) = 2^5 \cdot 3 \cdot 5 = 480$

T-14C/H-14

Have participants find LCM(24, 64) and LCM(32, 48) using prime factorization.

T-15A

The characterization of GCF and LCM in terms of prime factorization leads to the following important fact:

$$\textbf{GCF}(a, b) \cdot \textbf{LCM}(a, b) = a \cdot b.$$

Having already found that GCF(30, 96) = 6, we could now find LCM(30, 96) as

$$\text{LCM}(30, 96) = \frac{30 \text{ x } 96}{\text{GCF}(30, 96)} = \frac{30 \text{ x } 96}{6} = 5 \text{ x } 96 = 480.$$

T-15B

Explanation for this formula.

T-15C/H-15

After recalling that GCF(24, 64) = 8, ask participants to find LCM(24, 64) by applying GCF(a, b) • LCM($\hat{a}$, b) using H-15.

T-16

LCM's play an important role in the addition of fractions. In order to add $\frac{1}{12}+\frac{3}{8}$, we have to rewrite this problem in terms of equivalent fractions with a common denominator.

One possible choice for a common denominator is 8 • 12, leading to $\frac{8}{96}+\frac{36}{96}=\frac{44}{96}=\frac{11}{24}$.

A more convenient choice is the *least* common denominator, which is LCM(12, 8) = 24. This leads to $\frac{2}{24}+\frac{9}{24}=\frac{11}{24}$.

T-16A/H-16

Have participants add fractions by finding least common denominator on worksheet H-16.

T-16B

Practice word problems for LCM and GCF. Cover the answers.

T-17

In order to show that a given number is prime, it is necessary to show that its only factors are 1 and the number itself. For example,

The factors of 113 are: 1, 113

and for that reason 113 is prime.

Enrichment – Determining Whether a Number is Prime
A number that is a perfect square is surely not prime. Recalling that the factors of all other non-prime numbers occur in pairs, it is not necessary to check **all** smaller numbers as possible factors.

For example, to determine whether 137 is prime, we note that 137 < 144, where 144 is a perfect square. For any pair of factors whose product is 137, one of the factors would be less that $\sqrt{137}$. Since

$$\sqrt{121} < \sqrt{137} < \sqrt{144}$$
$$11 < \sqrt{137} < 12$$

it is sufficient to confirm that the numbers 2, 3, 4, 5, 6, 7, 8, 9, 10, and 11 are not factors of 137. Having done this, we can conclude that 137 is prime.

T-18

In fact, it is not even necessary to check **all** whole numbers between 1 and 11. If 6 were a factor of 137, its prime factors (notably 2 and 3) would be factors of 137 as well. On this basis, we need only check that the primes 2, 3, 5, 7, and 11 are not factors of 137.

More generally, to establish that *a* is prime, it is sufficient to confirm that all primes less than $\sqrt{a}$ fail to be factors of *a*.

T-19/H-19

Ask participants to determine whether the numbers 223 and 179 are prime. What is the smallest list of numbers that must be checked as possible factors of 223 and of 179?

Appendix

T-20
T-21
T-22
T-23

These slides give more detailed justification for the rules for determining divisibility by 2 and 3. The facilitator may use these slides at his or her discretion. They may be used immediately following slide T-6A.

Standards covered in this module:

Grade 4 Number Sense

4.0 Students know how to factor small whole numbers.

4.1 Understand that many whole numbers break down in different way.

4.2 Know that numbers such as 2, 3, 5, 7, and 11 do not have factors except 1 and themselves and that such numbers are called prime numbers.

Grade 5 Number Sense

1.4 Determine the prime factors of all numbers through 50 and write the numbers as the product of their prime factors using exponents to show multiples of a factor.

Grade 6 Number Sense

2.4 Determine the least common multiple and greatest common divisor of whole numbers; use them to solve problems with fractions.

Factors of Whole Numbers
Pre- Post-Test

1. List all the prime numbers less than 25:

2. Write the prime factorization for each number:

   a. 30 =
   b. 48 =
   c. 37 =
   d. 112 =

3. List <u>all</u> factors of

   a) 30
   b) 48
   c) 37
   d) 112

4. What is the greatest common factor of 48 and 112?

5. What is the least common multiple of 128 and 96?

6. State the Fundamental Theorem of Arithmetic.

Factors of Whole Numbers
Pre- Post-Test Answer Key

1. List all the prime numbers less than 25:

   2, 3, 5, 7, 11, 13, 17, 19, 23

2. Write the prime factorization for each number:

   a. 30 = **2 • 3 • 5**

   b. 48 = **2 • 2 • 2 • 2 • 3**
   = **2^4 • 3**

   c. 37 = **37**

   d. 112 = **2 • 2 • 2 • 2 • 7**
   = **2^4 • 7**

3. List <u>all</u> factors of

   a) 30 **1, 2, 3, 5, 6, 10, 15, 30**

   b) 48 **1, 2, 3, 4, 6, 8, 12 16, 24, 48**

   c) 37 **1, 37**

   d) 112 **1, 2, 4, 7, 8, 14, 16, 28, 56, 112**

4. What is the greatest common factor of 48 and 112?

   16

5. What is the least common multiple of 128 and 96?

   384

6. State the Fundamental Theorem of Arithmetic

   Every number has a unique prime factorization.

Divisibility

A whole number a is divisible by another (smaller) counting number b if $a = bq$ where q is a whole number.

In this case we write $a \div b = q$

b is a **factor** or **divisor** of a

Example: $a = 15$ is divisible by $b = 3$

because $15 = 3q$ where $q = 5$

Equivalently, $15 \div 3 = 5$

Notice that $a = bq$ is a special case of $a = bq + r$ when $r = 0$. So a is divisible by b means that the remainder is zero when a is divided by b.

Zero and One

Any whole number *a* is divisible by 1.

$a \div 1 = a$ because $a = 1 \cdot a$

0 is divisible by any number a ≠ 0

$0 \div a = 0$ because $0 = a \cdot 0$

Be careful!

$0 \bullet a = 0$

means that you can divide a number into zero

What about dividing a number <u>by</u> zero?

$a \div 0$ does not make sense

suppose $a \div 0 = q$
then $a = 0 \bullet q$

This is impossible if $a \neq 0$.

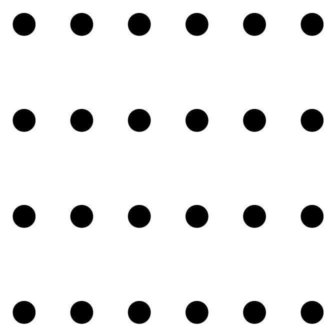
If $a = 0$ then any value of q works,
but for division there can only be one answer.

Visualization of $a = bq$

$$24 = 4 \cdot 6$$

$q = 6$

$b = 4$



a) Is there another rectangular array for 24?

b) How many arrays are there for 7?

Answer: For 7 there is only one array.

7

1

$$7 = 1 \cdot 7$$

because 7 is a “prime” number

A whole number is called **prime** if it has exactly two factors,

One and itself.

The list of primes begins:

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37,……..

There is <u>no</u> largest prime

A number > 1 that is not prime is called

composite

Divisibility Rules

There are well-known shortcuts for testing if whole numbers are divisible by other whole numbers.

Number	Shortcut
2:	Ones digit is 0 or even (0, 2, 4, 6, 8)
3:	Sum of digits is divisible by 3
4:	Last two (tens and ones) digits are divisible by 4
5:	Ones digit is zero or 5
6:	Rules for 2 and 3 both work
8:	Last 3 digits (hundreds, tens, and ones) are divisible by 8
9:	Sum of digits are divisible by 9
10:	Ones digit is zero

Practice without calculator

1. Is 324 divisible by 4?

2. Is 324 divisible by 3?

3. Is 324 divisible by 10?

Why does the divisibility rule for 2 work?

$$354 \div 2 = \frac{1}{2}\left(3 \bullet 10^2 + 5 \bullet 10 + 4\right)$$

$$\frac{3 \bullet 10^2}{2} + \frac{5 \bullet 10}{2} + \frac{4}{2}$$

Since 10 is divisible by 2, so is 10^2

Therefore, $\frac{3 \bullet 10^2}{2} = 3 \bullet \frac{10^2}{2}$ and $\frac{5 \bullet 10}{2} = 5 \bullet \frac{10}{2}$ are both whole numbers.

This is true even if 3 in the hundred's column and 5 in the ten's column were replaced by other numbers.

So, 354 ÷ 2 is a whole number exactly when the number in the one's column is divisible by 2. The number in the one's column must be 0, 2, 4, 6 or 8.

Why does the divisibility by 3 rule work?

Example: Is 651 divisible by 3?

Yes, because 6 + 5 + 1 = 12 is divisible by 3. Why does this work?

$$651 \div 3 = \frac{1}{3}\left(6 \bullet 10^2 + 5 \bullet 10 + 1\right)$$

$$= \frac{1}{3}[6(99+1) + 5(9+1) + 1]$$

$$= \frac{1}{3}[6 \bullet 99 + 6 + 5 \bullet 9 + 5 + 1]$$

$$= \frac{1}{3}[(6 \bullet 99 + 5 \bullet 9) + 6 + 5 + 1]$$

$$= \frac{6 \bullet 99}{3} + \frac{5 \bullet 9}{3} + \frac{6+5+1}{3}$$

Since any multiple of 99 and any multiple of 9 are divisible by 3, 651 is divisible by 3 if and only if 6 + 5 + 1 is divisible by 3.

Every composite number > 1 can be written as the product of primes.

For example:

“Chipaway” technique for 24

$$\begin{aligned} 24 &= 2 \bullet 12 \\ &= 2 \bullet 2 \bullet 6 \\ &= 2 \bullet 2 \bullet 2 \bullet 3 \\ &= 2^3 \bullet 3 \end{aligned}$$

“Split Asunder” technique for 24

$$\begin{aligned} 24 &= 4 \bullet 6 \\ &= 2 \bullet 2 \bullet 2 \bullet 3 \\ &= 2^3 \bullet 3 \end{aligned}$$

“Chipaway” technique for 1200

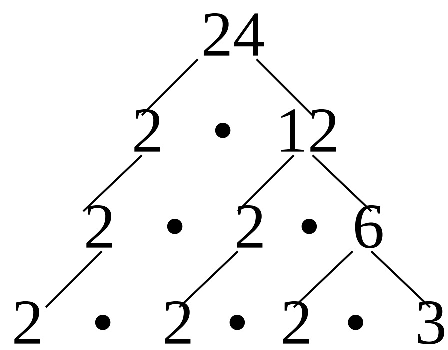
$$\begin{aligned} 1200 &= 2 \bullet 600 \\ &= 2 \bullet 2 \bullet 300 \\ &= 2 \bullet 2 \bullet 2 \bullet 150 \\ &= 2 \bullet 2 \bullet 2 \bullet 2 \bullet 75 \\ &= 2 \bullet 2 \bullet 2 \bullet 2 \bullet 3 \bullet 25 \\ &= 2 \bullet 2 \bullet 2 \bullet 2 \bullet 3 \bullet 5 \bullet 5 \\ &= 2^4 \bullet 3 \bullet 5^2 \end{aligned}$$

“Split Asunder” technique for 1200

$$\begin{aligned} 1200 &= 30 \bullet 40 \\ &= 5 \bullet 6 \bullet 5 \bullet 8 \\ &= 5 \bullet 2 \bullet 3 \bullet 5 \bullet 2 \bullet 2 \bullet 2 \\ &= 2^4 \bullet 3 \bullet 5^2 \end{aligned}$$

Factor trees can be useful to arrive at prime factorizations.

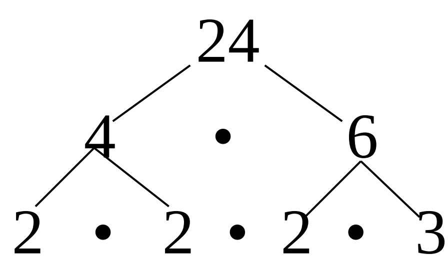
Using the "Chipaway" technique for 24



leading to:

$$24 = 2 \cdot 2 \cdot 2 \cdot 3$$
$$= 2^3 \cdot 3$$

Using the "Split Asunder" technique for 24



leading to:

$$24 = 2 \cdot 2 \cdot 2 \cdot 3$$
$$= 2^3 \cdot 3$$

Factors of Whole Numbers - Worksheet

Factor 60 and 500 by drawing factor trees for both techniques.

“Chipaway” Technique:

60 = 500 =

“Split Asunder” Technique:

60 = 500 =

Fundamental Theorem of Arithmetic:

Every composite number greater than one can be expressed as a product of prime numbers. Except for the order in which the prime numbers are written, this can only be done in one way.

Except for order, you will obtain the same list of primes regardless of the method used to arrive at a prime factorization.

Factors

Factors of 24

$$
\begin{aligned}
24 = \; & 1 \cdot 24 \\
& 2 \cdot 12 \\
& 3 \cdot 8 \\
& 4 \cdot 6
\end{aligned}
$$

So the list of all factors of 24 is

1, 2, 3, 4, 6, 8, 12, 24

Factors of 37

$37 = 1 \cdot 37 = 37 \cdot 1$

So the list of all the factors is

1, 37

Factors of 64

$$
\begin{aligned}
64 = \; & 1 \cdot 64 \\
& 2 \cdot 32 \\
& 4 \cdot 16 \\
& 8 \cdot 8
\end{aligned}
$$

So the list of factors of 64 is

1, 2, 4, 8, 16, 32, 64

(Note 8 is only listed once)

Factors

Factors of 24 are: 1, 2, 3, 4, 6, 8, 12, 24.

Factors of 37 are: 1, 37.

Factors of 64 are: 1, 2, 4, 8, 16, 32, 64.

Except for perfect squares, factors appear in pairs whose product is the number being factored.

1, 2, 3, 4, 6, 8, 12, 24

1, 37

1, 2, 4, 8, 16, 32, 64

Practice

Find the prime factors of 32 and 80.

Find all factors of 32 and 80.

Common Factors

Recalling the following lists of factors

24: 1, 2, 3, 4, 6, 8, 12, 24

64: 1, 2, 4, 8, 16, 32, 64

We see that the common factors of 24 and 64 are:

1, 2, 4 and 8

This leads to the **greatest common factor (GCF)**:

The GCF of 24 and 64 is 8

or GCF(24, 64) = 8

Practice:

a) Find the GCF(30, 96)

b) Find the GCF(90, 75)

GCF by Prime Factorization

It is also possible to determine GCF(a,b) from each number's prime factorization.

Example: Find GCF (360, 270)

$$360 = 2^3 \bullet 3^2 \bullet 5^1$$

$$270 = 2^1 \bullet 3^3 \bullet 5^1$$

Choose the smallest power of each prime that occurs in either list.

$$\text{GCF}(360, 270) = 2^1 \bullet 3^2 \bullet 5^1 = 90$$

Question: How does this work if there are different primes for each number?

Answer: Use the zero exponent.

GCF and the Zero Exponent

Example: Find GCF(84, 90)

$$84 = 2^2 \cdot 3 \cdot 7$$

$$90 = 2 \cdot 3^2 \cdot 5$$

Rewrite so that all primes appear in both lists. Use the zero exponent.

$$84 = 2^2 \cdot 3^1 \cdot 5^0 \cdot 7^1$$
$$90 = 2^1 \cdot 3^2 \cdot 5^1 \cdot 7^0$$

Now choose each prime raised to the lowest power.

$$\begin{aligned} GCF(84, 90) &= 2^1 \cdot 3^1 \cdot 5^0 \cdot 7^0 \\ &= 2 \cdot 3 \cdot 1 \cdot 1 \\ &= 6 \end{aligned}$$

GCF Practice

Find the GCF of 30 and 96 by both methods.

Find the GCF of 90 and 75 by both methods.

Find the GCF of 4500 and 4050 by prime factorization.

Least Common Multiple

One way to find LCM(30, 96) is to write lists of multiples of 30 and of 96.

The least common multiple is the **smallest** number common to both lists of multiples.

Multiples of 30:
30, 60, 90, 120, 150,…450, 480,
510,…,2880, 2910,…

Multiples of 96:
96, 192, 288, 384, 480, 776,…,2880, 2976,…

From these lists we see that LCM(30, 96) = 480.

However, LCM(30, 96) can also be found using prime factorizations.

$$30 = 2^2 \cdot 3^1 \cdot 5^1$$

$$96 = 2^5 \cdot 3^1 \cdot 5^0$$

Notice that $5^0 = 1$, so the prime factorization of 96 is really $2^5 \cdot 3$.

To find LCM's, make sure that each prime occurs in both lists.

To find the **LCM**, choose each prime raised to the **highest** power from either list.

$$\text{LCM}(30, 96) = 2^5 \cdot 3^1 \cdot 5^1$$

$$= 480$$

Least Common Multiple

Worksheet

Use prime factorization to find

LCM(24, 64)

LCM(32, 48)

The characterization of GCF and LCM in terms of prime factorization leads to:

$$\text{GCF}(a, b) \bullet \text{LCM}(a, b) = a \bullet b\ .$$

Example: GCF(30, 96) • LCM(30, 96) = 30 • 96

GCF(30, 96) = 6

LCM(30, 96) = 480

$$6 \bullet 480 = 30 \bullet 96$$

$$2880 = 2880$$

Using GCF(30, 96) = 6, we **could** have found LCM (30, 96) as

$$\text{LCM}(30,96) = \frac{30 \bullet 96}{\text{GCF}(30,\ 96)} = \frac{30 \bullet 96}{6}$$

$$= 5 \bullet 96$$

$$= 480$$

$$\text{LCM}(a,b) = \frac{a \bullet b}{\text{GCF}(a,\ b)}$$

Why does

GCF(a,b) • LCM(a,b) = $a \cdot b$?

Consider an example:

$a = 4500 = 2^2 \cdot 3^2 \cdot 5^3$

$b = 4050 = 2^1 \cdot 3^4 \cdot 5^2$

GCF(*a*, *b*) $= 2^1 \cdot 3^2 \cdot 5^2$

(Choose the smallest powers)

LCM(*a*, *b*) $= 2^2 \cdot 3^4 \cdot 5^3$

(Choose the largest powers)

But $a \cdot b = 2^2 \cdot 3^2 \cdot 5^3 \cdot 2^1 \cdot 3^4 \cdot 5^2$,

and this is the same as

GCF(*a*, *b*) • LCM (*a*, *b*)

Both are 18,225,000

Least Common Multiples Worksheet

Find LCM(24, 64) by applying

GCF(a, b) • LCM(a, b) = a • b.

Find LCM(32, 48) by applying

GCF(a, b) • LCM(a, b) = a • b

LEAST COMMON MULTIPLE AND ADDITION OF FRACTIONS

Least common multiples help with the addition of fractions.

For example

$$\frac{1}{12} + \frac{3}{8} = ?$$

We rewrite this problem in terms of equivalent fractions with a common denominator.

One possible choice for a common denominator is 8 • 12,

$$\frac{8}{96} + \frac{36}{96} = \frac{44}{96} = \frac{11}{24}$$

Or,

by using LCM(12, 8) = 24

$$\frac{2}{24} + \frac{9}{24} = \frac{11}{24}$$

Addition of Fractions Worksheet

Find the sums for the following fractions by using the least common denominator.

1. $\frac{2}{3} + \frac{4}{5} =$

2. $\frac{1}{3} + \frac{5}{12} =$

3. $\frac{2}{34} + \frac{5}{24} =$

4. $\frac{1}{30} + \frac{3}{96} =$

5. $\frac{5}{16} + \frac{3}{56} =$

6. $\frac{2}{51} + \frac{5}{68} =$

Sample Word Problems

Try these problems:

1. Two joggers are running around a track. One jogger runs a lap in 6 minutes and the other takes 10 minutes. If they start at the same time and place, how long will it take for them to meet at the starting place at the same time.

2. A 48 member band follows a 54 member band in a parade. If the same number of players must be in each row of both bands, what is the largest number of players that can be placed in each row? Assume each row has the same number of players.

Answers:

1. LCM(6, 10) = 30
2. GCF(48, 54) = 6

Enrichment
Determining Whether a Number is Prime

For a number that is not a perfect square, factors occur in pairs. To determine whether a number is prime, it is not necessary to check **all** smaller numbers as possible factors.

Is 137 prime?

For any pair of factors whose product is 137, one of the factors would be less than $\sqrt{137}$.

$$\sqrt{121} < \sqrt{137} < \sqrt{144}$$

$$11 < \sqrt{137} < 12$$

It is sufficient to confirm that the numbers

2, 3, 4, 5, 6, 7, 8, 9, 10, and 11

are <u>not</u> factors of 137

to show that 137 is prime.

In fact, it is not necessary to check **all** whole numbers between 1 and 11.

E.g., if 6 were a factor of 137,
its prime factors (2 and 3)
would also be factors of 137.

We only need to check that the **primes** $< \sqrt{137}$
2, 3, 5, 7, and 11
are <u>not</u> factors of 137.

To establish that *a* is prime,
it is sufficient to confirm that

all primes less than or equal to $\sqrt{a}$
<u>fail</u> to be factors of *a*.

Checking Whether a Number is Prime

Worksheet

What is the smallest list of numbers that must be checked as possible factors of 223?

Is 223 prime?

What is the smallest list of numbers that must be checked as possible factors of 179?

Is 179 prime?

Divisibility Rules

Basic ingredients:
"A|B" means "A divides B or in other words, there is an integer *k* such that
$B = A \bullet k$.

Example: 2|10 because 10 = 2 • 5

Theorem: If A|B and A|C , then A|(B+C).

Example: If 2|10 and 2|8, then 2|18

Proof: If A|B , there is an integer k_1, such that $B = A \bullet k_1$. If A|C, there is an integer k_2 such that $C = A \bullet k_2$.

Therefore: $B + C = A \bullet k_1 + A \bullet k_2$

$$= A\ (k_1 + k_2)$$

So, A|(B + C)

Here is another result

Theorem 2: If A|(B + C) and A|C, then A|B

Example: If 2|18 and 2|10, then 2|(18-10) or 2|8.

Proof: If A|(B+C), B + C = A • k_1 , for some integer k_1 . If A|C, C = A • k_2 for some integer k_2. Then

$$B = (B + C) – C = A \bullet k_1 – A \bullet k_2$$

$$= A (k_1 – k_2)$$

Therefore A|B.

Divisibility by 2

$2|(a \bullet 10^2 + b \bullet 10 + c)$ if and only if $2|c$.

Proof: Let $B = a \bullet 10^2 + b \bullet 10$. Then 2|B because $B = 2 \bullet (A \bullet 50 + 6 \bullet 5)$. If $2|c$, then $2|(B + c)$ by Theorem 1. If $2|(B + c)$, then $2|c$ by Theorem 2.

The only single digit numbers divisible by 2 are 0, 2, 4, 6, and 8.

Therefore, $2|(a \bullet 10^2 + b \bullet 10 + c)$ if and only if $c = 0, 2, 4, 6,$ or 8.

Divisibility by 3

$$3|(a \bullet 10^2 + b \bullet 10 + c) \text{ if and only if}$$
$$3|(a + b + c)$$

Proof:

$a \bullet 10^2 + b \bullet 10 + c = a \bullet (99 + 1) + b \bullet (9 + 1) + c$

$= (a \bullet 99 + b \bullet 9) + (a + b + c)$

Let $B = a \bullet 99 + b \bullet 9$. Then $3|B$
Because $B = 3 \bullet (a \bullet 33 + b \bullet 3)$.

Let $C = a + b + c$. If $3|C$, then $3|(B + C)$ by Theorem 1. If $3|(B + C)$ then $3|C$ by Theorem 2.

Example: 3|129 because 3|(1+2+9)

